On the positive weak almost limited operators

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Received 8 November 2014; accepted 5 February 2015
Available online 14 February 2015

Abstract. Using the concept of approximately order bounded sets with respect to a lattice seminorm, we establish some new characterizations of positive weak almost limited operators on Banach lattices. Consequently, we derive some results about the weak Dunford–Pettis* and the Dunford–Pettis* property of \(\sigma\)-Dedekind complete Banach lattices.

Keywords: Weak almost limited operator; The weak Dunford–Pettis* property; Banach lattice

2010 Mathematics Subject Classification: primary 46B42; secondary 46B50; 47B65

1. INTRODUCTION AND NOTATIONS

Throughout this paper \(X, Y\) will denote real Banach spaces, and \(E, F\) will denote real Banach lattices. \(E^+\) denotes the positive cone of \(E\) and \(\text{sol}(A)\) denotes the solid hull of a subset \(A\) of a Banach lattice. The notation \(x_n \perp x_m\) will mean that the sequence \((x_n)\) of a Banach lattice is disjoint, that is, \(|x_n| \wedge |x_m| = 0\), \(n \neq m\). An operator \(T : E \to F\) is positive if \(T(x) \geq 0\) in \(F\) whenever \(x \geq 0\) in \(E\). A lattice seminorm \(\varrho\) on a Banach lattice \(E\) is a seminorm such that for every \(x, y \in E\), \(|x| \leq |y|\) implies \(\varrho(x) \leq \varrho(y)\). The closed unit ball associated to a lattice seminorm \(\varrho\) is defined by \(B_\varrho = \{x \in E : \varrho(x) \leq 1\}\).

The lattice operations in a Banach lattice \(E\) (resp. \(E'\)) are weakly (resp. weak*) sequentially continuous if for every weakly null sequence \((x_n)\) in \(E\) (resp. weak* null sequence \((f_n)\) in \(E'\)), \(|x_n| \to 0\) for \(\sigma(E, E')\) (resp. \(|f_n| \to 0\) for \(\sigma(E', E)\)). Finally, we will use the term

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier
operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. We refer to [1,6] for unexplained terminology of Banach lattice theory and positive operators.

Several types of the Dunford–Pettis property are considered in the theory of Banach lattices. Namely, a Banach lattice $E$ has

- the Dunford–Pettis property, whenever $x_n \xrightarrow{w} 0$ in $E$ and $f_n \xrightarrow{w} 0$ in $E'$ imply $f_n(x_n) \to 0$.
- the Dunford–Pettis* property, whenever $x_n \xrightarrow{w} 0$ in $E$ and $f_n \xrightarrow{w*} 0$ in $E'$ imply $f_n(x_n) \to 0$.
- the weak Dunford–Pettis property (abb. wDP property) [7], whenever $x_n \perp x_m$, $x_n \xrightarrow{w} 0$ in $E$ and $f_n \xrightarrow{w} 0$ in $E'$ imply $f_n(x_n) \to 0$.
- the weak Dunford–Pettis* property (abb. wDP* property), whenever $x_n \xrightarrow{w} 0$ in $E$ and $f_n \perp f_m$, $f_n \xrightarrow{w*} 0$ in $E'$ imply $f_n(x_n) \to 0$.

The wDP* property, introduced recently by J. X. Chen et al. [3], is a weak version of the Dunford–Pettis* property and stronger than the wDP property. Note that the weak Dunford–Pettis property is related to the so called weak almost limited operators. An operator $T : E \to F$ between Banach lattices is said to be weak almost limited [4], whenever

$$x_n \xrightarrow{w} 0 \text{ in } E \text{ and } f_n \perp f_m, f_n \xrightarrow{w*} 0 \text{ in } E' \text{ imply } f_n(T(x_n)) \to 0.$$ 

Clearly, a Banach lattice $E$ has the weak Dunford–Pettis* property if and only if the identity operator on $E$ is weak almost limited.

Let us recall that an operator $T : X \to Y$ is said to be limited if $\|T^*(f_n)\| \to 0$ for every weak* null sequence $(f_n) \subset Y^*$. Furthermore, an operator $T : X \to E$ from a Banach space into a Banach lattice is said to be almost limited [5], if $\|T^*(f_n)\| \to 0$ for every disjoint weak* null sequence $(f_n) \subset E^*$. Accordingly, a Banach lattice $E$ is said to have the Schur property (resp. dual Schur property [5]), if weakly null sequences in $E$ are norm null (resp. disjoint weak* null sequences in $E'$ are norm null). For a $\sigma$-Dedekind complete Banach lattice $E$ (see [5, Theorem 3.3]), the dual Schur property coincide with the so called dual positive Schur property [2], that is, weak* null sequences in $(E')^+$ are norm null. Clearly, a Banach lattice $E$ has the dual Schur property if and only if the identity operator on $E$ is almost limited. For an operator $T : E \to F$ between Banach lattices the following implications are clear:

$$T \text{ is limited } \Rightarrow T \text{ is almost limited } \Rightarrow T \text{ is weak almost limited.}$$

However, there is a weak almost limited operator which needs not to be almost limited (and hence limited). Indeed, the identity operator $I : \ell^1 \to \ell^1$ is weak almost limited as $\ell^1$ has the Schur (wDP*) property. But, as $\ell^1$ does not have the dual positive Schur property [8, Proposition 2.1], $I : \ell^1 \to \ell^1$ is not almost limited. On the other hand, the identity operator on the Banach lattice $c$ is not weak almost limited. Indeed, let $f_n \in c^* = \ell^1$ be such that $f_n = (0, \ldots, 0, 1_{(2n)}, -1_{(2n+1)}, 0, \ldots)$. Then $(f_n)$ is a disjoint weak* null sequence in $c^*$ [3, Example 2.1(2)], and clearly, the sequence $(x_n)$ defined by $x_n = (0, \ldots, 0, 1_{(2n)}, 0, \ldots) \in c$ is weakly null, but $f_n(x_n) = 1$ for all $n$.

In this paper, using the concept of approximately order bounded sets with respect to a lattice seminorm, we establish a characterization of positive weak almost limited operators
(Theorem 2.5), and give consequently in terms of sequences in $E$ and $F'$, several characterizations of positive weak almost limited operators from $E$ into a $\sigma$-Dedekind complete Banach lattice $F$ (Theorem 2.7). As consequences we derive some new characterizations of the wDP* property of a $\sigma$-Dedekind complete Banach lattice (Corollary 2.10). Finally, we establish some sufficient conditions under which the wDP* and the Dunford–Pettis* properties coincide (Corollary 2.12).

2. Main results

The following lemmas will be used throughout this paper.

**Lemma 2.1.** Let $E$ be a Banach lattice, let $\{x_n\} \subset E^+$ be a norm bounded sequence and let $x = \sum_{n=1}^{\infty} 2^{-n} x_n$. Then the sequences $(u_n)$ and $(v_n)$ defined for every $n \geq 2$ by

$$u_n = \left( x_n - 2^n \sum_{i=1}^{n-1} x_i - x \right)^+$$

and

$$v_n = \left( x_n - 4^n \sum_{i=1}^{n-1} x_i - 2^{-n} x \right)^+$$

are a disjoint sequences.

**Proof.** Note that the proof is similar for the two sequences. If $n > m \geq 2$, then we have

$$0 \leq u_n \leq (x_n - 2^n x_m)^+$$. $$0 \leq 2^n u_m \leq 2^n (x_m - 2^{-n} x_n)^+ = (x_n - 2^n x_m)^-.$$ 

So, from $(x_n - 2^n x_m)^+ \perp (x_n - 2^n x_m)^-$ we see that $u_n \perp u_m$ as desired. \hfill \Box

**Lemma 2.2** ([1, Theorem 4.34]). If $A$ is a relatively weakly compact subset of a Banach lattice $E$, then every disjoint sequence in the solid hull of $A$ converges weakly to zero. In particular, for every sequences $(x_n), (y_n) \subset E$ such that $|y_n| \leq |x_n|, y_n \perp y_m$ and $x_n \overset{w}{\rightarrow} 0$ we have $y_n \overset{w}{\rightarrow} 0$.

**Lemma 2.3** ([3, Lemma 2.2]). Let $E$ be a $\sigma$-Dedekind complete Banach lattice. Then for every sequences $(f_n), (g_n) \subset E'$ such that $|g_n| \leq |f_n|, g_n \perp g_m$ and $f_n \overset{w^*}{\rightarrow} 0$ we have $g_n \overset{w^*}{\rightarrow} 0$.

Let us recall that for a lattice seminorm $\varrho$ on a Banach lattice $E$, a subset $A$ of $E$ is said to be approximately order bounded with respect to $\varrho$ if for every $\varepsilon > 0$ there exists $u \in E^+$ such that $A \subset [-u, u] + \varepsilon B_{\varrho}$ (see [6, Remark, p. 73]). Note that from [6, Remark, p. 73], it follows that $A \subset E$ is approximately order bounded with respect to $\varrho$ if and only
if for every $\varepsilon > 0$ there exists $u \in E^+$ such that $\varrho \left( (|x| - u)^+ \right) \leq \varepsilon$ for every $x \in A$. Moreover, if $A \subset E$ is a norm bounded subset, and $T : E \to F$ is a positive operator, then it is easy to see that $\varrho_{T,A} (f) := \sup \{ |f| (T (|x|)) : x \in A \}$ defines a lattice seminorm on $F'$. For the identity operator $I : E \to E$, we get the lattice seminorm on $E'$ defined by $\varrho_A (f) = \sup \{ |f| (|x|) : x \in A \}$.

We shall need the following lemma which characterizes approximately order bounded sequences with respect to a lattice seminorm.

**Lemma 2.4.** A sequence $(x_n)$ of a Banach lattice $E$ is approximately order bounded with respect to a lattice seminorm $\varrho$, if and only if for every $\varepsilon > 0$ there exist $u \in E^+$ and a natural number $k$ such that $\varrho \left( (|x_n| - u)^+ \right) \leq \varepsilon$ for every $n > k$.

**Proof.** The “only if” part is obvious. For the “if” part, let $\varepsilon > 0$. There exist $u \in E^+$ and a natural number $k$ such that $\varrho \left( (|x_n| - u)^+ \right) \leq \varepsilon$ for every $n > k$. Put $v_k = \sqrt[4]{\sum_{n=1}^{k} |x_n|}$ and $v = u + v_k$. So $\varrho \left( (|x_n| - v)^+ \right) \leq \varepsilon$ holds for every $n$. In fact,

- if $n \leq k$ then $\varrho \left( (|x_n| - v)^+ \right) = \varrho (0) = 0 \leq \varepsilon$;
- if $n > k$ then $(|x_n| - v)^+ \leq (|x_n| - u)^+$ and hence $\varrho \left( (|x_n| - v)^+ \right) \leq \varrho \left( (|x_n| - u)^+ \right) \leq \varepsilon$.

This ends the proof. □

Our following result characterizes positive weak almost limited operators from $E$ into $\sigma$-Dedekind complete Banach lattice $F$ through weak* null sequences in $F'$ that are approximately order bounded with respect to a lattice seminorm.

**Theorem 2.5.** Let $E$ and $F$ be two Banach lattices such that $F$ is $\sigma$-Dedekind complete. Then, a positive operator $T : E \to F$ is a weak almost limited if, and only if, each weak* null sequence $(f_n) \subset F'$ is approximately order bounded with respect to the lattice seminorm $\varrho_{T,A}$ for every relatively weakly compact set $A \subset E$.

**Proof.** For the “only if” part, assume by way of contradiction that there exist a weak* null sequence $(f_n) \subset F'$, a relatively weakly compact subset $A \subset E$, such that $(f_n)$ is not approximately order bounded with respect to $\varrho_{T,A}$. That is by Lemma 2.4, there is some $\varepsilon > 0$ so that for each $g \in (F')^+$ and each natural number $k$ we have

$$\varrho_{T,A} \left( (|f_n| - g)^+ \right) > \varepsilon$$

for at least one $n > k$ and thus, $(|f_n| - g)^+ (T |x_n|) > \varepsilon$ for at least one $x_n \in A$. In particular, an easy inductive argument shows that there exist a subsequence of $(f_n)$ (which we still denote $(f_n)$) and a sequence $(x_n) \subset A$ such that

$$\left( |f_n| - 4^n \sum_{i=1}^{n-1} |f_i| \right)^+ (T |x_n|) > \varepsilon$$
holds for all $n \geq 2$. Let $f = \sum_{n=1}^{\infty} 2^{-n} |f_n|$ and 

$$g_n = \left( |f_n| - 4^n \sum_{i=1}^{n-1} |f_i| - 2^{-n} f \right)^+ \quad (n \geq 2).$$

Clearly, $0 \leq g_n \leq |f_n|$ holds for every $n$, and note that from Lemma 2.1 ($g_n$) is a disjoint sequence. Then by Lemma 2.3, $g_n \overset{w^*}{\to} 0$. Hence, as $T$ is weak almost limited we see that $T$ (sol $(A)$) is an almost limited set ([4, Theorem 2.4 (5)]), and then $g_n (T |x_n|) \to 0$. On the other hand, we have for every $n \geq 2$

$$0 < \varepsilon < \left( |f_n| - 4^n \sum_{i=1}^{n-1} |f_i| \right)^+ (T |x_n|) \leq g_n (T |x_n|) + 2^{-n} f (T |x_n|) \to 0,$$

which is impossible.

Now, for the “if” part, let $(x_n) \subset E, (f_n) \subset F'$ be respectively a disjoint weakly null and a disjoint weak* null sequences. We shall see by [4, Theorem 2.4 (3)] that $f_n (Tx_n) \to 0$. To this end, put $A = \{x_n : n \in \mathbb{N}\}$ and let $\varepsilon > 0$. By hypothesis there exists some $g \in (F')^+$ so that $(|f_n| - g)^+ (T |x_n|) \leq g_{T,A} (|f_n| - g)^+ \leq \varepsilon$ holds for all $n$. As $|x_n| \overset{w^*}{\to} 0$ (Lemma 2.2), choose some natural number $m$ such that $g (T |x_n|) \leq \varepsilon$ holds for every $n \geq m$. Thus, for every $n \geq m$ we get

$$|f_n (Tx_n)| \leq |f_n (T |x_n|)|$$

$$\leq (|f_n| - g)^+ (T |x_n|) + g (T |x_n|)$$

$$\leq 2\varepsilon.$$

This show that $f_n (Tx_n) \to 0$, and then $T$ is a weak almost limited operator. \qed

Consequently, $\sigma$-Dedekind complete Banach lattices with the wDP* property enjoy the following lattice approximation property.

**Corollary 2.6.** A $\sigma$-Dedekind complete Banach lattice $E$ has the wDP* property if, and only if, each weak* null sequence $(f_n) \subset E'$ is approximately order bounded with respect to the lattice seminorm $\varrho_A$ for every relatively weakly compact set $A \subset E$.

The following main result gives some characterizations of positive weak almost limited operators (related to sequences with positive terms in statements (6)–(8)).

**Theorem 2.7.** Let $E$ and $F$ be two Banach lattices such that $F$ is $\sigma$-Dedekind complete. Then for a positive operator $T : E \to F$, the following assertions are equivalent:

1. $T$ is weak almost limited.
2. $f_n (Tx_n) \to 0$ for every weakly null sequence $(x_n) \subset E^+$ and every disjoint weak* null sequence $(f_n) \subset F'$.
3. $f_n (Tx_n) \to 0$ for every disjoint weakly null sequence $(x_n) \subset E^+$ and every disjoint weak* null sequence $(f_n) \subset F'$.
4. $f_n (Tx_n) \to 0$ for every disjoint weakly null sequence $(x_n) \subset E^+$ and every disjoint weak* null sequence $(f_n) \subset (F')^+$.
5. $f_n (Tx_n) \to 0$ for every disjoint weakly null sequence $(x_n) \subset E$ and every weak* null sequence $(f_n) \subset F'$. 
(6) \(f_n(Tx_n) \to 0\) for every weakly null sequence \((x_n) \subset E^+\) and every weak* null sequence \((f_n) \subset F'\).

(7) \(f_n(Tx_n) \to 0\) for every weakly null sequence \((x_n) \subset E\) and every weak* null sequence \((f_n) \subset (F')^+\).

(8) \(f_n(Tx_n) \to 0\) for every weakly null sequence \((x_n) \subset E^+\) and every weak* null sequence \((f_n) \subset (F')^+\).

**Proof.** (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (4) Obvious.

(4) \(\Rightarrow\) (1) Follows from ([4], Theorem 2.4 (1 \(\Leftrightarrow\) 7)).

(1) \(\Rightarrow\) (6) Let \((x_n) \subset E^+\), \((f_n) \subset F'\) be respectively a weak null and weak* null sequences, and let \(\varepsilon > 0\). Put \(A = \{x_n : n \in N\}\). From Theorem 2.5, pick some \(g \in (F')^+\) so that \((|f_n| - g)^+(Tx_n) \leq g_{T,A}((|f_n| - g)^+) \leq \varepsilon\) holds for all \(n\), and choose some natural number \(m\) such that \(g(Tx_n) < \varepsilon\) holds for every \(n \geq m\). Now, for every \(n \geq m\) we have

\[|f_n(Tx_n)| \leq |f_n|(Tx_n) \leq (|f_n| - g)^+(Tx_n) + g(Tx_n) \leq 2\varepsilon.\]

This shows that \(f_n(Tx_n) \to 0\).

(6) \(\Rightarrow\) (4) Obvious.

(6) \(\Rightarrow\) (5) If \((x_n) \subset E\) is a disjoint weakly null sequence then by Lemma 2.2, we have \(x_n^+ \rightarrow 0\) and \(x_n^− \rightarrow 0\) and the result follows from the equality \(f_n(Tx_n) = f_n(Tx_n^+) - f_n(Tx_n^-)\).

(5) \(\Rightarrow\) (4) Obvious.

(5) \(\Rightarrow\) (7) Let \((x_n) \subset E\), \((f_n) \subset (F')^+\) be respectively a weak null and weak* null sequences, and let \(\varepsilon > 0\). We claim in this case that there exist \(\varepsilon \in E^+\) and a natural number \(k\) such that

\[f_n\left(T\left(\left|x_n\right| - z\right)^+\right) < \varepsilon\]  

(\#) holds for all \(n > k\). To see this, assume by way of contradiction that (\#) is false. That is, for each \(z \in E^+\) and each \(k\) we have \(f_n\left(T\left(\left|x_n\right| - z\right)^+\right) \geq \varepsilon\) for at least one \(n > k\). An easy inductive argument shows that there exist a subsequence of \((x_n)\) and a subsequence of \((f_n)\) (which we still denote \((x_n)\) and \((f_n)\)) such that

\[f_n\left(T\left(\left|x_n\right| - 2^n \sum_{i=1}^{n-1} |x_i|\right)^+\right) \geq \varepsilon\]

holds for all \(n \geq 2\). Let \(x = \sum_{n=1}^{\infty} 2^{-n} |x_n|\) and \(y_n = \left(\left|x_n\right| - 2^n \sum_{i=1}^{n-1} |x_i| - x\right)^+\). Clearly, \(0 \leq y_n \leq |x_n|\) holds for every \(n \geq 2\), and note that from Lemma 2.1 \((y_n)\) is a disjoint sequence. Then by Lemma 2.2 we get \(y_n \rightarrow 0\). Now, from our hypothesis we have \(f_n(Ty_n) \to 0\). Or for every \(n \geq 2\) we have

\[0 < \varepsilon \leq f_n\left(T\left(\left|x_n\right| - 2^n \sum_{i=1}^{n-1} |x_i|\right)^+\right) \leq f_n(Ty_n) + f_n(Tx) \to 0,\]

which is impossible. Therefore, (\#) is true.
Now, let \( z \in E^+ \) and let \( k \) be such that \((\ast)\) is valid, and choose \( m > k \) such that 
\[
f_n(T(z)) < \varepsilon \text{ holds for every } n \geq m.
\]
Thus, for every \( n \geq m \) we have 
\[
|f_n(Tx_n)| \leq f_n(T|x_n|) \leq f_n(T(|x_n| - z)^+) + f_n(Tz) \leq 2\varepsilon.
\]
This shows that \( f_n(Tx_n) \to 0 \).

(7) \Rightarrow (4) Obvious.

(6) \Rightarrow (8) \Rightarrow (4) Obvious. \( \Box \)

From the statements (6) or (7) or (8) of Theorem 2.7, it follows easily the following corollaries.

**Corollary 2.8.** Let \( E, F \) and \( G \) be a Banach lattices such that both \( F \) and \( G \) are \( \sigma\)-Dedekind complete. If for the scheme of positive operators \( E \overset{T}{\rightarrow} F \overset{R}{\rightarrow} G \), \( T \) or \( R \) is weak almost limited then, so is the product \( RT \). In particular if \( E \) is a \( \sigma\)-Dedekind complete Banach lattice, then the square of each positive weak almost limited operator \( T : E \to E \) is likewise weak almost limited.

**Corollary 2.9.** If \( E \) and \( F \) are a \( \sigma\)-Dedekind complete Banach lattices such that \( E \) or \( F \) has the wDP* property, then each positive operator \( T : E \to F \) is weak almost limited.

The following corollary gives some new characterizations of the wDP* property of a \( \sigma\)-Dedekind complete Banach lattice, other than those established in [3, Theorem 3.2].

**Corollary 2.10.** Let \( E \) be a \( \sigma\)-Dedekind complete Banach lattice. Then the following assertions are equivalent:

1. \( E \) has the wDP* property.
2. \( f_n(x_n) \to 0 \) for every disjoint weak null sequence \( (x_n) \subset E \) and every weak* null sequence \( (f_n) \subset E' \).
3. \( f_n(x_n) \to 0 \) for every weakly null sequence \( (x_n) \subset E^+ \) and every weak* null sequence \( (f_n) \subset E' \).
4. \( f_n(x_n) \to 0 \) for every weakly null sequence \( (x_n) \subset E \) and every weak* null sequence \( (f_n) \subset (E')^+ \).
5. \( f_n(x_n) \to 0 \) for every weak null sequence \( (x_n) \subset E^+ \) and every weak* null sequence \( (f_n) \subset (E')^+ \).

**Corollary 2.11.** Let \( T : E \to F \) be a positive operator from a Banach lattice \( E \) into a \( \sigma\)-Dedekind complete Banach lattice \( F \). If the lattice operations of \( E \) are sequentially weakly continuous (resp. the lattice operations of \( F' \) are sequentially weak* continuous), then the following statements are equivalent:

1. \( T \) is weak almost limited.
2. \( f_n(Tx_n) \to 0 \) for every weakly null sequence \( (x_n) \subset E \) and every weak* null sequence \( (f_n) \subset F' \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( (x_n) \subset E \) and \( (f_n) \subset F' \) be respectively weak null and weak* null sequences. We shall see that \( f_n(Tx_n) \to 0 \).
If the lattice operations of $E$ are sequentially weakly continuous, then the sequences $(x_n^+)$ and $(x_n^-)$ are both weak null. Thus, since $T$ is weak almost limited, by Theorem 2.7(6) we have $f_n(Tx_n^+) \to 0$ and $f_n(Tx_n^-) \to 0$. Now, the result follows from the equality $f_n(Tx_n^+) = f_n(Tx_n^-) - f_n(Tx_n^-)$.

If the lattice operations of $E'$ are sequentially weak* continuous, then the sequences $(f_n^+)$ and $(f_n^-)$ are both weak* null. Thus, since $T$ is weak almost limited, by Theorem 2.7(7) we have $f_n^+(Tx_n) \to 0$ and $f_n^-(Tx_n) \to 0$, and the result follows from the equality $f_n(Tx_n) = f_n^+(Tx_n) - f_n^-(Tx_n)$.

(2) $\Rightarrow$ (1) Obvious.

Note that a Banach lattice which has the wDP* property needs not to have the DP* one (eg $L^1 [0, 1]$, see [3, Proposition 3.3]). However, from the preceding theorem, another corollary can be derived easily.

**Corollary 2.12.** Let $E$ be a $\sigma$-Dedekind complete Banach lattice such that the lattice operations of $E$ are sequentially weakly continuous, or the lattice operations of $E'$ are sequentially weak* continuous. Then $E$ has the wDP* property if and only if it has the DP* property.

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