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Bifurcations and Trajectories Joining Critical Points

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1. INTRODUCTION

There are many physical circumstances that give rise to the question about an ordinary differential equation: Is there a trajectory that, as $t \to +\infty$ and $t \to -\infty$, tends to a given pair of critical points of the equation? A well-known example comes from the equations of gas dynamics, which, under certain assumptions, can be reduced to ordinary differential equations [1]. The critical points of these equations represent uniform flow, and a trajectory between them is a "shock structure" that describes the transition between the two uniform flows.

Unlike questions about the behavior of solutions near critical points, questions concerning trajectories between critical points involve the global behavior of solutions. For this reason, most techniques for investigating their existence are topological (e.g., [1–5]). (See [3] for further references. Exceptions are [6–8].) In this paper, we explore analytic techniques for finding the trajectories. More specifically, we investigate one-parameter families of ordinary differential equations; we show that the same conditions on the equations that imply that there is a pair of critical points bifurcating from a single one are often enough to guarantee the existence of a trajectory joining those two points. This

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technique is complementary to the topological techniques. It gives more information about the nature of trajectory. However, unlike the topological methods, it is useful only for ranges of the parameter near the bifurcation value.

The theorems of this paper are in the spirit of the so-called Hopf bifurcation theorem [9–11]. Their hypotheses are local and easily verifiable assumptions about a one-parameter (or sometimes a two-parameter) family of equations. The conclusions are global: the existence of trajectories joining critical points. The relationship of this paper to the Hopf theorem is closest in Section 7, which deals with a two-parameter family of equations and proves the existence of a one-parameter family of homoclinic orbits. (These are trajectories that approach the same critical point as $t \to +\infty$ and as $t \to -\infty$.)

Section 2 spells out the assumptions on a one-parameter family of vector fields that lead to a bifurcation of a single critical point into exactly two critical points (for a given range of the parameter μ). The hypotheses are weaker than those usually assumed; in particular, it is not assumed that, at the bifurcation value, the critical point has a *simple* zero eigenvalue. In fact, the multiplicity k of the zero determines the qualitative nature of the desired trajectory near the endpoints. This extension was motivated in large part by an example involving shock structures in reaction–diffusion equations in which $k = 3$ [12].

Section 3 discusses the reduction of the problem to a one-parameter family of differential equations on R^k, where k is as above. In most of the paper after this section, it is assumed that this reduction has been made and the resulting equations

$$\dot{X} = F_\mu(X) \tag{1.1}$$

satisfy the hypotheses introduced in Section 3.

Section 4 contains a further reduction of the problem. We show that for each value (in some range) of the bifurcation parameter μ, the k-dimensional system (1.1) can be written as a perturbation of a canonical system:

$$\dot{y}_j = y_{j+1}, \qquad j \neq k, \tag{1.2a}$$

$$\dot{y}_k = \tfrac{1}{2}(y_1^2 - 1). \tag{1.2b}$$

The significance of this reduction is that it reduces the general k-dimensional problem to proving that (1.2) has a locally unique trajectory joining to two critical points and that this trajectory persists under perturbations of the system. That is, if it can be proved for any given k that (1.2) has such a structurally stable trajectory, then for each value of μ (in some range), (1.1) has a locally unique trajectory joining its critical points. The technique of this section involves a change of variables that is μ-dependent and that is singular at $\mu = 0$. Such singular scalings are also in [6–8, 11, 13].

Section 5 uses the results of Section 4 to describe the behavior of trajectories near the critical points of (1.1). For k odd, it emerges that the stable and unstable manifolds of the critical points (for each μ in some range) have the right dimensions for the existence of a unique structurally stable trajectory. For k even, the dimensions cannot be determined without further information.

For $k = 1$, it is clear that (1.2) has a structurally stable trajectory joining its critical points $y_1 = \pm 1$. Section 6 concerns the case $k = 3$ and shows that the stable manifold of $(1, 0, 0)$ transversely intersects the unstable manifold of $(-1, 0, 0)$; hence, there is a trajectory between the critical points that persists under perturbations. We conjecture that, for all odd k, (1.2) is structurally stable with a unique trajectory joining the critical points.

For even k, (1.2) is not structurally stable; indeed, one of the two critical points has a pair of pure imaginary eigenvalues. In Section 7, we discuss the case $k = 2$. Since, for fixed μ in some range, Eqs. (1.1) are perturbations of (1.2), little can be concluded about (1.1). However, there is a one-dimensional "unfolding" of (1.2) whose properties are invariant under small perturbations. This leads to the theorem, referred to above, concerning a two-parameter family of equations $\dot{X} = F_{\mu,\nu}(X)$. It is shown that, under hypotheses close to those used in the previous sections, there is a curve $f(\mu, \nu) = 0$ in parameter-space such that if $f(\mu_0, \nu_0) = 0$, then $\dot{X} = F_{\mu_0,\nu_0}(X)$ has a homoclinic orbit. Furthermore, the one-parameter family of homoclinic orbits bounds a two-parameter family of periodic orbits in (X, μ, ν) space. The existence of a trajectory joining the two critical points of $\dot{X} = F_{\mu,\nu}(X)$ is also discussed.

The techniques of Section 7 include singular change of coordinates as in Section 4. Results related to this section have been obtained independently by Takens [13] and Bogdanov (see [14]).

Section 8 is concerned with iterative techniques for calculating the shock trajectories without first making the reductions done in Section 3. The iteration is closely related to the technique that Foy [6] used to calculate shock structures for hyperbolic systems of conservation laws with viscosity.

Many of the known results [6] about "weak" shocks for hyperbolic systems of conservation laws can be easily rederived and even extended using the analysis of this paper for $k = 1$. (For a system of dimension m, one applies Theorem 2.1 and the reductions of Sections 3 and 4 to each of m families of shocks. Hypothesis 3 of the theorem is guaranteed by the Lax condition of "genuine nonlinearity" [15].) The $k = 1$ case, (as well as the $k = 3$ case), is also used to calculate shock structures in reaction–diffusion equations [12]. The $k = 2$ case gives a tool for finding homoclinic orbits, which, in many physical situations, represent travelling pulses [4, 7, 8, 16, 17]. We do not know of applications for the cases $k > 3$.

2. Curves of Critical Points

Before tackling the question of trajectories joining critical points, we first extend the results in [18] and discuss the existence of critical points for a one-parameter family of ordinary differential equations on R^n. We are especially interested in conditions under which, for each value (in some range) of a parameter μ, there are exactly two critical points near $X \equiv (x_1, ..., x_n) = (0, ..., 0)$. This situation can arise from a bifurcation in either of two ways. In the first, there are two critical points for each sufficiently small $\mu \neq 0$ and these critical points coalesce at $\mu = 0$. In the second, for $\mu < 0$ (resp. $\mu > 0$) there are no critical points near $X = 0$ and for $\mu > 0$ (resp. $\mu < 0$) there are two; i.e., after coalescing, the critical points disappear.

These two situations are described in Theorems 2.1 and 2.2. Neither theorem assumes that, at $\mu = 0$, there is a simple zero eigenvalue for the unique critical point. However, they do assume (Hypothesis 1) that there is a unique eigenvector for the eigenvalue 0. Along with an assumption about the dependence on μ (Hypothesis 2), this is enough to give the existence of a curve of critical points in (X, μ) space for which $X = 0$ only when $\mu = 0$. (For Theorem 2.1, this is proved in [18]; a joint proof for both will be given below.)

These first two hypotheses of each theorem give the weakest conditions on the linear part of the equations, with respect to X and μ, such that for all possible higher-order terms in X and μ, there is exactly one curve of critical points (not counting the μ-axis in Theorem 2.1). (The higher-order terms are $o(\mu, \mu x_i, x_i x_j)$ for Theorem 2.1 and $o(\mu, x_i x_j)$ for Theorem 2.2.) This is shown in [18] for Theorem 2.1; the proof in the case of Theorem 2.2 is similar but easier.

The last assumption in each case, which concerns only the vector field at $\mu = 0$, guarantees that there are, at most, two critical points near $X = 0$ for any μ small enough. This third hypothesis is generic: It is satisfied by an open dense set of vector fields satisfying Hypothesis 1.

THEOREM 2.1. *Let $\dot{X} = F_\mu(X)$ be a one-parameter family of autonomous differential equations on R^n, such that F_μ is C^2 smooth in all its $n + 1$ arguments. Suppose that for each μ sufficiently small, $F_\mu(0) = 0$, so that $\dot{X} = F_\mu(X)$ may be written in the form*

$$\dot{X} = (A + \mu A_1) X + Q(X, X) + R_1(X, \mu), \tag{2.1}$$

where A and A_1 are $n \times n$ matrices. The vector $Q(X, X)$ contains the terms quadratic in the x_i and independent of μ and

$$R_1(X, \mu) = o(\mu x_i, x_i x_j).$$

Assume further, that

1. *The rank of A is $n - 1$. We denote by e the eigenvector with eigenvalue zero.*

2. *The matrix $[A, A_1 e]$, gotten by augmenting A using the vector $A_1 e$, has rank n.*

3. *The matrix $[A, Q(e, e)]$ has rank n.*

Then, there is a curve of critical points $x_i = \bar{x}_i(s)$, $\mu = \bar{\mu}(s)$, (s sufficiently small) such that $\bar{X}(0) = 0$, $\bar{\mu}(0) = 0$, $(d\bar{x}/ds)(0) = e$, and $(d\bar{\mu}/ds)(0) \neq 0$. These are the only critical points in (X, μ) space near $X = 0$, $\mu = 0$ other than additional points on the μ axis. Hence, for each μ sufficiently small, $\mu \neq 0$, there are exactly two critical points (including $X = 0$).

Remark. The first two hypotheses are equivalent to the conditions $(d/d\mu) \det(dF_\mu(0))|_{\mu=0} \neq 0$ and $\det A = 0$. [18].

THEOREM 2.2. *Let $\dot{X} = G_\mu(X)$ be a one-parameter family of autonomous differential equations on R^n such that G_μ is C^2 smooth in all its $n + 1$ arguments. Suppose that $G_0(0) = 0$, so that $\dot{X} = G_\mu(X)$ may be written in the form*

$$\dot{X} = AX + \mu g + Q(X, X) + R_2(X, \mu), \tag{2.2}$$

where A is an $n \times n$ matrix and g is an n-dimensional vector. Here, $Q(X, X)$ is as before and $R_2(X, \mu) = o(\mu, x_i x_j)$.

Assume further that

1. *The rank of A is $n - 1$.*
2. *The matrix $[A, g]$ has rank n.*
3. *The matrix $[A, Q(e, e)]$ has rank n.*

Then, there is a curve of critical points $x_i = \bar{x}_i(s)$, $\mu = \bar{\mu}(s)$, (s sufficiently small) such that $\bar{X}_i(0) = 0$, $\bar{\mu}(0) = 0$, $(d\bar{X}/ds)(0) = e$, $(d\bar{\mu}/ds)(0) = 0$, and $(d^2\bar{\mu}/ds^2)(0) \neq 0$. These are the only critical points near $X = 0$, $\mu = 0$. Hence, for each μ sufficiently small and positive (if $d^2\bar{\mu}/ds^2 > 0$), or negative (if $d^2\bar{\mu}/ds^2 < 0$), there are exactly two critical points, one (denoted $X^+(\mu)$) for some $s > 0$ and another ($X^-(\mu)$) for some $s < 0$.

Proof of Theorems 2.1 *and* 2.2. For later use, we shall first prove a lemma somewhat stronger than needed to prove Theorems 2.1 and 2.2.

LEMMA 2.1. *Let $X = h(Y, \bar{\mu})$, $\mu = \bar{\mu}$ be a smooth (μ-dependent, level preserving) change of coordinates in $R^n \times R$, such that the vectors $h(0, 0) = 0$, and $(\partial h/\partial\bar{\mu})(0, 0) = 0$. Then, the hypotheses of Theorems* 2.1 *and* 2.2 *still hold in the new coordinates.*

Proof. First, we consider only linear changes of variables. Let $X = LY$ where L is a nonsingular $n \times n$ matrix. In terms of the Y variable, Eqs. (2.1) and (2.2) become

$$\dot{Y} = L^{-1}(A + \mu A_1)LY + L^{-1}Q(LY, LY) + L^{-1}R_1(LY, \mu) \tag{2.3}$$

and

$$\dot{Y} = L^{-1}ALY + L^{-1}\mu g + L^{-1}Q(LY, LY) + L^{-1}R_2(LY, \mu). \tag{2.4}$$

Hypothesis 1 is clearly satisfied. Let l be the left eigenvector (with eigenvalue 0) of the matrix A. Hypotheses 2 and 3 of Theorems 2.1 and 2.2 may be rephrased:

2. $l \cdot A_1 e \neq 0$ (resp. $l \cdot g \neq 0$),
3. $l \cdot Q(e, e) \neq 0$.

The left eigenvector of $L^{-1}AL$ is lL and the right eigenvector is $L^{-1}e$. Now, it is easy to check that Eqs. (2.3) and (2.4) satisfy Hypotheses 2 and 3.

To complete the proof of the lemma, we consider a change of variables $X = h(Y, \mu)$, where $h(Y) = Y + H(Y, \mu)$ and $H(Y, \mu) = O(y_i y_j, \mu^2, \mu y_i)$. (This is sufficient since an arbitrary nonsingular change of variables satisfying the hypotheses may be factored into transformations of the above two types.) This transforms (2.1) and (2.2) into

$$\dot{Y} = dh^{-1}[(A + \mu A_1)(h(Y, \mu)) + Q(h(Y, \mu), h(Y, \mu)) + R_1(h(Y, \mu), \mu)], \quad (2.5)$$

$$\dot{Y} = dh^{-1}[A(h(Y, \mu) + \mu g + Q(h(Y, \mu), h(Y, \mu)) + R_2((h(Y, \mu), \mu)]. \quad (2.6)$$

The linear part of each right-hand side is A, so the right and left eigenvectors are e and l. Hypothesis 2 is satisfied since the constant matrix (A_1) multiplying the μ in (2.5) and the constant vector (g) multiplying the μ in (2.6) remain the same. To see that Hypothesis 3 still holds, let $h(Y) \equiv h(Y, 0)$; we note that the quadratic terms of interest in (2.5) and (2.6) come from $dh^{-1}(A(h(Y) + Q(h(Y), h(Y))$. Now, let $Y = Ke$ for any constant K. Then, $Ah(Ke) = AH(Ke)$. That is, in the direction of the eigenvector e, $Ah(Ke)$ and $Q(h(Ke), h(Ke))$ are (at least) quadratic functions of the length of the vector. Therefore, the quadratic terms of (2.5) and (2.6), in the direction of the eigenvector, are obtained by considering only the linear terms of dh^{-1}, the linear terms of h in $Q(h(e), h(e))$, and the quadratic terms of h in $Ah(e)$. Since the linear part of dh^{-1} and h is the identity, to prove Hypothesis 3 it suffices to show that $l \cdot [AH(e) + Q(e, e)] \neq 0$. But $l \cdot AH(e) = 0$ because l is a left eigenvector of A and $l \cdot Q(e, e) \neq 0$.

Theorems 2.1 and 2.2 follow from Lemma 2.1: Let k be the multiplicity of the zero eigenvalue of A. Hypothesis 1 implies that a linear change of coordinates may be made such that, in these coordinates, A has the form

$$A = \begin{pmatrix} A_* & 0 \\ 0 & B \end{pmatrix},$$

where A_* is the $k \times k$ matrix that is identically zero except for the entries above the diagonal, which are one; and B is nonsingular. In these coordinates, $e = (1, 0, \ldots, 0)$. Hypothesis 2 then implies in the case of Theorem 2.1 (resp. 2.2) that the kth entry of the vector $A_1(e)$ (resp. g) is nonzero. Let $F_\mu{}^j(x)$ (resp. $G_\mu{}^j(X)$) be the jth component of $F_\mu(X)$ (resp. $G_\mu(X)$). Then, Hypothesis 3 implies that the coefficient of $x_1{}^2$ in $F_\mu{}^k(X)$ (resp. $G_\mu{}^k(X)$) is nonzero.

Now, the desired curve of critical points is the set of solutions to the n equations $F_\mu(X) = 0$ (resp. $G_\mu(X) = 0$). Using the implicit function theorem, we see that the $n - 1$ equations $F_\mu{}^j(X) = 0$ (resp. $G_\mu{}^j(X) = 0$), $j \neq k$, may be solved for $x_2, \ldots, x_n$ in terms of x_1 and μ; these solutions

$x_j(x_1, \mu)$ are at least quadratic in their dependence on x_1. Substituting these solutions into the kth equation, we see that Hypothesis 2 implies that the equations may be solved for μ in terms of x_1. (For Theorem 2.2 this is straightforward; for Theorem 2.1 one must first divide the equation by x_1.) The parameter s of the conclusion may be chosen to be x_1. Then, the curve of solutions satisfies $\bar{X}(0) = 0$, $\bar{\mu}(0) = 0$, $(d\bar{X}/ds)(0) \neq 0$ in Theorem 2.1; for Theorem 2.2 it implies that $(d\bar{\mu}/ds)(0) = 0$, but that $(d^2\bar{\mu}/ds^2)(0) \neq 0$. This completes the proof.

3. Reduction of the Problem, Center Manifolds

The existence of the trajectory joining critical points is an issue that really depends only on a k-dimensional subsystem of the original equations, where k is the dimension of the Jordan block of the zero eigenvalue of the matrix A. In this section, we show that this is implied by the so-called center manifold theorem [9, 19, 20]. Our hypotheses are those of Theorem 2.1 or 2.2 of the previous section; we also assume that the other $n - k$ eigenvalues of A have nonzero real part.

The center manifold theorem says

THEOREM. *Suppose that a system of ordinary differential equations may be written as*

$$\begin{aligned}\dot{U} &= \mathcal{A}U + \tilde{U}(U, V, W),\\ \dot{V} &= \mathcal{B}V + \tilde{V}(U, V, W),\\ \dot{W} &= \mathcal{C}W + \tilde{W}(U, V, W),\end{aligned} \tag{3.1}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are square matrices whose eigenvalues have their real parts, respectively, >0, $=0$ and <0. *$\tilde{U}$, $\tilde{V}$, and $\tilde{W}$ are C^r differentiable* $(r \geqslant 2)$ *and vanish, along with their derivatives, at* $(U, V, W) = 0$. *Then: There is a smooth manifold $\mathcal{M}^*$ of the same dimension as the vector V, which passes through* $(U, V, W) = 0$, *is tangent there to* $U = 0$, $W = 0$, *and which is invariant under the above equations (i.e., is composed of solution curves of those equations). $\mathcal{M}^*$ is defined only locally in a neighborhood of* $(U, V, W) = 0$ *and is called a "center manifold" for Eqs.* (3.1). *$\mathcal{M}^*$ is parameterized by V; i.e., U and W are given as functions $U_1(V)$, $W_1(V)$. Since $\mathcal{M}^*$ is tangent to* $U = 0$, $W = 0$, *these functions are at least quadratic at the origin.*

Center manifolds are not, in general, uniquely defined. (See [19] for an example.) However, sometimes there are points near the origin that must belong to any such center manifold. These are the points in some neighborhood of $(U, V, W) = 0$ whose trajectories are bounded for all positive and negative times [9, 20]. In particular, all critical points that are close enough to $(U, V, W) = 0$ to belong to such a neighborhood must be on any center manifold for (3.1). Hence:

PROPOSITION 3.1. *Suppose that there is a curve* $(U(s), V(s), W(s))$ *such that* $(U(0), V(0), W(0)) = 0$ *and for each* s, $(U(s), V(s), W(s))$ *is a critical point for* (3.1). *Then, for all sufficiently small* s, $(U(s), V(s), W(s))$ *lies in any center manifold* $\mathscr{M}^*$.

We now go back to Eq. (2.1) or (2.2). We consider these as $(n + 1)$-dimensional systems whose last equation is $d\mu/dt = 0$. Under the hypotheses of Theorem 2.1 or 2.2, $X = 0$, $\mu = 0$ is a critical point of the system (2.1) or (2.2). There are $n - k$ eigenvalues of A that do not belong to the k-dimensional Jordan block of the zero eigenvalue. If these $n - k$ eigenvalues have nonzero real part, then there are exactly $k + 1$ eigenvalues of the linearization (of the $(n + 1)$-dimensional system) at $X = 0$, $\mu = 0$, that have zero real part. Hence, by the center manifold theorem and Proposition 3.1, there is a $(k + 1)$-dimensional invariant subsystem that contains all the critical points of (2.1) or (2.2) for μ sufficiently small. This invariant space is tangent to the μ-axis and the generalized eigenspace of the zero eigenvalue of A.

One may choose coordinates for the $(k + 1)$-dimensional center manifold, (one of them being μ) and extend these coordinates to R^{n+1}. Suppose the old coordinates on R^n are (U, V, W), where U, V, and W are as in (3.1) for the system (2.1) or (2.2) at $\mu = 0$. (This may be achieved by a μ-independent linear change of variables.) If the center manifold in R^{n+1} of (2.1) or (2.2), $\dot{\mu} = 0$, is given by $U = U_1(V, \mu)$, $W = W_1(V, \mu)$, the new coordinates may be chosen as $U' = U - U_1(V, \mu)$, $V' = V$, $W' = W - W_1(V, \mu)$, $\mu' = \mu$. This change of coordinates satisfies the hypotheses of Lemma 2.1. By Lemma 2.1, the hypotheses of Theorems 2.1 and 2.2 are invariant under such changes of coordinates. Hence, for the next few sections, we shall assume $n = k$ and look for the trajectories joining the critical points in the center manifold. In Section 8, we return to the case $n \geqslant k$ to discuss how these trajectories may be calculated without first calculating the center manifold.

4. Scaling: Reduction to Canonical Problems

As we saw in Section 2, the distance between the critical points at any "level" μ is related to the size of μ. More specifically, in the situation of Theorem 2.1, $|X| = O(\mu)$ for X a critical point, $X \neq 0$; for that of Theorem 2.2, the critical points are $O(\mu^{1/2})$. In each of the two cases, we shall make a μ-dependent change of variables that is singular for $\mu = 0$ and that keeps the critical points separated. To first order, this change of variables scales out the dependence on μ. Under these changes of variables, (2.1) or (2.2) may be written

$$\dot{y}_j = y_{j+1} + \epsilon r_j(y_i, \epsilon), \qquad j < k, \tag{4.1a}$$

$$\dot{y}_k = \tfrac{1}{2}(y_1{}^2 - 1) + \epsilon r_k(y_i, \epsilon), \tag{4.1b}$$

where $r_j = O(1)$ and $\epsilon \to 0$ as $\mu \to 0$. (As we shall see below, this is strictly true only for the range of μ in which there is a pair of critical points, i.e., for (2.2) it holds for $\mu > 0$ or $\mu < 0$.) For $\epsilon = 0$, (4.1) is (1.2), which corresponds to the single equation $y^{(k)} = \frac{1}{2}(y^2 - 1)$. The change of variables expresses (2.1) or (2.2) (for fixed $\mu \neq 0$) as (1.2) plus a perturbation that is small when μ is small.

Now, we make the changes of variables referred to above. We assume that the linear parts of (2.1) and (2.2) are in Jordan normal form, i.e., A is a matrix with 1's above the diagonal and zeros everywhere else.

For (2.1): let a be the $(k, 1)$ entry of A_1 and b the kth entry in $Q(e, e)$, where $e = (1, 0,..., 0)$. Let $\epsilon = |(a/b)\,\mu|^{1/k}$. (As seen in Section 2, $a \neq 0$, $b \neq 0$.)

For (2.2): let a be the kth entry in g and b the kth entry in $Q(e, e)$. Let $\epsilon = |(a/b)\,\mu|^{1/2k}$.

For both: Let $t = |2b|^{-1/k}\,\epsilon^{-1}\bar{t}$ and $x_j = b_j \epsilon^{k+j-1} y_j$, where $b_j = |2b|^{(j-1)/k}$.

Note that the variables x_i are scaled differently from each other. This scaling is designed to leave only the terms of (1.2) when $\epsilon = 0$. Up to multiplication by constants such as a and b, this scaling is unique in this respect.

If the variables $\bar{t}$ and $\{y_j\}$ are substituted in (2.1) and (2.2) for t and $\{x_j\}$ and the jth equation is divided by $|2b|^{j/k}\epsilon^{k+j}$, $1 \leqslant j \leqslant k$, one arrives at equations the first $(k - 1)$ of which have the form (4.1a). The last equation becomes

$$\dot{y}_k = \tfrac{1}{2} y_1(\operatorname{sign}(b)\, y_1 + \operatorname{sign}(a\mu)) + \epsilon r_k(y_i, \epsilon) \tag{4.2}$$

in the case of (2.1) and

$$\dot{y}_k = \tfrac{1}{2}(\text{sign}(b)\, y_1^2 + \text{sign}(a\mu)) + \epsilon r_k(y_i\,,\epsilon) \tag{4.3}$$

for (2.2). (The differentiation is now with respect to the new time variable $\bar{t}$, but we suppress the bar.) Note that for (2.2), or equivalently (4.1a) plus (4.3), there is a pair of critical points at level μ (small enough) only when sign $(a\mu) = -\text{sign}(b)$; hence, we assume this.

Except for this restriction on signs, (4.1a) plus (4.2) or (4.3) is indeed equivalent to (4.1). It is easy to see that we may assume, e.g., that $b > 0$ and that $a\mu < 0$, by changing the signs of t and/or some of the $\{y_i\}$. Thus, (2.2) is equivalent to (4.1) for $\mu \neq 0$. It is also easy to make a transformation that takes (4.1a) plus (4.2) into (4.1a) plus (4.3). (It is merely a question of whether we want the "standard" positions of the critical points to be 0 and 1 or -1 and 1.)

For use in Section 7, we note that (4.1) is also equivalent to

$$\begin{aligned} \dot{y}_j &= y_{j+1} + \epsilon r_j(y_i\,,\epsilon), \qquad j < k, \\ \dot{y}_k &= y_1(1 - y_1) + \epsilon r_k(y_i\,,\epsilon). \end{aligned} \tag{4.4}$$

5. Behavior Near the Critical Points

If there is to be a trajectory joining each pair of critical points found in Section 2, the stable manifold [See, e.g., 21] of one of the critical points must intersect the unstable manifold of the other. For this to happen in a manner that cannot be destroyed by small perturbations, these manifolds must intersect transversely, i.e., "in general position." This requires that the sum of their dimensions be greater than k.

For k odd, these dimensions may be deduced almost immediately from the scaled versions of (2.1) and (2.2). The two critical points of (1.2) are $(1, 0,..., 0)$ and $(-1, 0,..., 0)$ and the eigenvalues are, respectively, the roots of $+1$ and the roots of -1. These roots are far away from the imaginary axis. If $k = 2q + 1$, q even, then $q + 1$ (resp. q) of the roots of $+1$ (resp. -1) have positive real part. If q is odd, the number of roots with positive real part is q (resp. $q + 1$). In either case, the number of positive real part eigenvalues of one of the two critical points, plus the number of negative real part eigenvalues of the other equals $2(q + 1) = k + 1$. This is exactly the situation necessary for the stable manifold of one of the critical points to transversely intersect the unstable manifold of the other along a one-dimensional curve.

Now, (4.1) is a perturbation of (1.2). For ϵ sufficiently small, the dimensions of the stable and unstable manifolds of its critical points are the same as those of (1.2). Furthermore, for $\mu \neq 0$, these dimensions are not affected by the scaling (with the exception that the stable and unstable manifolds might have been interchanged if t was changed to $-t$). Hence, for μ sufficiently small, $\mu \neq 0$, we know the dimensions of the stable and unstable manifolds of the critical points of (2.1) and (2.2).

Now, suppose that k is even. Then, if $k = 2q$, q odd, a pair of the kth roots of -1 are pure imaginary; if q is even, two of the kth roots of $+1$ are pure imaginary. It is easy to show that, depending on the nonlinear terms in ϵ and Y, the real part of the eigenvalues near the imaginary axis may have either sign. Hence, the dimensions of the stable and unstable manifolds of one of the critical points of (4.1) (and so (2.1) or (2.2)) are not determined by the hypotheses of Theorem 2.1 or 2.2.

Remark 5.1. The conclusions of this section may be derived without the scaling of Section 5 by using the propositions of [18, pp. 280, 281].

6. The Canonical Problem, $k = 3$

In this section, we prove

Theorem 6.1. *For $k = 3$, (1.2) has a locally unique trajectory that goes from the critical point $(-1, 0, 0)$ to the critical point $(1, 0, 0)$. A trajectory joining two critical points remains even if (1.2) is perturbed.*

We do not show directly that the stable manifold of $(1, 0, 0)$ transversely intersects the unstable manifold of $(-1, 0, 0)$. Instead, we make use of the symmetry of the equations. Eqs. (1.2) are invariant under the transformation $y_1 \to -y_1$, $y_3 \to -y_3$, $t \to -t$. Hence, if there is a trajectory $Y(t)$ satisfying $y_1(0) = 0$, $y_3(0) = 0$, and $Y(t) \to (1, 0, 0)$ as $t \to \infty$, then $Y(t) \to (-1, 0, 0)$ as $t \to -\infty$. Thus, to find a trajectory joining the critical points, it suffices to find a trajectory that passes through the line $y_1 = y_3 = 0$ and that belongs to the stable manifold S of $(1, 0, 0)$. Most of this section is devoted to showing that the two-dimensional manifold T of trajectories passing through $y_1 = y_3 = 0$ has a transverse intersection with S. A further argument (Lemma 6.8) then establishes the transverse intersection of S with the unstable manifold U of $(-1, 0, 0)$.

To show that T transversely intersects S, we shall show that both T

and S transversely intersect the plane $y_1 = 1$ and that $T_1 = T \cap \{y_1 = 1\}$ and $S_1 = S \cap \{y_1 = 1\}$ have a transversal intersection in the plane $y_1 = 1$. We consider T first and give some estimates on the trajectories of T; these estimates imply that T transversely intersects $\{y_1 = 1\}$. They are also used to calculate where T intersects $\{y_1 = 1\}$, which is useful in showing that T_1 and S_1 intersect.

Consider the trajectory $Y(t)$ with initial conditions $(0, v, 0)$. By (1.2b), $\dot{y}_3 \geqslant -1/2$. Hence, for $t \geqslant 0$:

$$\begin{aligned} y_3 &\geqslant -t/2, \\ y_2 &\geqslant v - t^2/4, \\ y_1 &\geqslant vt - t^3/12 \equiv p_1(t). \end{aligned} \tag{6.1}$$

From these, we see that since $p_1(t)$ is monotonically increasing for $0 \leqslant t < 2v^{1/2}$, y_1 is too. Also, if $v > (3/4)^{2/3}$, the value of p_1 at $t = 2(v^{1/2})$ exceeds 1, so that of y_1 must too. Thus, y_1 must cross 1 for some $t = t_0 < 2(v^{1/2})$ and be monotonically increasing, at least on $0 \leqslant t \leqslant 2v^{1/2}$; in particular, $\dot{y}_1(t_0) > 0$. This is true for all v exceeding some minimum value that is not greater than $(3/4)^{2/3} = 0.825...$ (computer calculations suggest that the actual least v is about 0.77). These observations demonstrate

LEMMA 6.1. *T transversely intersects the plane $y_1 = 1$.*

The above idea can be pursued to obtain closer estimates on Y which will enable us to estimate more closely where T intersects $y_1 = 1$. We define the sequence $p_m(t)$, starting with the above p_1, by $\dddot{p}_{m+1} = \frac{1}{2}(p_m{}^2 - 1)$, $p_{m+1}(0) = \ddot{p}_{m+1}(0) = 0$, $\dot{p}_{m+1}(0) = v$. Thus, $p_2 = vt - t^3/12 + v^2t^5/120 - vt^7/2520 + t^9/145{,}152$. We shall be interested in these functions only over subintervals of $I_1 = \{0 \leqslant t \leqslant (12v)^{1/2}\}$, on which $p_1 \geqslant 0$. Since $\dddot{p}_{m+2} - \dddot{p}_{m+1} = \frac{1}{2}(p_{m+1} - p_m)(p_{m+1} + p_m)$ for $m \geqslant 0$ (if we set $p_0 \equiv 0$) and all the p_m for $m \geqslant 1$ have the same initial conditions at $t = 0$, we see inductively (by integrating repeatedly over $(0, t)$) that on I_1, we always have $\ddot{p}_{m+1} - \ddot{p}_m \geqslant 0$, $\dot{p}_{m+1} - \dot{p}_m \geqslant 0$, $p_{m+1} - p_m \geqslant 0$. Thus, the sequences p_m, $\dot{p}_m$, $\ddot{p}_m$ are monotone increasing for each t on I_1. Furthermore, each member of the sequence provides lower bounds for y_1 or its derivatives on I_1. (This is true for p_1 and if it is true for p_m, then $\dddot{y} - \dddot{p}_{m+1} = \frac{1}{2}(y - p_m)(y + p_m) \geqslant 0$ on I_1. As before, this implies that $\ddot{y} \geqslant \ddot{p}_{m+1}$, $\dot{y} \geqslant \dot{p}_{m+1}$, and $y > p_{m+1}$.)

For future reference, we note that since $\dot{p}_1 \geqslant 0$ for $0 \leqslant t \leqslant 2v^{1/2}$, the same is true of $\dot{p}_2$; if one calculates p_2 from the above formula at $v = 0.8$ and $t = 2v^{1/2}$ one obtains $p_2 = 1.0437... > 1$, so in fact, the least v for which y_1 reaches 1 is less than 0.8. Soon, we shall estimate Y fairly closely for this value, 0.8, of v and later use this to show that S_1 intersects T_1.

In a similar way, upper bounds for the components of Y can be obtained. We let $P_1 = vt$ and define P_m by $\ddot{P}_m = \frac{1}{2}(P_m{}^2 - 1)$ with the same initial conditions (those of y_1) as before. (Thus, $P_2 = vt - t^3/12 + v^2t^5/120$.) We have seen that for sufficiently large v, in particular for $v \geqslant 0.8$, y_1 is monotone increasing up to and somewhat past the value 1. Now, so long as $0 \leqslant y_1 \leqslant 1$, we have $\frac{1}{2}(y_1{}^2 - 1) \leqslant 0$; hence, integrating (1.2b), $\ddot{y}_1 \leqslant 0 = \ddot{P}_1$, $\dot{y}_1 \leqslant v = \dot{P}_1$, and $y_1 \leqslant vt = P_1$. Thus, P_1 and its derivatives provide upper bounds for y_1 and its derivatives, on $0 \leqslant t \leqslant t_0$. But also, $\ddot{P}_{m+1} - \ddot{y}_1 = \frac{1}{2}(P_m{}^2 - y_1{}^2) = \frac{1}{2}(P_m - y_1)(P_m + y_1)$. Thus, if $P_m \geqslant y_1$ on $(0, t_0)$, (as is true for $m = 1$), we see, as before, that P_{m+1}, $\dot{P}_{m+1}$, and $\ddot{P}_{m+1}$ also provide upper bounds on y_1, $\dot{y}_1$, and $\ddot{y}_1$; hence, this holds for all m.

Finally, we note that the P_m form a monotone decreasing sequence for a limited but sufficiently extensive range of t. For, $P_2 - P_1 = -t^3(1 - v^2t^2/10)/12 \leqslant 0$ for $v^2t^2 < 10$ and from this, it follows as above that $P_{m+2} \leqslant P_{m+1}$ (together with the first two derivatives) for $m \geqslant 1$ so long as $v^2t^2 < 10$ and $0 \leqslant t \leqslant t_0$. Direct use of the formulas shows that $\dot{P}_2 \leqslant \dot{P}_1$ for $(vt)^2 \leqslant 6$ and $\ddot{P}_2 \leqslant \ddot{P}_1$ for $(vt)^2 \leqslant 3$. These last inequalities are in fact satisfied for $0 \leqslant t \leqslant t_0$; e.g., for $v = 0.8$ we have seen that $t_0 < 2(0.8)^{1/2}$, which gives $(vt_0)^2 < 4(0.8)^3 = 2.048 < 3$.

To obtain fairly accurate estimates of the point on T_1 corresponding to $v = 0.8$ we note first that $\ddot{P}_3 - \ddot{p}_2 = \frac{1}{2}(P_2 - p_1)(P_2 + p_1) \leqslant (v^2t^5/5!)vt$, from which it follows that $\dot{P}_3 - \dot{p}_2 \leqslant 6v^3t^7/7!$, $\dot{P}_3 - \dot{p}_2 \leqslant 6v^3t^8/8!$, and $P_3 - p_2 \leqslant 6v^3t^9/9!$. Direct calculation with $v = 0.8$ gives $p_2(1.647) = 1.00017...$, $p_2(1.643) = 0.998995...$, and $6v^3(1.643)^9/9! = 0.000738...$. The first of these shows that $y_1(1.647) > 1$, while the second two give $P_3(1.643) \leqslant 0.999735$ and hence, $y_1(1.643) < 1$. Thus, y_1 crosses 1 at a value t_0 between 1.643 and 1.647. Furthermore, since $\ddot{p}_2 < 0$ in this interval, $\dot{p}_2$ is decreasing and the same is true of $\dot{P}_3$. Thus, $\dot{p}_2(1.647) \leqslant \dot{p}_2(t_0) \leqslant \dot{y}_1(t_0) \leqslant \dot{P}_3(1.643) \leqslant \dot{p}_2(1.643) + 6v^3(1.643)^8/8!$. Putting in the numbers, this says that

$$0.2770 \leqslant \dot{y}_1(t_0) \leqslant 0.2831. \tag{6.2}$$

Since $\ddot{y}_1(t_0) = 0$ with $\dddot{y}_1 < 0$ for t somewhat less than t_0 and $\dddot{y}_1 > 0$ for t a bit larger than t_0, $\ddot{y}_1$ evidently has a (negative) local minimum at t_0; thus, $\ddot{y}_1(t_0) \leqslant \ddot{y}_1(1.643) \leqslant \ddot{P}_3(1.643) \leqslant \ddot{p}_2(1.643) + 6v^3(1.643)^7/7!$ On the other hand, since p_1 is increasing on the interval and $p_1(1.647) = 0.945\cdots < 1$, we see that $\dddot{p}_2 = (1/2)(p_1{}^2 - 1) < 0$ on the whole interval, so $\ddot{p}_2$ is decreasing (slowly) on it. Thus, $\ddot{y}_1(t_0) \geqslant \ddot{p}_2(t_0) \geqslant \ddot{p}_2(1.647)$. Putting in the numbers here gives the bounds

$$-0.493 \leqslant \ddot{y}_1(t_0) \leqslant -0.472. \tag{6.3}$$

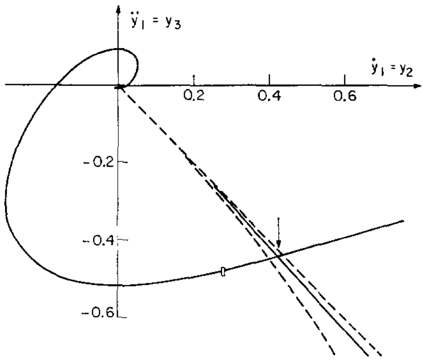

FIG. 1. The lower portion of the spiral curve in the quadrant $y_2 > 0$, $y_3 < 0$ represents the first intersection of trajectories of T with $\{y_1 = 1\}$. There is such a first intersection for all $v \geqslant v_* \sim 0.77$. Some of these trajectories (approximately $0.77 < v < 0.842$) change direction and intersect $\{y_1 = 1\}$ again (from above). The points where these trajectories intersect $\{y_1 = 1\}$ for the second time are drawn in the half plane $y_2 < 0$. A still smaller subset of these trajectories intersect $\{y_1 = 1\}$ for a third time; these intersection points are those in the upper part of the graph, with $y_2 > 0$. T_1 probably contains an infinite spiral whose center is $y_2 = y_3 = 0$. The dashed straight line is the line $y_3 = -y_2$, and the dashed curve tangent to it is the cubic $y_3 = -y_2 - 0.175y_2{}^2 - 0.3y_2{}^3$. S_1 (the solid curve) lies between these two curves, at least for $y_2 \leqslant 0.642$. The intersection ($\downarrow$) of T and S is at $y_2 = 0.428$, $y_3 = -0.441$. The small rectangle indicates bounds given in the text for a point on T.

A graph of T_1 in the plane $y_1 = 1$ is given in Fig. 1. This graph (the spiral curve) was obtained by numerically integrating (1.2) from $(0, v, 0)$ for a number of values of v. The small rectangle indicates the rigorous bounds calculated above for $v = 0.8$. (The actual values obtained numerically for 0.8 are (0.2814, $-$ 0.4781).)

It is fairly clear from the numerical evidence of Fig. 1, considering only the portion of T_1 representing the first intersection with $\{y_1 = 1\}$, that the slope dy_3/dy_2 of the curve is positive. This can be seen directly from the equations. Let $\mathscr{P}_i(v)$, $i = 2, 3$, give the second and third coordinates of the trajectory (beginning at $(0, v, 0)$) at the value of t for which $y_1(t) = 1$ for the first time. (Here, $\mathscr{P}_i$ is analogous to a Poincaré map; it looks at the image of some trajectories as they reach a specified region and it ignores the parameterization of the trajectories.)

LEMMA 6.2. *$d\mathscr{P}_i/dv > 0$ for each v for which $\mathscr{P}_i$* is defined.

Proof. It is useful to write down the variational equations for (1.2). That is, let $z_i(t) = (\partial y_i/\partial v)(t)$. Then, along an orbit $Y(t)$, $Z(t) = (z_1, z_2, z_3)$ satisfies the (nonautonomous) system

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = z_3, \qquad \dot{z}_3 = y_1 z_1. \tag{6.4}$$

To compare orbits starting at $(0, v, 0)$ and $(0, \bar{v}, 0)$ with "nearby" $v, \bar{v}$, we consider the initial conditions $Z(0) = (0, 1, 0)$. We see that if $y_1 > 0$ along an orbit for $t \leqslant \tilde{t}$, then $z_i(t) > 0$ for $0 < t \leqslant \tilde{t}$. First, we conclude that if $v > \bar{v}$ then, along the trajectories $y(t)$ and $\bar{y}(t)$ starting at $(0, v, 0)$ and $(0, \bar{v}, 0)$, $\dot{y}_2(t) > \dot{\bar{y}}_2(t)$. If t_0 and $\bar{t}_0$ are the times at which these orbits intersect $\{y_1 = 1\}$, this implies that $t_0 < \bar{t}_0$.

We wish to show that

$$\lim_{\bar{v} \to v} \frac{\mathscr{P}_i(v) - \mathscr{P}_i(\bar{v})}{v - \bar{v}} > 0.$$

$$\frac{\mathscr{P}_i(v) - \mathscr{P}_i(\bar{v})}{v - \bar{v}} = \frac{y_i(t_0) - \bar{y}_i(t_0)}{v - \bar{v}} + \frac{\bar{y}_i(t_0) - \bar{y}_i(\bar{t}_0)}{v - \bar{v}}.$$

Now,

$$\lim_{v \to \bar{v}} \frac{y_i(t_0) - \bar{y}_i(t_0)}{v - \bar{v}} = z_i(t_0),$$

where $Z(0) = (0, 1, 0)$. As previously described, $z_i(t_0) > 0$. Hence, it remains to show that the second term is nonnegative. Suppose, for definiteness, that $v > \bar{v}$, so $t_0 \leqslant \bar{t}_0$. From (1.2), we see that $\dot{\bar{y}}_3 = \frac{1}{2}(\bar{y}_1^2 - 1) < 0$ for $\bar{y}_1 < 1$. Also, $\dot{\bar{y}}_2 = \bar{y}_3 < 0$ (at least for $0 < \bar{y}_1 < 1$). Hence, since $t_0 < \bar{t}_0$, $\bar{y}_i(t_0) - \bar{y}_i(\bar{t}_0) \geqslant 0$ and we are done.

Since the slope dy_3/dy_2 along T_1 is $(\partial\mathscr{P}_3/\partial v)/(\partial\mathscr{P}_2/\partial v)$, it is positive.

Remark. It is also easy to see from the equations that there is a least value v^* of v such that, if $Y(0) = (0, v, 0)$, then $y_1(t)$ travels monotonically from $y_1 = 0$ to $y_1 = 1$. For $v = v_*$, the trajectory $Y(t)$ hits $\{y_1 = 1\}$ nontransversely, i.e., with $\dot{y}_1 = y_2 = 0$.

Now, we turn to S. It is very easy to calculate the tangent plane to S at $(1, 0, 0)$. The linearization of (1.2) at $(1, 0, 0)$ has the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

whose eigenvalues are the cube roots of unity. Thus, the unstable manifold is tangent to the eigenvector of $\lambda = 1$, which is $(1, 1, 1)$. Since M is orthogonal, the other invariant plane, which is the tangent space to S, is perpendicular to $(1, 1, 1)$; i.e., the plane is $(y_1 - 1) + y_2 + y_3 = 0$. This plane is clearly transversal to the plane $y_1 = 1$.

The intersection of the two planes is the line $y_1 = 1$, $y_3 = -y_2$. This line transversely cuts the curve T_1. To show that S_1 cuts T_1, it suffices to show that S stays close to its tangent plane for a sufficient distance from the critical point, i.e., far enough out to intersect T_1. More precisely, we show that S_1 lies between the line $y_3 = -y_2$ and the cubic $y_3 = -y_2 - 0.175y_2^2 - 0.3y_2^3$ for $y_2 \leqslant 0.641$. Furthermore, along S_1, $dy_3/dy_2 < 0$, at least for $y_2 \leqslant 0.566$.

This will suffice to prove that S_1 and T_1 transversely intersect, since previous calculations showed that T_1 passes through the rectangle $0.277 < y_2 < 0.2831$, $-0.493 \leqslant y_3 \leqslant -0.472$ and has positive slope. (It also extends to arbitrarily large y_2; for $y_1(2/v) \geqslant p_1(2/v) = 2 - (2/3v^3)$, so $t_0 < 2/v$ if $v^3 > 3/2$. Thus, $y_2(t_0) \geqslant p_1(t_0) = v - (t_0^2/4) > v - (1/v^2)$, which $\to \infty$ as v does.) For small y_2, S_1 lies above T_1, but for $0.522 < y_2 < 0.566$ the points of S_1 have $y_3 < -0.522$ and so this part of S_1 lies below T_1. Hence, there is a transversal (and so locally unique) intersection.

The calculations required to obtain the necessary estimates for S_1 are quite long. First, we describe an iterative procedure for calculating the trajectories of the stable manifold of the critical point $y_1 = 1$, $y_2 = 0$, $y_3 = 0$. To find the stable manifold, we set $y = 1 + u$ and look for those solutions of

$$\dddot{u} - u = \tfrac{1}{2}u^2 \tag{6.5}$$

that $\to 0$ for $t \to \infty$. We may normalize the t origin so that the asymptotic form for $t \to \infty$ of a particular such solution is $\kappa e^{-t/2} \sin (3^{1/2}/2)\, t$; thus, κ and t may be regarded as coordinates on the stable manifold. For each κ in $0 \leqslant \kappa \leqslant 1$, we construct a solution iteratively by starting with $u_1 = \kappa e^{-t/2} \sin (3^{1/2}/2)\, t$ and determining u_{n+1} as the (unique) solution of

$$\ddot{u}_{n+1} - u_{n+1} = \tfrac{1}{2} u_n^2, \tag{6.6}$$

for which

$$\lim_{t\to\infty} e^{t/2}(u_{n+1} - u_1) = 0. \tag{6.7}$$

(Since u_1 is the only solution of $\ddot{u} - u = 0$ with $e^{t/2}(u - u_1) \to 0$, the uniqueness of u_{n+1} is clear. Its existence is evident from the variation of constants formula, assuming $u_n \sim u_1$. Note that (6.6) is an ordinary differential equation, not a system, and u_n means the nth iterate; y_2 and y_3 refer to the second and third components of a third-order system.)

We show below that this sequence converges and we obtain estimates for the error $| u - u_n |$. We also consider the functions $w = \partial u/\partial\kappa$ and get error estimates on $| w_n - w |$, where $w_n = \partial u_n/\partial\kappa$. The rest of the calculations involve estimating the intersection of the stable manifold with $\{y_1 = 1\}$, i.e., $\{u = 0\}$. First we show that for each κ, $0 \leqslant \kappa \leqslant 1$, there is a unique value of t, for $0 \leqslant t \leqslant 0.3_\kappa$, at which this intersection takes place. This estimate is used to show that the curve S_1, in the (y_2, y_3) plane, exists for $y_2 \leqslant 0.641$. Further estimates show that S_1 is bounded above by $y_3 = -y_2$ and below by the cubic previously mentioned. These estimates are enough to show the existence of the intersection. The transversality follows from estimates involving $\dot{w}$ and $\ddot{w}$, which show that S_1 has a negative slope, at least for $y_2 \leqslant 0.566$.

To prove the convergence, we use the following:

LEMMA 6.3. *If* $| \ddot{\delta} - \delta | \leqslant e^{-nt/2}$ $(n > 1)$ *and* $e^{t/2}\delta \to 0$ *for* $t \to \infty$, *then*

$$| \delta | \leqslant K_n e^{-nt/2},$$

$$| \dot{\delta} | \leqslant K_n' e^{-nt/2},$$

$$| \ddot{\delta} | \leqslant K_n'' e^{-nt/2},$$

where

$$K_n = \coth [(n-1)\, \pi/2(3^{1/2})]/(1 + (n/2)^3),$$

$$K_n' = \frac{4}{3^{1/2}(1 + (n/2))(n-1)},$$

$$K_n'' = \frac{1}{1 + (n/2)} \left(1 + \frac{4/(3^{1/2})}{n-1}\right).$$

Slightly simpler bounds on K_n *are* $K_2 \leqslant 0.6948$, $K_n \leqslant 8.48/n^3$ *for* $n \geqslant 3$.

Proof. Let $\dddot{\delta} - \delta = \zeta$, or $(D-1)[\ddot{\delta} + \dot{\delta} + \delta] = \zeta$. Since the quantity in the square brackets $\to 0$ for $t \to \infty$, we have $\ddot{\delta} + \dot{\delta} + \delta = -e^t \int_t^\infty e^{-t'} \zeta(t')\, dt' \equiv \xi(t)$. Similarly, solving this equation for δ in terms of ξ, using the conditions at ∞, we get

$$\delta = -\frac{2}{3^{1/2}} \int_t^\infty e^{-(1/2)(t-t')} \sin \frac{3^{1/2}}{2}(t-t')\, \xi(t')\, dt',$$

from which we also get

$$\dot{\delta} = -\frac{2}{3^{1/2}} \int_t^\infty e^{-(1/2)(t-t')} \left[-\frac{1}{2} \sin \frac{3^{1/2}}{2}(t-t') + \frac{3^{1/2}}{2} \cos \frac{3^{1/2}}{2}(t-t') \right] \xi\, dt'.$$

Letting $t' = t + \theta$ in these integrals we find

$$\delta(t) = \frac{2}{3^{1/2}} \int_0^\infty e^{\theta/2} \sin \frac{3^{1/2}}{2} \theta \xi(\theta + t)\, d\theta, \tag{6.8}$$

$$\dot{\delta}(t) = \frac{2}{3^{1/2}} \int_0^\infty e^{\theta/2} \left[-\frac{1}{2} \sin \frac{3^{1/2}}{2} \theta - \frac{3^{1/2}}{2} \cos \frac{3^{1/2}}{2} \theta \right] \xi(\theta + t)\, d\theta. \tag{6.9}$$

Since $\ddot{\delta} + \dot{\delta} + \delta = \xi$, we also get

$$\ddot{\delta}(t) = \xi + \frac{2}{3^{1/2}} \int_0^\infty e^{\theta/2} \left[-\frac{1}{2} \sin \frac{3^{1/2}}{2} \theta + \frac{3^{1/2}}{2} \cos \frac{3^{1/2}}{2} \theta \right] \xi(\theta + t)\, d\theta. \tag{6.10}$$

Since $|\zeta| \leqslant e^{-nt/2}$,

$$|\xi| \leqslant e^t \int_t^\infty e^{-t'(1+n/2)}\, dt' = \frac{e^{-nt/2}}{1 + n/2}$$

and using this in (6.8), we find

$$|\delta| \leqslant \frac{e^{-nt/2}}{1+n/2} \frac{2}{3^{1/2}} \int_0^\infty e^{(1-n)\theta/2} \left| \sin \frac{3^{1/2}}{2} \theta \right| d\theta.$$

Evaluation of the integral, which is elementary, gives the first estimate in the lemma, with the value of K_n quoted. A similar procedure can be applied to (6.9) and (6.10), but it is simpler (and does not lose too much in these cases) to estimate

$$\left| -\frac{1}{2} \sin \frac{3^{1/2}}{2} \theta \pm \frac{3^{1/2}}{2} \cos \frac{3^{1/2}}{2} \theta \right|$$

by its maximum, namely, 1. This gives the other estimates stated and finishes the proof of the lemma. The simpler bound for $n \geqslant 3$ comes immediately from the monotone decrease of coth and the calculation $8 \coth(\pi/3^{1/2}) = 8.43, 689, 148$.

Now, we return to our iteration scheme (6.6), (6.7). Also, we shall be interested in the function $w = \partial u/\partial \kappa$, which will be the limit of the sequence $w_n = \partial u_n/\partial \kappa$. From (6.6), we see that the w_n can be computed iteratively from

$$\dddot{w}_{n+1} - w_{n+1} = u_n w_n, \qquad (n \geqslant 1), \tag{6.11}$$

with the condition $w_{n+1} \sim w_1$ for $t \to \infty$. Thus, for the start of these sequences, we have

$$u_1 = \kappa e^{-t/2} \sin(3^{1/2}/2)\, t, \qquad w_1 = e^{-t/2} \sin(3^{1/2}/2)\, t, \tag{6.12}$$

$$u_2 = u_1 - (\kappa^2/8)\, e^{-t}(1 + \tfrac{2}{7} \cos 3^{1/2}t), \qquad w_2 = w_1 - (\kappa/4)\, e^{-t}(1 + \tfrac{2}{7} \cos 3^{1/2}t). \tag{6.13}$$

It is clear that each u_n and w_n is an elementary function of t and a polynomial in κ. We now show inductively that on $t \geqslant 0$ and $0 \leqslant \kappa \leqslant 1$, there are the bounds

$$| u_n | \leqslant (5/4)\, \kappa e^{-t/2}, \qquad (n \geqslant 1), \tag{6.14}$$

$$| w_n | \leqslant (5/3)\, e^{-t/2}, \qquad (n \geqslant 1), \tag{6.15}$$

$$| u_n - u_{n-1} | \leqslant \frac{9}{7\beta^2} \frac{(\beta\kappa e^{-t/2})^n}{(n!)^3}, \qquad (n \geqslant 2), \tag{6.16}$$

$$| w_n - w_{n-1} | \leqslant \frac{6}{7\kappa\beta^2} \frac{(\beta\kappa e^{-t/2})^n}{(n!)^3} (2n-1), \qquad (n \geqslant 2). \tag{6.17}$$

Here, the constant β is 10.6.

It is apparent from the explicit forms (6.12) and (6.13) that (6.14) and (6.15) hold for $n = 1$ and 2 and that (6.16) and (6.17) hold for $n = 2$. Suppose that they hold up to a specific value of $n (\geqslant 2)$. Then, from (6.6), using (6.14) and (6.16), we have

$$|(\dddot{u}_{n+1} - \dddot{u}_n) - (u_{n+1} - u_n)| = \tfrac{1}{2} | u_n + u_{n-1} | \, | u_n - u_{n-1} |$$
$$\leqslant \frac{5}{4\beta} \cdot \frac{9}{7\beta^2} \frac{(\beta\kappa e^{-t/2})^{n+1}}{(n!)^3}.$$

Applying Lemma 6.3 (with the estimate $K_{n+1} \leqslant 8.48/(n+1)^3$ for $n \geqslant 2$), we obtain

$$| u_{n+1} - u_n | \leqslant \frac{5}{4} \cdot \frac{8.48}{\beta} \cdot \frac{9}{7\beta^2} \frac{(\beta\kappa e^{-t/2})^{n+1}}{((n+1)!)^3},$$

which (since $(5/4)(8.48) = 10.6 = \beta$), verifies that (6.16) also holds at $n + 1$. Similarly, using (6.11), (6.15), and (6.17) we have

$$|(\dddot{w}_{n+1} - \dddot{w}_n) - (w_{n+1} - w_n)| = | u_n(w_n - w_{n-1}) + (u_n - u_{n-1})\, w_{n-1} |$$
$$\leqslant \frac{5}{4\beta} \cdot \frac{6}{7\kappa\beta^2} \frac{(\beta\kappa e^{-t/2})^{n+1}}{(n!)^3} (2n+1).$$

Again applying the lemma, this verifies (6.17) at $n + 1$. To check that (6.14) continues to hold at $n + 1$, we have

$$| u_{n+1} | = | u_1 + (u_2 - u_1) + \cdots + (u_{n+1} - u_n)|$$
$$\leqslant \kappa e^{-t/2} \left[1 + (9/7) \sum_0^\infty \beta^m((m+2)!)^{-3} \right].$$

The sum of the series is easily calculated accurately and is found, for $\beta = 10.6$, to be 0.18292620... . Since $1 + (9/7)(0.18292610...) = 1.23519... < 5/4$, (6.14) is verified at $n + 1$. Similarly,

$$| w_{n+1} | \leqslant e^{-t/2} \left[1 + (6/7) \sum_0^\infty (2m+3)\, \beta^m((m+2)!)^{-3} \right]$$
$$= e^{-t/2}(1.58616...) < (5/3)\, e^{-t/2}.$$

Thus, (6.15) also continues to hold and the induction is complete.

Now, $\lim u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + \cdots$. Since, by (6.16), the terms of this series are, uniformly on $0 \leqslant \kappa \leqslant 1$ and $t \geqslant 0$, less in absolute value than the terms of a convergent series of positive numbers, the u_n converge uniformly to a limit function u. Using this uniformity and an integral equation equivalent to (6.5) with $u \sim u_1$ at ∞ (which is obtained by inverting $(D^3 - 1)$ as in the proof of the lemma), it is easy to check that u is the desired solution. It is equally clear that $w \equiv \lim w_n$ exists and is $\partial u/\partial\kappa$. Hence, we have:

LEMMA 6.4. *The iteration* (6.6), (6.7) *converges for* $0 \leqslant \kappa \leqslant 1$, *as does the iteration* (6.11) *for* $\partial u/\partial\kappa$.

An estimate of the error in approximating u by u_n is obtained (for $n \geqslant 1$) from

$$(u - u_n) = |(u_{n+1} - u_n) + (u_{n+2} - u_{n+1}) + \cdots|$$
$$\leqslant \frac{9}{7\beta^2} \cdot \frac{(\beta\kappa e^{-t/2})^{n+1}}{((n+1)!)^3} \cdot \left[1 + \frac{\beta}{(n+2)^3} + \frac{\beta^2}{(n+2)^3(n+3)^3} + \cdots\right].$$

For $n \geqslant 1$ the sum of the series is

$$\leqslant 1 + \frac{\beta}{3^3} + \frac{\beta^2}{3^3 4^3} + \cdots = 8(0.18292610...) \leqslant 1.46341.$$

Thus, for $n \geqslant 1$,

$$|u - u_n| \leqslant \frac{1.882}{(10.6)^2} \frac{(10.6\kappa e^{-t/2})^{n+1}}{((n+1)!)^3}. \tag{6.18}$$

Similarly,

$$|w - w_n| \leqslant \frac{1.5631}{\kappa(10.6)^2} \frac{(10.6\kappa e^{-t/2})^{n+1}}{((n+1)!)^3}(2n+1), \qquad (n \geqslant 1). \tag{6.19}$$

To obtain error estimates on the derivatives, we note that

$$|(\dddot{u} - \dddot{u}_{n+1}) - (u - u_{n+1})| = \tfrac{1}{2}\,|u + u_n|\,|u - u_n| \leqslant \tfrac{5}{4}\,\kappa e^{-t/2}\,|u - u_n|$$
$$\leqslant \frac{2.36}{(10.6)^3} \frac{(10.6\kappa e^{-t/2})^{n+2}}{((n+1)!)^3}$$

using (6.18). Applying the lemma to this gives, for $n \geqslant 1$,

$$|u - u_{n+1}| \leqslant K_{n+2}\left[\frac{2.36}{(10.6)^3} \frac{(10.6\kappa e^{-t/2})^{n+2}}{((n+1)!)^3}\right],$$
$$|\dot{u} - \dot{u}_{n+1}| \leqslant K'_{n+2}\left[\frac{2.36}{(10.6)^3} \frac{(10.6\kappa e^{-t/2})^{n+2}}{((n+1)!)^3}\right], \tag{6.20}$$
$$|\ddot{u} - \ddot{u}_{n+1}| \leqslant K''_{n+2}\left[\frac{2.36}{(10.6)^3} \frac{(10.6\kappa e^{-t/2})^{n+2}}{((n+1)!)^3}\right].$$

In particular, for $n = 1$, since $K_3 \leqslant 0.242$, $K_3' \leqslant 0.462$, and $K_3'' \leqslant 0.862$, this becomes

$$|u - u_2| \leqslant 0.0714\,\kappa^3 e^{-3t/2},$$
$$|\dot{u} - \dot{u}_2| \leqslant 0.1363\,\kappa^3 e^{-3t/2}, \tag{6.21}$$
$$|\ddot{u} - \ddot{u}_2| \leqslant 0.2543\,\kappa^3 e^{-3t/2}.$$

In the same way, one finds

$$
\begin{aligned}
| w - w_2 | &\leqslant 0.273\, \kappa^2 e^{-3t/2}, \\
| \dot{w} - \dot{w}_2 | &\leqslant 0.520\, \kappa^2 e^{-3t/2}, \\
| \ddot{w} - \ddot{w}_2 | &\leqslant 0.970\, \kappa^2 e^{-3t/2}.
\end{aligned} \tag{6.22}
$$

Now, we make some calculations concerning the intersection of S with $\{u = 0\}$.

LEMMA 6.5. *For each κ in $0 \leqslant \kappa \leqslant 1$, $u(t, \kappa) = 0$ exactly once for $0 \leqslant t \leqslant 0.3\kappa$. Furthermore, if we denote by $(y_2(\kappa), y_3(\kappa))$ the value of $(\dot{u}, \ddot{u})$ at that t, then $y_2(1) \geqslant 0.64196$.*

Proof. First, we show that $u(0, \kappa) < 0$ and that $u(0.3\kappa, \kappa) > 0$. We do this by estimating the time at which $u_2(t, \kappa)$ (given explicitly by (6.13)) crosses $u = 0$ and then using (6.21).

At $t = 0$, $u_1 = 0$ and $u_2 = -9\kappa^2/56 < 0$. Using (6.21) we see that $u(0, \kappa) \leqslant 0.0714\kappa^3 - 9\kappa^2/56 \leqslant -0.089\kappa^2$ for all κ on $[0, 1]$. Thus, $u(0, \kappa) < 0$ for $0 < \kappa \leqslant 1$. On the other hand, since $\sin x \geqslant x - x^3/6$ $(x \geqslant 0)$, we have

$$
u_2 \geqslant \kappa e^{-t/2} \left(\frac{3^{1/2}}{2} t\right)\left(1 - \frac{t^2}{8}\right) - \frac{9}{56} \kappa^2 e^{-t}, \qquad \text{on} \quad 0 \leqslant t \leqslant \frac{\pi}{2(3^{1/2})} = 0.9068\ldots.
$$

Hence, on $0 \leqslant \kappa \leqslant 1$,

$$
u_2(0.3\kappa) \geqslant \kappa^2 e^{-0.15\kappa} \left[(0.3)\frac{3^{1/2}}{2}\left(1 - \frac{(0.3)^2}{8}\right) - \frac{9}{56}\right] \geqslant 0.09617\kappa^2 e^{-0.15\kappa}.
$$

Using (6.21), this implies that $u(0.3\kappa) \geqslant 0.09617\kappa^2 e^{-0.15\kappa} - 0.0714\kappa^3 e^{-0.3\kappa} > 0$.

To see that there is a *unique* value $t_0(\kappa)$, in $[0, 0.3\kappa]$, such that $u(t_0(\kappa), \kappa) = 0$, we now show that $\dot{u} > 0$:

$$
\dot{u}_2(t, \kappa) = \kappa e^{-t/2} \cos\left(\frac{3^{1/2}}{2} t + \frac{\pi}{6}\right) + \frac{\kappa^2}{8} e^{-t}\left(1 + \frac{4}{7}\cos\left(3^{1/2} t - \frac{\pi}{3}\right)\right). \tag{6.23}
$$

On $0 \leqslant t \leqslant 0.3$,

$$
\cos\left(\frac{3^{1/2}}{2} t + \frac{\pi}{6}\right) \leqslant 0.7085 \qquad \text{and} \qquad \cos\left(3^{1/2} t - \frac{\pi}{6}\right) \geqslant \frac{1}{2}.
$$

Thus, on $0 \leqslant t \leqslant 0.3\kappa$, $\dot{u}_2(t, \kappa) \geqslant 0.7085\kappa e^{-t/2} + (9/56)\, \kappa^2 e^{-t} > 0$. Using (6.21), on this interval we have

$$\dot{u}(t, \kappa) \geqslant 0.7085\kappa e^{-t/2} + (9/56)\, \kappa^2 e^{-t} - 0.1363\kappa^3 e^{-3t/2}. \tag{6.24}$$

Since $\kappa e^{-t/2} \leqslant 1$, we have $\kappa^n e^{-nt/2} < \kappa e^{-t/2}$ if $n > 1$. Hence, (6.24) implies that $\dot{u}(t, \kappa) \geqslant \kappa e^{-t/2}(0.7085 + (9/56) - 0.1363)$, or

$$\dot{u}(t, \kappa) \geqslant 0.7329\kappa e^{-t/2} > 0. \tag{6.25}$$

Now, $\dot{u}(0.3, 1)$ is at least as large as the value of the right-hand side of (6.24), namely, 0.64196. Also,

$$y_2(1) \equiv \dot{u}(t_0(1), 1) \geqslant 0.64196, \tag{6.26}$$

since the polynomial in $\kappa e^{-t/2}$ in (6.24) is increasing on $0 \leqslant \kappa e^{-t/2} \leqslant 1$ and $t_0(1) \leqslant 0.3$. Hence, the intersection curve $S_1 = (y_2(\kappa), y_3(\kappa))$ extends from $y_2 = 0$ to at least $y_2 = .64196$.

LEMMA 6.6. *For* $0 \leqslant \kappa \leqslant 1$, $(y_2(\kappa), y_3(\kappa))$ *lies between the line* $y_3 = -y_2$ *and the cubic* $y_3 = -y_2 - 0.175y_2^2 - 0.3y_2^3$.

Proof. Eq. (6.5) can be written:

$$(d/dt)\, e^{-t}(\ddot{u} + \dot{u} + u) = \tfrac{1}{2} e^{-t} u^2.$$

Also, $u \to 0$ as $t \to \infty$, so

$$\ddot{u} + \dot{u} + u = -\tfrac{1}{2} e^{t} \int_t^\infty e^{-t'} u^2(t')\, dt'. \tag{6.27}$$

Thus

$$\begin{aligned} 0 \geqslant \ddot{u}(t_0) + \dot{u}(t_0) &= -\tfrac{1}{2} e^{t_0} \int_{t_0}^\infty e^{-t'} u^2\, dt' \\ &\geqslant -\tfrac{1}{2} e^{t_0} \int_{t_0}^\infty e^{-t'} u_1^2\, dt' - \tfrac{1}{2} e^{t_0} \int_{t_0}^\infty e^{-t'} \mid u^2 - u_1^2 \mid dt'. \end{aligned} \tag{6.28}$$

Using (6.18) with $n = 1$ and (6.14), we estimate the second integral on the right in (6.28). Since

$$\frac{1}{2} \mid u^2 - u_1^2 \mid \leqslant \frac{5}{4} \kappa e^{-t/2} \cdot \frac{1.882}{8} \kappa^2 e^{-t},$$

we have

$$-\frac{1}{2}e^{t_0}\int_{t_0}^{\infty} e^{-t'} \mid u^2 - u_1{}^2 \mid dt' \geqslant -\frac{5}{4}\cdot\frac{1.882}{8}\cdot\kappa^3\cdot\frac{2}{5}e^{-3t_0/2} \geqslant -0.118\,\kappa^3 e^{-3t_0/2}.$$

Calculating the first integral,

$$-\frac{1}{2}e^{t_0}\int_{t_0}^{\infty} e^{-t'} u_1{}^2\, dt' = -\frac{\kappa^2}{8}e^{-t_0}\left[1 - \frac{2}{7}\cos(3^{1/2})\,t_0 - 3^{1/2}\sin(3^{1/2})\,t_0)\right].$$

It is easily checked that the quantity in the square bracket here is a positive increasing function of t_0 on $0 \leqslant t_0 \leqslant 0.3$; hence, it is less than its value at 0.3, namely, 0.74972... . Thus, the first integral is $\geqslant -\kappa^2 e^{-t_0}(0.09372)$ and (6.27) now tells us that

$$0 \geqslant \ddot{u}(t_0\,,\kappa) + \dot{u}(t_0\,,\kappa) \geqslant -0.09372\,\kappa^2 e^{-t_0} - 0.118\,\kappa^3 e^{-3t_0/2}. \tag{6.29}$$

By (6.25), $\dot{u}(t_0\,,\kappa) \geqslant 0.7329\kappa e^{-t_0/2}$. Thus,

$$\begin{aligned} -\dot{u}(t_0\,,\kappa) &\geqslant \ddot{u}(t_0\,,\kappa) \\ &\geqslant -\dot{u}(t_0\,,\kappa) - \frac{0.09372}{(0.7329)^2}\dot{u}^2 - \frac{0.118}{(0.7329)^3}\dot{u}^3 \\ &\geqslant -\dot{u} - 0.175\dot{u}^2 - 0.3\dot{u}^3. \end{aligned} \tag{6.30}$$

LEMMA 6.7. *For* $0 \leqslant \kappa \leqslant 0.856$, S_1 *slopes downward, i.e.,* $\partial y_2/\partial\kappa > 0$ *and* $\partial y_3/\partial\kappa < 0$.

Note. The estimates here are not sharp; the slope of S_1 is probably negative for $0 \leqslant \kappa \leqslant 1$.

Proof. Since $u(t_0\,,\kappa) = 0$, $\dot{u}(t_0\,,\kappa)(dt_0/d\kappa) + w(t_0\,,\kappa) = 0$. Hence,

$$\frac{d}{d\kappa}(\dot{u}(t_0\,,\kappa)) = \dot{w}(t_0\,,\kappa) - \frac{\ddot{u}(t_0\,,\kappa)}{\dot{u}(t_0\,,\kappa)}w(t_0\,,\kappa). \tag{6.31}$$

We want to show that this quantity is positive, at least in the range where the intersection may occur. Now, from (6.25) and (6.29), we know that

$$\begin{aligned} 0 \leqslant -\frac{\ddot{u}(t_0\,,\kappa)}{\dot{u}(t_0\,,\kappa)} - 1 &\leqslant \frac{0.09372}{0.7329}\kappa e^{-t_0/2} + \frac{0.118}{0.7329}\kappa^2 e^{-t_0} \\ &\leqslant 0.128\,\kappa e^{-t_0/2} + 0.161\,\kappa^2 e^{-t_0}. \end{aligned} \tag{6.32}$$

Also, from (6.13), $-(9/28)\,\kappa e^{-t_0} \leqslant w_2 \leqslant ((0.3)\,3^{1/2}/2)\,\kappa e^{-t_0/2} - (\kappa/4)\,e^{-t_0}$; since $((0.3)\,3^{1/2}/2)\,e^{0.15} \leqslant 0.302 < 9/28$, this means that $|\,w_2\,| \leqslant (9/28)\,\kappa e^{-t_0}$. Using (6.22) we have

$$|\,w(t_0\,,\kappa)| \leqslant (9/28)\,\kappa e^{-t_0} + 0.273\,\kappa^2 e^{-3t_0/2}. \tag{6.33}$$

Now,

$$\dot{w}_2(t_0\,,\kappa) = e^{-t_0/2}\cos\left(\frac{3^{1/2}}{2}\,t_0 + \frac{\pi}{6}\right) + \frac{\kappa}{4}\,e^{-t_0}\left(1 + \frac{4}{7}\cos\left(3^{1/2}t_0 - \frac{\pi}{3}\right)\right)$$

$$\geqslant e^{-t_0/2}\cos\left(\frac{3^{1/2}}{2}\,(0.3) + \frac{\pi}{6}\right) + \frac{9}{28}\,\kappa e^{-t_0} \geqslant 0.7085\,e^{-t_0/2} + \frac{9}{28}\,\kappa e^{-t_0}\,;$$

using this with (6.22), (6.32), and (6.33) in (6.31) we have

$$\begin{aligned}
&(d/d\kappa)(\dot{u}(t_0\,,\kappa))\\
&\quad\geqslant 0.7085\,e^{-t_0/2} + (9/28)\,\kappa e^{-t_0} - 0.529\,\kappa^2 e^{-3t/2}\\
&\qquad - (1 + 0.128\,\kappa e^{-t_0/2} + 0.161\,\kappa^2 e^{-t_0})((9/28)\,\kappa e^{-t_0})(1 + 0.85\,\kappa e^{-t_0/2})\\
&\quad\geqslant 0.7085\,e^{-t_0/2} - 0.520\,\kappa^2 e^{-3t_0/2} - (9/28)\,\kappa e^{-t_0}\\
&\qquad\cdot [0.978\,\kappa e^{-t_0/2} + 0.270\,\kappa^2 e^{-t_0} + 0.137\,\kappa^3 e^{-3t_0/2}]\\
&\quad\geqslant 0.7085\,e^{-t_0/2} - 0.9652\,\kappa^2 e^{-3t_0/2} > 0,
\end{aligned}$$

for $\qquad \kappa^2 < 0.7085/0.9652 = 0.7340 = (0.8567...)^2.$

Thus, S_1 moves to the *right* as κ increases, at least out to $\kappa = 0.856$. To see that it also goes down as κ increases, we have

$$(d/d\kappa)(\ddot{u}(t_0\,,\kappa)) = \dddot{u}(t_0\,,\kappa)(dt_0/d\kappa) + \ddot{w}(t_0\,,\kappa).$$

Since $\dddot{u} = u + 1/2u^2$ and $u(t_0\,,\kappa) = 0$, we have

$$(d/d\kappa)(\ddot{u}(t_0\,,\kappa)) = \ddot{w}(t_0\,,\kappa). \tag{6.34}$$

Now,

$$\ddot{w}_2(t_0\,,\kappa) = -e^{-t_0/2}\cos\left(\frac{3^{1/2}}{2}\,t_0 - \frac{\pi}{6}\right) - \frac{\kappa}{4}\,e^{-t_0}\left(1 - \frac{8}{7}\cos\left(3^{1/2}t_0 + \frac{\pi}{3}\right)\right)$$

$$\leqslant -(3^{1/2}/2)\,e^{-t_0/2} - (3/28)\,\kappa e^{-t_0}.$$

Using (6.22), we then have

$$\ddot{w}(t_0, \kappa) \leqslant -(3^{1/2}/2)\, e^{-t_0/2} - (3/28)\, \kappa e^{-t_0} + 0.970\, \kappa^2 e^{-3t_0/2}$$

$$\leqslant -(3^{1/2}/2)\, e^{-t_0/2} + \kappa e^{-t_0}(0.863) \leqslant e^{-t_0/2}(-(3^{1/2}/2) + 0.863).$$

Since $-(3^{1/2}/2) + 0.863 < -0.003$, we find that also, out to $\kappa = 1$, $(d/d\kappa)(\ddot{u}(t_0, \kappa)) < 0$. This shows that S_1 slopes down at least out to $\kappa = 0.856$. Since $t_0 \leqslant 0.3\kappa$, $\kappa e^{-t_0/2} \geqslant 0.75285...$ at $\kappa = 0.856$ and using this in (6.24), as we did previously at $\kappa = 1$, we find that at $\kappa = 0.856$, $y_2 \geqslant 0.5663$.

Finally, we finish Theorem 6.1 by showing:

LEMMA 6.8. *S and U interest transversely.*

Proof. Recall the Poincaré maps $\mathscr{P}_i(v)$. These maps may be extended to a neighborhood of the y_2 axis in $\{y_1 = 0\}$. That is, for v sufficiently large and v' sufficiently small, the trajectory $y(t)$ starting at $(0, v, v')$ travels monotonically to $\{y_1 = 1\}$. Let $\mathscr{P}_i(v, v')$, $i = 2, 3$ be the second and third components of the point of intersection, and let $\mathscr{P} = (\mathscr{P}_2, \mathscr{P}_3)$. Where $\mathscr{P}$ is defined, it is a local diffeomorphism from a domain in $\{y_1 = 0\}$ to $\{y_1 = 1\}$. On its image, which contains $S_1 \cap T_1$, $\mathscr{P}$ has an inverse (which is obtained by integrating backward). Consider the two transverse curves S_1 and T_1 in $\{y_1 = 1\}$. Then, $S_0 = \mathscr{P}^{-1}(S_1)$ and $\mathscr{P}^{-1}(T_1)$ intersect transversely in $\{y_1 = 0\}$. $\mathscr{P}^{-1}(T_1)$ is just a subset of the y_2 axis ($y_1 = y_3 = 0$).

We now use the symmetry of Eqs. (1.2). It is easy to show that S transversely intersects $\{y_1 = 0\}$ (at S_0) and similarly, U intersects $\{y_1 = 0\}$ at U_0. To show S intersects U transversely, it suffices to show that S_0 intersects U_0 transversely in $\{y_1 = 0\}$. By symmetry, these curves U_0 and S_0 are symmetric around the y_2 axis (in $\{y_1 = 0\}$). Hence, since S_0 transversely intersects $y_3 = 0$ (in $\{y_1 = 0\}$), the only way the intersection of S_0 and U_0 can fail to be transverse is if they are both perpendicular to the y_2 axis at the point of intersection, i.e., if S_0 has tangent vector $(0, 0, 1)$. But the variational Eqs. (6.4) show that the image under $\mathscr{P}$ of a curve with tangent $(0, 0, 1)$ must satisfy $\partial\mathscr{P}_i/\partial v > 0$. This is impossible since $\mathscr{P}(S_0) = S_1$, which has a negative slope (i.e., $((\partial\mathscr{P}_3/\partial v)/(\partial\mathscr{P}_2/\partial v)) < 0$).

Remark. The trajectory joining $(-1, 0, 0)$ to $(1, 0, 0)$ may be calculated by an iterative method that does not consider intersections in the plane $y_1 = 1$. Again, we look for a trajectory that passes through $y_1 = y_3 = 0$ and that tends to $(1, 0, 0)$. Let $u = y - 1$ so that u satisfies $u''' - u = (1/2)\, u^2$ or $(D - 1)(D^2 + D + 1)\, u = \frac{1}{2}u^2$. Integrating this once, we see that every trajectory in S satisfies the integral equation $(D^2 + D + 1)\, u = e^t \int_t^\infty e^{-t'} u^2(t')\, dt'$. There is a two-parameter family of such solutions and we look for the one satisfying

$$u(0) = -1, \qquad u''(0) = 0. \tag{6.35}$$

To calculate the solution, let $u_1(t) = e^{-t/2}(\alpha_1 \cos (3^{1/2}/2)\, t + \beta_1 \sin (3^{1/2}/2)\, t)$, where α_1 and β_1 are chosen to satisfy the conditions at $t = 0$. (Here, $u_1(t)$ is in the tangent plane to S.) Let $u_{n+1}(t)$ be the solution to $(D^2 + D + 1)\, u_{n+1} = e^{t'} \int_t^\infty e^{-t} u_n^2(t')\, dt'$, which satisfies (6.35). (To compute the right-hand side, it is convenient to calculate numerically for small values of t' and approximate $u_n(t')$ by the appropriate function of the form $e^{-t}(\bar{\alpha} \cos (3^{1/2}/2)\, t + \bar{\beta} \sin (3^{1/2}/2)\, t)$ for t large.) The resulting iteration appears to converge very quickly to the same function computed by considering the intersection of S and T. However, we have not proved the convergence for this method.

7. The Unfolding of the Canonical Example, $k = 2$

We saw in Section 4 that for $\mu \neq 0$, the general system (2.1) or (2.2) could be expressed as (4.1), where $\epsilon \to 0$ as $\mu \to 0$. This was useful for $k = 3$, since we could show that (1.2) has a structurally stable trajectory joining the critical points.

For this section, we work with an equivalent version of (1.2), with $k = 2$, namely,

$$\dot{y}_1 = y_2\,; \qquad \dot{y}_2 = y_1(1 - y_1). \tag{7.1}$$

This system has a first integral: The values of the function $I(y_1, y_2) = (y_2^2/2) - (y_1^2/2) + (y_1^3/3)$ are constant along the trajectories. The trajectories are sketched in Fig. 2. A perturbation of (7.1) need no longer have a homoclinic orbit or any closed orbits; for example, it could have a trajectory joining the critical points that spirals out from (1,0) and goes to (0, 0). However, there is a one-dimensional unfolding of (7.1) whose properties persist under perturbations. The theorem to be given below is an unfolded version of Theorem (2.1); there is an analogue for Theorem (2.2).

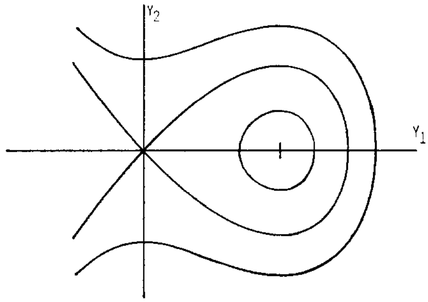

FIG. 2. Integral curves for Eq. (7.1).

Recalling that (7.1) is the scaled limit (as $\mu \to 0$) of (2.1), we go back to (2.1) and put in another (unfolding) parameter ν related to the trace of the system linearized at (0, 0); the scaled limit (as $\mu \to 0$) of this two-parameter family will essentially be the one-parameter unfolding of (7.1). We will show that for each $\mu \neq 0$, there is a locally unique $\nu(\mu)$ for which the system has a homoclinic orbit. Furthermore, there are parameter values (μ, ν) for which there are periodic trajectories and other parameter values for which $\dot{X} = F_{\mu,\nu}(X)$ has a trajectory joining the critical points.

THEOREM 7.1. *Let*

$$\dot{X} = F_{\mu,\nu}(X) \tag{7.2}$$

be a two-parameter family of ordinary differential equations on R^2, F smooth in all of its four arguments, such that $F_{\mu,\nu}(0) = 0$. Also assume:

1. *$dF_{0,0}(0) \equiv A$ has a double zero eigenvalue and a single eigenvector e.*

2. *The mapping $(\mu, \nu) \to (\det dF_{\mu,\nu}(0), \operatorname{tr} dF_{\mu,\nu}(0))$ has a nonzero Jacobian at $(\mu, \nu) = (0, 0)$.*

3. *If $Q(e, e)$ is as in Theorem 2.1, then $[dF_{(0,0)}(0), Q(e, e)]$ has rank 2 (same as Hypothesis 3 of Theorems 2.1 and 2.2).*

Then: There is a curve $f(\mu, \nu) = 0$ such that if $f(\mu_0, \nu_0) = 0$, then $\dot{X} = F_{\mu_0,\nu_0}(X)$ has a homoclinic orbit. This one-parameter family of homoclinic orbits (in (X, μ, ν) space) is on the boundary of a two-parameter family of periodic solutions. For all $|\mu|$, $|\nu|$ sufficiently small, if $\dot{X} = F_{\mu,\nu}(X)$ has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.

Proof. We assume, as usual, that $A = dF_{0,0}(0)$ is in Jordan normal form, with entries $a_{12} = 1$, $a_{ij} = 0$ for $(i, j) \neq (1, 2)$. We shall make two nonsingular changes of variables before scaling the equations $\dot{X} = F_{\mu,\nu}(X)$. The first is a μ, ν-dependent transformation of x_1 and x_2 designed to put $dF_{\mu,\nu}(0)$ into a more convenient form. Suppose that $dF_{\mu,\nu}(0)$ has the matrix (a_{ij}), where the a_{ij} are functions of μ and ν. Let

$$\bar{x}_1 = x_1, \qquad \bar{x}_2 = a_{11}x_1 + a_{12}x_2 .$$

For μ, ν sufficiently small, this is a nonsingular change of variables. In terms of $\bar{x}_1$ and $\bar{x}_2$, Eqs. (7.2) are

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ -\det & \mathrm{tr} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \cdots,$$

where $-\det = -(a_{11}a_{22} - a_{12}a_{21})$ and $\mathrm{tr} = a_{11} + a_{22}$. We assume this transformation has been made and drop the bars over the x_i.

The next change of variables is in the parameter space alone. We let $\bar{\mu}(\mu, \nu) = -\det$ and $\bar{\nu}(\mu, \nu) = \mathrm{tr}$. By Hypotheses 2, this change of variables is also nonsingular for μ, ν sufficiently small. Thus, dropping the bars over the μ, ν, we may assume the equations have the form

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ \mu & \nu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + Q(X, X) + R(X, \mu, \nu). \tag{7.3}$$

(Note that the Q and R terms of (7.3) are different from those of (7.2), but Hypothesis 3 still holds (by Section 3).)

Now we use the scaling procedure discussed in Section 4. Let b be the second component of $Q(e, e)(b \neq 0$ by Hypothesis 3) and let $\epsilon = | \mu/b |^{1/2}$. As before, $x_1 = \epsilon^2 y_1$, $x_2 = | b |^{1/2} \epsilon^3 y_2$, and $t = | b |^{-1/2} \epsilon^{-1} \bar{t}$. Also, $\nu = | b |^{1/2} \epsilon\tau$. Then (e.g., if $b < 0$, $\mu > 0$), (7.3) becomes

$$\begin{aligned} \dot{y}_1 &= y_2 + \epsilon q_1(y_i, \tau, \epsilon), \\ \dot{y}_2 &= y_1 + \tau y_2 - y_1^2 + \epsilon q_2(y_i, \tau, \epsilon), \end{aligned} \tag{7.4}$$

where $q_i = O(1)$. Here, the $\cdot$ means differentiation with respect to $\bar{t}$ and we shall now drop the bar.

For $\epsilon = 0$, (7.4) is the unfolding of (7.1), with τ as the (scaled) unfolding parameter. The rest of the proof follows from the next three lemmas.

LEMMA 7.1. *For each ϵ sufficiently small, there is a $\tau(\epsilon)$ such that Eqs.* (7.4) *have a homoclinic orbit.* $\tau(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof. For ϵ, τ sufficiently small, the point (0, 0) of (7.4) is a saddle. Let $S_{\tau,\epsilon}$ and $U_{\tau,\epsilon}$ be the stable and unstable manifolds of this critical point. Recall that $I(y_1, y_2) = (y_2^2/2) - (y_1^2/2) + (y_1^3/3)$ is a first integral for (7.4) when $\tau = \epsilon = 0$. Let $I_+(\tau, \epsilon)$ be the value of $I(y_1, y_2)$ at the place where $U_{\tau,\epsilon}$ hits $y_2 = 0$; similarly, $I_-(\tau, \epsilon)$ is the place where $S_{\tau,\epsilon}$ hits $y_2 = 0$. (These functions exist for τ, ϵ sufficiently small, since $S_{\tau,\epsilon}$ and $U_{\tau,\epsilon}$ can be made close to $S_{0,0}$ and $U_{0,0}$ for any finite portions of these manifolds.) Also, let $\oint_{\tau,\epsilon,+} d\mathscr{F}(y)$ denote the integral of $d\mathscr{F}(y_1, y_2)$ over the portion of $U_{\tau,\epsilon}$ with $y_1 > 0$ and before the manifold intersects $y_2 = 0$. Integral curves of autonomous differential equations have a natural parameterization by the time variable t, up to translation in t. It is convenient to have the notation $\oint_{\tau,\epsilon,+} (d\mathscr{F}/dt)\, dt$ to mean the above-line integral parameterized by any translation of t. Similarly, $\oint_{\tau,\epsilon,-}$ denotes the integral over the portion of $S_{\tau,\epsilon}$ with $y_1 > 0$ and before the manifold intersects $y_2 = 0$; this integration is done from $y_1 = 0$ to $y_1 > 0$, i.e., backward in time.

If $I_+(\tau, \epsilon) = I_-(\tau, \epsilon)$, the system (7.4) has a homoclinic orbit. Now,

$$\left.\frac{\partial I_+}{\partial \tau}\right|_{\tau=\epsilon=0} = \frac{\partial}{\partial \tau} \oint_{0,0,+} \frac{dI}{dt}\, dt = \oint_{0,0,+} y_2^2\, dt > 0.$$

Also, by symmetry, $(\partial I_+/\partial \tau)|_{\tau=\epsilon=0} = -(\partial I_-/\partial \tau)|_{\tau=\epsilon=0}$. This implies that the equations $I_+(\tau, \epsilon) = I_-(\tau, \epsilon)$ may be solved for $\tau(\epsilon)$, if ϵ is sufficiently small.

LEMMA 7.2. *Let ϵ sufficiently small be fixed. Then the one-parameter family of Eqs.* (7.4) (*with parameter τ*) *has, among them, a one-parameter family of periodic orbits. That is, if* (7.4) *is considered as a level-preserving equation on* (y_1, y_2, τ)*-space* (*the last equation being* $\dot{\tau} = 0$), *then there is a two-dimensional invariant surface in R^3 consisting of periodic orbits. This surface is parameterized by t and c, the y_1-coordinate of the periodic orbit as it passes through $y_2 = 0$, with $\dot{y}_2 > 0$. Furthermore, as $c \to 0$, this surface tends to the homoclinic orbit given by the previous lemma. The other boundary of the surface of periodic solutions is the critical point of* (7.4) *near* (1, 0).

Proof. This lemma is proved in a manner analogous to the previous one.

Let $I_+(c, \tau, \epsilon)$ be the value of I where the trajectory of (7.4) starting at $y_1 = c, y_2 = 0$ ($\dot{y}_2 > 0$) is integrated forward in time until it intersects $y_2 = 0$ again (with $\dot{y}_2 < 0$). (Here, I_+ is defined for τ, ϵ sufficiently small since the trajectories of (7.4) are close to those of (7.4) with $\tau = \epsilon = 0$; the latter system has a one-parameter family of periodic orbits.) Similarly, $I_-(c, \tau, \epsilon)$ is the value obtained by integrating backward. Also, $\oint_{c,0,0,+}$ denotes the integral over the trajectory of (7.4), which starts at $y_1 = c$, $y_2 = 0$ and ends when $y_2 = 0$ again; $\oint_{c,0,0,-}$ is similarly defined by integrating backward.

An orbit of (7.4) which passes through $y_1 = c$, $y_2 = 0$ is periodic if and only if it satisfies $I_+(c, \tau, \epsilon) - I_-(c, \tau, \epsilon) = 0$. Now,

$$(\partial I_+/\partial\tau)(c, 0, 0) = \oint_{c,0,0,+} y_2^2(t)\, dt = -(\partial I_-/\partial\tau)(c, 0, 0).$$

We wish to solve $I_+ - I_- = 0$ for $\tau = \tau(\epsilon, c)$ for ϵ sufficiently small, but uniformly in c. This cannot be done in general for $0 \leqslant c \leqslant 1$ since, for $\epsilon \neq 0$, the critical points of (7.4) may be moved. However, an ϵ, τ dependent change of the y_i's can be made so that the critical points are at $(0, 0)$ and $(1, 0)$ for all ϵ, τ and we assume such a change has been made. Since $(\partial I/\partial\tau)|_{\tau=\epsilon=0} \to 0$ as $c \to 1$, we consider

$$\hat{I}_+ - \hat{I}_- = (1/(c-1)^2)(I_+ - I_-)$$

instead of $I_+ - I_-$. We shall show that $\hat{I}_+ - \hat{I}_-$ is bounded for $0 \leqslant c \leqslant 1$ and that $(\partial/\partial\tau)(\hat{I}_+ - \hat{I}_-)$ is bounded away from zero. Then, $\hat{I}_+ - \hat{I}_- = 0$ may be solved for $\tau = \tau(c, \epsilon)$, for sufficiently small ϵ, uniformly in c, $0 \leqslant c \leqslant 1$.

Since the critical points of (7.4) are $(0, 0)$ and $(1, 0)$ for all ϵ, τ, the functions $q_i(y_1, y_2, \epsilon, \tau)$ satisfy $q_i(1, 0, \epsilon, \tau) = 0$. Hence, they may be written as $q_i = (y_1 - 1)\, q_{i1} + y_2 q_{i2}$. Now, along the trajectories of (7.4), $dI/dt = \tau y_2^2 + \epsilon[y_2 q_2 + (y_1 - y_1^2)\, q_1]$. For y in some compact region, e.g., $-1 \leqslant y_1 \leqslant 2$, $-1 \leqslant y_2 \leqslant 1$, the coefficient of ϵ in the above expression is bounded above by $\bar{K}_{11}(y_1 - 1)^2 + \bar{K}_{12}(y_1 - 1)\, y_2 + \bar{K}_{22} y_2^2$ and below by a similar expression $\underline{K}_{11}(y_1 - 1)^2 + \underline{K}_{12}(y_1 - 1)\, y_2 + \underline{K}_{22} y_2^2$. Thus, dI/dt is bounded above and below by quadratic forms in $(y_1 - 1)$ and y_2, where the coefficients K_{ij} are all $O(\tau) + O(\epsilon)$. Also, $(I + \frac{1}{6}) = y_2^2/2 + \frac{1}{6}(2y_1 + 1)(y_1 - 1)^2$, which is bounded above and below by quadratic forms in $(y_1 - 1)$ and y_2. Thus, there are functions $\bar{\lambda}(\tau, \epsilon) > 0$, $\underline{\lambda}(\tau, \epsilon) < 0$ such that $\underline{\lambda}(\tau, \epsilon)(I + \frac{1}{6}) \leqslant dI/dt \leqslant \bar{\lambda}(\tau, \epsilon)(I + \frac{1}{6})$ for y_i in the above compact region.

Recall that $I_+ = I(c, 0) + \oint_{c,\tau,\epsilon,+} (dI/dt)\, dt$. Since $\underline{\lambda}(\tau, \epsilon) \leqslant (d/dt)(\ln(I + \frac{1}{6})) \leqslant \bar{\lambda}(\tau, \epsilon)$, we have

$$\oint_{c,\tau,\epsilon,+} \underline{\lambda}(\tau, \epsilon)\, dt \leqslant \ln((I_+ + \tfrac{1}{6})/I(c, 0) + \tfrac{1}{6}) \leqslant \oint_{c,\tau,\epsilon,+} \bar{\lambda}(\tau, \epsilon)\, dt.$$

Now, for ϵ, τ small, the time taken to traverse the path is bounded, say, by $\hat{t}$. Thus,

$$e^{\underline{\lambda}(\tau,\epsilon)\hat{t}} \leqslant (I_+ + \tfrac{1}{6})/(I(c, 0) + \tfrac{1}{6}) \leqslant e^{\bar{\lambda}(\tau,\epsilon)\hat{t}}.$$

Similarly,

$$e^{\underline{\lambda}(\tau,\epsilon)\hat{t}} \leqslant (I_- + \tfrac{1}{6})/(I(c, 0) + \tfrac{1}{6}) \leqslant e^{\underline{\lambda}(\tau,\epsilon)\hat{t}}.$$

Hence,

$$\left| \frac{I_+ - I_-}{I(c, 0) + \frac{1}{6}} \right| \leqslant e^{\bar{\lambda}(\tau,\epsilon)\hat{t}} - e^{\underline{\lambda}(\tau,\epsilon)\hat{t}} = O(\tau) + O(\epsilon).$$

Now, $I(c, 0) + \frac{1}{6} = \frac{1}{6}(2c + 1)(c - 1)^2$, which shows that $(I_+ - I_-)/(c - 1)^2$ is bounded for $0 \leqslant c \leqslant 1$ and the bounds are small for ϵ and τ small.

Also we must show that $(\oint_{c,0,0,+} y_2^2\, dt)/(c - 1)^2$ is nonzero uniformly in c. But for (7.4), with $\epsilon = \tau = 0$, $dy_1/dt = y_2$ so $\oint_{c,0,0,+} y_2^2\, dt = \oint_{c,0,0,+} y_2\, dy_1 = \frac{1}{2}$ area inside the closed orbit through $y_1 = c$, $y_2 = 0$. Near $c = 1$, these orbits are approximately the curves

$$\frac{y_2^2}{2} + \frac{(y_1 - 1)^2}{2} = I(c, 0) + \frac{1}{6} \approx \frac{(c - 1)^2}{2};$$

so the full area is approximately $\pi(c - 1)^2$.

The above calculations show that for each fixed ϵ sufficiently small, there is a one-parameter family of periodic orbits, parameterized by c. As $c \to 1$ (if the critical points have been normalized) these periodic orbits approach the critical point $(1, 0)$.

To complete the lemma, it remains to show that the homoclinic orbit of Lemma 7.1 lies on a boundary of this family. Let $h_+(c, \tau, \epsilon)$, $0 < c < 1$, be the value of y_1 where the orbit of (7.4) starting at $y_1 = c$, $y_2 = 0$ hits $y_2 = 0$ again; similarly, define $h_-(c, \tau, \epsilon)$ by integrating backward. Let $h_+(0, \tau, \epsilon)$ be the value of $y_1 > 0$, where $U_{\tau,\epsilon}$ hits $y_2 = 0$ and similarly for $h_-(0, \tau, \epsilon)$. The equations for the periodic orbits and the homoclinic orbits could be written $h_+(c, \tau(c, \epsilon), \epsilon) = h_-(c, \tau(c, \epsilon), \epsilon)$.

We will show that h_+ and h_- are continuous in their dependence on c at $c = 0$. This will imply that $\tau(c, \epsilon)$ is continuous in c at $c = 0$ and, therefore, so is $h_+(c, \tau(c, \epsilon), \epsilon)$. (This suffices, since the homoclinic orbit passes through $y_1 = h_+(0, \tau(0, \epsilon), \epsilon)$, $y_2 = 0$ and the periodic solutions through $y_1 = h_+(c, \tau(c, \epsilon), \epsilon)$, $c > 0$.)

The continuity of h_+ cannot be concluded immediately from the continuous dependence of orbits on initial conditions since, as $c \to 0$, the time required to reach $y_2 = 0$ tends to ∞. However, near the critical point (0, 0), the system (7.4) behaves like its linearization. For a linear map, forward trajectories starting near the critical point (and off the stable manifold) stay close (as an embedded manifold, ignoring the parametric dependence on time) to the unstable manifold. That is, for the linearization of (7.4) around (0, 0), the point at which the unstable manifold intersects any vertical line $y_1 = \text{const}$ is the limit, as $c \to 0$, of the intersection point for the trajectory starting at $y_1 = c$, $y_2 = 0$. The continuity of h_+ at $c = 0$ now follows by using the closeness of (7.4) to its linearization for the part of each orbit with y_1 small and the continuous dependence of trajectories on initial conditions (for finite time) for the rest of the orbit.

Most of Theorem 7.1 follows immediately from Lemmas 7.1 and 7.2 by "descaling" the variables, especially the parameters τ and ϵ. Recall that $\mu = -b\epsilon^2 \operatorname{sign} \mu$ ($b < 0$ for definiteness). Then, $\nu = |b|^{1/2} \epsilon\tau$ and $x_1 = \epsilon^2 y_1$. The function $\tau(c, \epsilon)$ gives the value of τ for which there is a periodic orbit of (7.4) through $y_1 = c$, $y_2 = 0$, ($\dot{y}_2 > 0$), or a homoclinic orbit, if $c = 0$. In terms of the unscaled variables, there is a periodic orbit (or homoclinic orbit) through

$$\begin{aligned} x_1 = \epsilon^2 c, \qquad x_2 = 0, \qquad \dot{x}_2 > 0, \qquad \text{if} \quad \nu &= \nu(x_1, \mu) \\ &\equiv \mu^{1/2}\tau(|bx_1/\mu|, |\mu/b|^{1/2}). \end{aligned}$$

To finish the theorem, we must show

LEMMA 7.3. *For all $|\mu|$, $|\nu|$ sufficiently small, if* (7.2) *has neither a periodic orbit nor a homoclinic orbit, then there is a unique trajectory joining the critical points.*

Proof. Consider (7.2) with the x_i and μ scaled as before, but the ν unscaled. Also, assume the critical points are normalized at (0, 0) and (1, 0) as before. The equations are then

$$\begin{aligned} \dot{y}_1 &= y_2 + \epsilon q_1(y_i, \nu, \epsilon), \\ \dot{y}_2 &= y_1 + (\nu/\epsilon) y_2 - y_1^2 + \epsilon q_2(y_i, \nu, \epsilon). \end{aligned} \tag{7.5}$$

For these parameters, $\nu(c, \epsilon) \equiv \epsilon\tau(c, \epsilon)$ gives the value of ν for which (7.5) has a homoclinic orbit or periodic orbit.

Let $|\mu|, |\nu|$ be sufficiently small so that the change of coordinates from the original μ, ν to $(-\det, \mathrm{tr})$ is nonsingular. Also, $|\mu|, |\nu|$ (or equivalently ϵ and $|\nu|$) are small enough so that (7.5) has exactly two critical points for $|y_i| \leqslant 2$. Further, we require $\epsilon, |\nu|$ to be sufficiently small so that, for each ϵ, there is exactly one ν for which the linearization at $(1, 0)$ has pure imaginary eigenvalues. This curve in ϵ, ν space easily can be found using the implicit function theorem. We note for future reference that this curve is given by $\nu = \nu(1, \epsilon)$; i.e., the linearization at $(1, 0)$ has pure imaginary eigenvalues when (7.5) has an "infinitesimal" periodic solution. Thus, if $\nu > \nu(1, \epsilon)$, the eigenvalues at $(1, 0)$ have positive real part and for $\nu < \nu(1, \epsilon)$, the eigenvalues have negative real part.

Consider any value of $\nu > 0$ with ν sufficiently small as above. First, we show that for ϵ sufficiently small, but depending on ν, the conclusion holds. Along an orbit of (7.5), $dI/dt = (\nu/\epsilon)\, y_2^2 + \epsilon(y_2q_2 - y_1q_1 - y_1^2q_1)$. This implies that, integrating backward, dI/dt decreases along an orbit except possibly in a thin strip along the y_1 axis. (Choose, e.g., the strip so that $(\nu/\epsilon)\, y_2^2 > |\epsilon(y_2q_2 - y_1q_1 - y_1^2q_1)|$ for $0 \leqslant y_1 \leqslant 2$.) In that strip, $\dot{y}_1$ can be made arbitrarily small by taking ϵ small. Also, except near $(0, 0)$ and $(1, 0)$, $|\dot{y}_2|$ is bounded from below, so if the trajectory passes through this strip, the change in the y_1 coordinate as it passes through can be made arbitrarily small by taking ϵ small. This implies that, for ϵ sufficiently small, the stable manifold of the critical point $(0, 0)$ (to be denoted $S(0, 0)$) remains inside the homoclinic orbit of (7.5) with $\epsilon = \nu = 0$. There is only one critical point $(1, 0)$ inside this curve, and since $\nu > \nu(c, \epsilon) \equiv \epsilon\tau(c, \epsilon)$ for ϵ sufficiently small, this critical point has eigenvalues with positive real part. Also, there are no periodic orbits or homoclinic orbits in this region for such parameter values ν, ϵ. Thus, as $t \to -\infty$, $S(0, 0)$ must tend toward $(1, 0)$; equivalently, $S(0, 0)$ is contained in $U(1, 0)$ (the unstable manifold of $(1, 0)$). Similarly, for $\nu < 0$ and ϵ sufficiently small, $U(0, 0)$ is contained in $S(1, 0)$.

The above ϵ depends on ν, so the argument proves the conclusion only for a neighborhood of (a finite portion of) the ν-axis and we want the conclusion for all $\epsilon, |\nu|$ sufficiently small. The conclusion holds if $\nu = \nu(c, \epsilon)$ for any c. The other values of ν and ϵ fall into two connected regions: $\nu > \nu(c, \epsilon)$ for all c, or $\nu < \nu(c, \epsilon)$ for all c; the curve $\nu = \nu(1, \epsilon)$ lies between these two regions. Consider some ν_0, ϵ_0, for which $\nu_0 > \nu(c, \epsilon_0)$. There is a path in ν, ϵ space joining ν_0, ϵ_0 to $\nu_0, \epsilon(\nu_0)$, where $\epsilon(\nu_0)$ is sufficiently small so the above argument holds, and such that all ν, ϵ along the path satisfy $\nu > \nu(c, \epsilon)$. Then, everywhere along this path, the critical point $(0, 0)$ is a saddle and $(1, 0)$ is a source. Furthermore, the transversal intersection of $S(0, 0)$ and $U(1, 0)$ known to exist for ν_0, $\epsilon(\nu_0)$ cannot be broken anywhere along the path: In order for the connection to be broken, there must be a first ν, ϵ for which $S(0, 0)$ and $U(1, 0)$ are not transversal. At such a place, the boundary of $S(0, 0)$ must intersect the boundary of $U(1, 0)$. Since there are no other critical points nearby, this can happen only if this intersection is $S(0, 0)$ itself (a homoclinic orbit) or a closed path around $S(0, 0)$ (a periodic orbit). But since $\nu > \nu(c, \epsilon)$ everywhere along the path, there are no periodic orbits or homoclinic orbits.

Theorem 7.1 is not strictly a two-dimensional theorem; an n-dimensional version of it is as follows:

COROLLARY 7.1. *Let $\dot{X} = F_{\mu,\nu}(X)$ be a two-parameter family of ordinary differential equations on R^n, $n \geqslant 2$, with F smooth in all its arguments and $F_{\mu,\nu}(0) = 0$. Also, assume*

1. *$dF_{0,0}(0)$ has rank $n - 1$ and a zero eigenvalue of multiplicity 2. Let e be the right eigenvector of the zero eigenvalue and l the left eigenvector.*

2. *The mapping $(\mu, \nu) \rightsquigarrow (\det dF_{\mu,\nu}(0), \sigma_{n-1} dF_{\mu,\nu}(0))$ has nonzero Jacobian at $(\mu, \nu) = (0, 0)$. (Here, $\sigma_{n-1} dF_{\mu,\nu}(0)$ is the coefficient of the linear term in the characteristic polynomial of $dF_{\mu,\nu}(0)$).*

3. *Same as Hypothesis 3 of Theorem 7.1.*

Then, the conclusion of Theorem 7.1 follows.

Proof. The equations $\dot{X} = F_{\mu,\nu}(X)$, $\dot{\mu} = 0$, $\dot{\nu} = 0$ have a four-dimensional center manifold at the critical point $X = 0$, $\mu = \nu = 0$. We choose coordinates on that center manifold; in the new coordinates, the equations are $\dot{\bar{X}} = \bar{F}_{\mu,\nu}(\bar{X})$, $\dot{\mu} = 0$, $\dot{\nu} = 0$, where $\bar{X}$ is two-dimensional. To prove Corollary 7.1, we show $\dot{\bar{X}} = \bar{F}_{\mu,\nu}(\bar{X})$ satisfies the hypotheses of Theorem 7.1.

Hypotheses 1 and 3 clearly hold (by Lemma 2.1). To see that Hypothesis 2 above implies the second hypothesis of Theorem 7.1, we assume that $dF_{0,0}(0)$ is in Jordan normal form, with the upper left-hand 2×2 block all zeros, except for a 1 in the $(1, 2)$ position, and the lower right-hand $(n - 2) \times (n - 2)$ matrix is nonsingular. Let $\bar{\mu}(\mu, \nu)$ be the $(2, 1)$ entry of $dF_{\mu,\nu}(0)$ and $\bar{\nu}(\mu, \nu)$ be the trace of the upper 2×2 block. Hypothesis 2 above implies that $\partial(\bar{\mu}, \bar{\nu})/\partial(\mu, \nu)$ is nonsingular at

$\mu = \nu = 0$. But the 2×2 matrix $d\bar{F}_{\mu,\nu}(0)$ is the upper left-hand 2×2 block of $dF_{\mu,\nu}(0)$. Hence, the second hypothesis of Theorem 7.1 holds.

Remark 7.1. There is an analog of Theorem 7.1 for the unfolding of Theorem 2.2. Namely:

THEOREM 7.2. *Let $\dot{X} = G_{\mu,\nu}(X)$ be a two-parameter family of ordinary differential equations on R^n, with G smooth in all its arguments and $G_{0,0}(0) = 0$. Also assume*

1. *$dG_{0,0}(0)$ has rank $n - 1$ and a zero eigenvalue of multiplicity 2. Let e and l be as in Corollary 7.1.*

2. *The mapping $(\mu, \nu) \rightsquigarrow (l \cdot G_{\mu,\nu}(0), \sigma_{n-1} dG_{\mu,\nu}(0))$ has nonzero Jacobian at $(\mu, \nu) = (0, 0)$. [σ_{n-1} is as above.]*

3. *Same as Hypothesis 3 of Theorem 7.1.*

Then, the conclusion of Theorem 7.1 follows.

This theorem is proved in a way similar to that of Theorem 7.1. As before, an application of the center manifold theorem reduces the proof to the case $n = 2$, which we shall assume.

We show below that, by scaling, one can write $\dot{X} = G_{\mu,\nu}(X)$ as

$$\begin{aligned} \dot{y}_1 &= y_2 + \epsilon q_1(Y, \tau, \epsilon), \\ \dot{y}_2 &= 1 - y_1^2 + \tau y_2 + \epsilon q_2(Y, \tau, \epsilon), \end{aligned} \tag{7.6}$$

where $\epsilon \to 0$ as $\mu \to 0$ and τ is a scaled replacement for ν. The rest of the proof is essentially identical to that of Theorem 7.1. (Eqs. (7.6) and (7.4) are equivalent.)

To see that there is such a scaling, we first normalize the variables. Again, we assume that $A = dG_{0,0}(0)$ is in Jordan normal form. We first make several μ, ν-dependent nonsingular changes of the x_i. A μ, ν-dependent translation of x_2 enables us to to assume that the critical points are at $x_2 = 0$. A second such translation can be made to normalize the position of the degenerate critical points, which are the solutions of the four equations $G_{\mu,\nu}(X) = 0$, det $[dG_{\mu,\nu}(X)] = 0$. Hypothesis 2 implies that grad $l \cdot G_{\mu,\nu}(0)$ is not zero at $\mu = \nu = 0$. This, plus Hypothesis 3, implies that the 4×4 matrix of the linearization of $G_{\mu,\nu}(X)$, det $[dG_{\mu,\nu}(X)]$ has rank 3 at $X = 0$, $\mu = \nu = 0$ and may be solved for X and μ or ν in terms of the other parameter, say ν (otherwise interchange μ and ν). The degenerate critical points are then at $x_1 = \tilde{x}_1(\nu)$, $x_2 = 0$,

$\mu = \tilde{\mu}(\nu)$. If we choose new coordinates $\bar{x}_1 = x_1 - \tilde{x}_1(\nu)$, $\bar{\mu} = \mu - \tilde{\mu}(\nu)$, the degenerate critical points are all at $\bar{X} = 0$, $\bar{\mu} = 0$. We assume this has been done and drop the bars. In these translated coordinates, $\det dG_{0,\nu}(0) = 0$ for all ν. Finally, by a ν-dependent linear change of the x_i, we may assume that the a_{11} and a_{21} entries of $dG_{0,\nu}(0)$ are zero.

As in the normalization done for Theorem 7.1, we now change coordinates in parameter space. Let $\bar{\nu}(\mu, \nu)$ be the a_{22} entry of $dG_{\mu,\nu}(0)$ and $\bar{\mu}(\mu, \nu) = l \cdot G_{\mu,\nu}(0)$. By Hypothesis 2, $\partial(\bar{\mu}, \bar{\nu})/\partial(\mu, \nu)$ is nonsingular at $\mu = \nu = 0$, so we may use $\bar{\mu}$ and $\bar{\nu}$ as parameters. We drop the bars over the μ and ν. The equations now have the form

$$\dot{X} = \begin{pmatrix} a_{11}(\mu, \nu) & a_{12}(\mu, \nu) \\ a_{21}(\mu, \nu) & a_{22}(\mu, \nu) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mu \end{pmatrix} + Q(X, X) + \cdots, \tag{7.7}$$

where $a_{11}(0, \nu) = a_{21}(0, \nu) = 0$, $a_{12}(0, \nu) = 1$.

We now scale: As before, let b be the second component of $Q(e, e)$; we can assume b is negative and we consider only μ positive. (Otherwise we consider only μ negative.) Let $\epsilon = |b|^{-1/2} \mu^{1/4}$, $x_1 = \epsilon^2 y_1$, $x_2 = |b|^{1/2} \epsilon^3 y_2$, $t = |b|^{-1/2} \epsilon^{-1} \bar{t}$, $\nu = |b|^{1/2} \epsilon\tau$. Then, Eqs. (7.7) take the form (7.6).

Remark 7.2. The function $\tau(c, \epsilon)$ obtained in Lemma 7.2 depends on the higher-order terms in the equation. For small ϵ, one can get an approximate formula for $\tau(c, \epsilon)$, which is fairly easy to evaluate.

Define $\hat{q}_i$ by $q_i(y, \tau, \epsilon) = \hat{q}_i(Y) + \epsilon O(1) + \tau O(1)$ (see (7.4)). If $\epsilon \neq 0$, then along any orbit of (7.4), $dI/dt = \tau y_2^2 + \epsilon(-y_1 q_1 + y_1^2 q_1 + y_2 q_2)$. Let $H(y_1, y_2) = -y_1\hat{q}_1 + y_1^2\hat{q}_1 + y_2\hat{q}_2$, so $dI/dt = \tau y_2^2 + \epsilon H(y_1, y_2)$ + higher-order terms in ϵ and τ.

The equations for the periodic and homoclinic orbits are $I_+(c, \tau, \epsilon) = I_-(c, \tau, \epsilon)$. By the above, this may be written

$$\tau = -\epsilon \left[\frac{\oint_{c,0,0,+} H(y_1, y_2)\, dt - \oint_{c,0,0,-} H(y_1, y_2)\, dt}{2 \oint_{c,0,0,+} y_2^2\, dt} \right] + \cdots, \tag{7.8}$$

where the terms omitted are higher order in ϵ. Note that

$$0 \geqslant \oint_{c,0,0,-} y_2^2\, dt = - \oint_{c,0,0,+} y_2^2\, dt$$

by the symmetry of the trajectories involved.

Note that the numerator is evaluated along the integral curves, $I =$ const, of (7.4), with $\tau = \epsilon = 0$. These paths can be found explicitly.

(There is a simple formula.) Hence, one does not need to know how to integrate the more complicated system with $\tau \neq 0$ or $\epsilon \neq 0$ to evaluate these line integrals.

Remark 7.3. Takens [13] and Bogdanov (see [14]) have independently proved theorems about two-parameter families of equations similar to the family considered in this section. Takens studies, essentially,

$$\dot{x}_1 = x_2 + x_1^2, \qquad \dot{x}_2 = \mu + \nu x_1 - x_1^2. \tag{7.9}$$

Bogdanov considers

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \mu + \nu x_1 + x_1^2 + x_1 x_2. \tag{7.10}$$

Takens and Bogdanov study these equations from a different point of view. They are interested in normal forms for local singularities (equivalence classes of vector fields) and their unfoldings, the (complete) unfolding of, e.g., $\dot{x}_1 = x_2 + x_1^2$, $\dot{x}_2 = -x_1^2$ (which we denote $(7.9)_0$) is a two-parameter family such as (7.9) with the property that any perturbation of $(7.9)_0$ has the same phase–plane portrait as (7.9) for some μ and ν. (See [14] for details.)

Our approach does not involve normal forms. Indeed, we are essentially studying the singularity $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^2$, which is more degenerate than $(7.9)_0$ or $(7.10)_0$; it does not have a (complete) *two*-dimensional unfolding, in the sense given above. (We have been using the word unfolding in Section 7 to mean any two-parameter family satisfying the conditions of Theorem 7.1 or 7.2. These unfoldings need not have the property that any perturbation of $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^2$ is equivalent to something in the family.) To get a vector field with a complete two-dimensional unfolding (i.e., whose codimension is two) one must add some higher-order terms to $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^2$ that reduce the degeneracy. However, one can still conclude about this singularity all the same qualitative behavior one knows about the particular normal forms $(7.9)_0$ and $(7.10)_0$ and their unfoldings (7.9) and (7.10). That is, for the two-parameter family $\dot{X} = G_{\mu,\nu}(X)$ (in Theorem 7.2), as for (7.9) or (7.10), there is a one-parameter family of homoclinic orbits bounding a two-parameter family of periodic orbits. The only conclusion one cannot draw is how many of these bounded orbits there are for $\mu = \text{const}$, $\nu = \text{const}$, for example, one cannot conclude from the hypotheses of Theorem 7.2 (it is in general false) that there is, at most, one periodic orbit for any parameter values μ, ν. But if more is known about the vector fields, this information can be deduced from (7.8) (see below).

It seems to us that this way of studying singularities is more useful for applications than the categorical approach of normal form theory, which states, e.g., that $(7.10)_0$ is a normal form for a codimension two singularity with (7.10) as its universal unfolding. The hypotheses of Theorem 7.2 are easy to check and are less restrictive than requiring the basic singularity to be equivalent to $(7.10)_0$; the conclusions directly describe the qualitative behavior of the systems. This is analogous to the use of the Hopf theorem in applications (see Remark 7.4).

As Arnold [14] remarks, the hard part of proving that a vector field such as $(7.9)_0$ has a (complete) two-dimensional unfolding is showing that for each μ and ν, there is at most one periodic orbit. Formula (7.8) may be used to obtain a condition on the higher-order terms $\hat{q}_i$, which insures that for each ϵ, τ (equivalently each μ, ν of the unscaled problem) there is at most one periodic solution. For this to hold, it is necessary and sufficient that, for fixed $\epsilon \neq 0$, $\tau(c, \epsilon)$ be strictly monotone in c. For ϵ small, a sufficient condition for this is that the coefficient of ϵ in (7.8) have a derivative with respect to c that is nonzero.

For (7.9) or (7.10), it can be shown that $\tau(c, \epsilon)$ is strictly monotone in c. For (7.10), $H(y_1, y_2) = y_1 y_2^2$. By symmetry, (7.8) is then

$$\tau = -\epsilon \left[\oint_{c,0,0,+} y_1 y_2 \, dy_1 \Big/ \oint_{c,0,0,+} y_2 \, dy_1 \right] + \cdots. \tag{7.11}$$

We note that the coefficient of $-\epsilon$ is the y_1 coordinate of the center of gravity of the interior of the orbit of (7.4), $\tau = \epsilon = 0$, which passes through $y_1 = c$, $y_2 = 0$, $\dot{y}_2 > 0$. (Eq. 7.9) leads to the evaluation of the same integrals (after an integration by parts). Here is a proof that $\partial^2\tau/\partial\epsilon\partial c \,|_{\epsilon=0} \neq 0$:

The curves of $2I(y_1, y_2) = \alpha$ are given by

$$y_2 = \mathscr{Y}(y_1) = (y_1^2 - \tfrac{2}{3} y_1^3 + \alpha)^{1/2};$$

for the curve passing through $y_1 = c$, $y_2 = 0$, $\alpha = c^2(\frac{2}{3}c - 1)$. Let $\frac{1}{2}m = \oint \mathscr{Y} \, dy_1$, where the integral is taken as above. Note that m is the area inside the curve $2I = \alpha$ and that $\mathscr{Y}$ vanishes at the endpoints of the integral. Let $\bar{y}_1$ be the y_1 coordinate of the center of gravity of the region inside $2I = \alpha$, so $\frac{1}{2}m\bar{y}_1 = \oint \mathscr{Y} y_1 \, dy_1$. Then,

$$\frac{1}{2} m \left(\frac{d\bar{y}_1}{d\alpha}\right) = \frac{d}{d\alpha}\left(\left(\frac{1}{2} m\bar{y}_1\right)\right) - \bar{y}_1 \frac{d}{d\alpha}\left(\frac{m}{2}\right) = \oint \frac{y_1 - \bar{y}_1}{2\mathscr{Y}} \, dy_1 .$$

Since $\mathscr{Y}\mathscr{Y}' = y_1(1 - y_1)$, we have $(\bar{y}_1 - 1)\, m/2 = \oint(y_1 - 1)\,\mathscr{Y}\, dy_1 = -\oint(\mathscr{Y}^2\mathscr{Y}'/y_1)\, dy_1 = -\oint(\mathscr{Y}^3/3y_1^2)\, dy_1 < 0$. Thus, $\bar{y}_1 < 1$; clearly, $\bar{y}_1 \to 1$ as $\alpha \to -\frac{1}{3}$ and a simple computation shows that $\bar{y}_1 = 6/7$ at $\alpha = 0$.

If we integrate by parts in $\oint(y_1 - \bar{y}_1)\,\mathscr{Y}\, dy_1 = 0$, we find that

$$\oint [(y - \bar{y}_1)^2/2][y_1(1 - y_1)/\mathscr{Y}]\, dy_1 = 0.$$

Let $\bar{\mathscr{Y}}^2 = \mathscr{Y}^2(\bar{y}_1)$. Then,

$$\begin{aligned}\bar{\mathscr{Y}}^2 \frac{m}{2}\frac{d\bar{y}_1}{d\alpha} &= \bar{\mathscr{Y}}^2 \oint \frac{y_1 - \bar{y}_1}{2\mathscr{Y}}\, dy_1 - \frac{1}{2}\oint (y_1 - \bar{y}_1)\, dy_1 \\ &\quad + \frac{2}{3}\left(\bar{y}_1 + \frac{1}{2}\right)\oint \frac{(y - \bar{y}_1)^2}{2}\frac{y_1(1 - y_1)}{\mathscr{Y}}\, dy_1 \\ &= -\oint \frac{(y_1 - \bar{y}_1)^2}{3\mathscr{Y}}\left[\left(\bar{y}_1 - \frac{1}{2}\right)(y_1 - 1)^2 + (1 - \bar{y}_1)\left(\bar{y}_1 + \frac{1}{2}\right)\right] dy_1 .\end{aligned}$$

Since $0 < \bar{y}_1 \leqslant 1$ for $-\frac{1}{3} \leqslant \alpha \leqslant 0$, this shows that

$$d\bar{y}_1/d\alpha \leqslant -q(\alpha)(\bar{y}_1 - \tfrac{1}{2}), \tag{7.12}$$

where $q(\alpha) \geqslant 0$. Hence, $(d/d\alpha)[(\bar{y}_1 - \frac{1}{2}) \exp (\int_0^\alpha q(\alpha')\, d\alpha')] \leqslant 0$. Since $\bar{y}_1 - (1/2) = 5/14 > 0$ at $\alpha = 0$, this shows that $\bar{y}_1 > 6/7$ for $-\frac{1}{3} \leqslant \alpha < 0$ and (7.12) then shows that $\bar{y}_1$ decreases monotonically from 1 to 6/7 as α goes from $-\frac{1}{3}$ to 0. Also, $d\bar{y}_1/d\alpha$ (and hence, $d\bar{y}_1/dc$) is less than zero except at $\alpha = -\frac{1}{3}$, where $d\bar{y}_1/d\alpha = 0$.

Remark 7.4. Theorem 7.1 is very close in spirit to the Hopf bifurcation theorem [9–11] a simple version of which can be stated as follows:

THEOREM. *Let $\dot{X} = F_\tau(X)$ be a one-parameter family of smooth vector fields on R^2, such that $F_\tau(0) = 0$ for all τ sufficiently small. Assume further that*

1. *$dF_0(0)$ has pure imaginary eigenvalues.*
2. *If $\lambda(\tau)$ is an eigenvalue of $dF_\tau(0)$, then $(d/d\tau)$* Re $\lambda(\tau)|_{\tau=0} \neq 0$.

Then, among them, the one-parameter family of equations has a one-parameter family of periodic solutions; i.e., there is an invariant surface in X, τ space consisting of periodic solutions to $\dot{X} = F_\tau(X)$. The parameter of this family of solutions may be taken to be a measure of the amplitude of the oscillation around the critical point. These periodic solutions are sometimes said to bifurcate off the critical point.

Andronov and Chaiken [22] note in passing the relationship between periodic solutions bifurcating off a critical point and those bifurcating off a separatrix. To make the analogy explicit, we now show how the techniques of this section may be used to prove the Hopf theorem. (We give only a sketch of such a proof.)

We assume that coordinates have been chosen so that

$$dF_\tau(0) = \begin{pmatrix} \operatorname{Re} \lambda(\tau) & \operatorname{Im} \lambda(\tau) \\ -\operatorname{Im} \lambda(\tau) & \operatorname{Re} \lambda(\tau) \end{pmatrix}. \qquad \text{(See [9] for details.)}$$

We now scale the variables: $x_i = \epsilon y_i$, where ϵ is not defined. This transforms the one-parameter system $\dot{X} = F_\tau(X)$ into the two-parameter (ϵ and τ) system

$$\dot{Y} = A_\tau Y + \epsilon O(1), \qquad \text{where } A_\tau = \begin{pmatrix} \operatorname{Re} \lambda(\tau) & \operatorname{Im} \lambda(\tau) \\ -\operatorname{Im} \lambda(\tau) & \operatorname{Re} \lambda(\tau) \end{pmatrix}. \tag{7.13}$$

Let $(7.13)_0$ denote (7.13) with $\epsilon = 0$. The one-parameter (τ) family $(7.13)_0$ has a one-parameter family of periodic solutions, all of which occur for $\tau = 0$. These solutions are $Y = c(\cos qt, -\sin qt)$, where $iq = \lambda(0)$; they are parameterized by the amplitude c.

The idea of the proof is to show that for $c = 1$, there is an $\epsilon_0 > 0$ such that for each $\epsilon < \epsilon_0$, there is a $\tau(c, \epsilon)$ for which (7.13) has a periodic orbit through $y_1 = c$, $y_2 = 0$. Upon descaling the variables, we get a one-parameter family of periodic solutions, parameterized by ϵ, which passes through the initial conditions $x_1 = \epsilon$, $x_2 = 0$. These periodic solutions are the orbits in the conclusion of the Hopf theorem. Note that we may use $c = c_0$, for any $c_0 > 0$. Thus, (7.13) really has a two-parameter family of periodic solutions. This idea was essentially used by Hopf in his analytic version of the theorem.

The rest of this proof, like that of Theorem 7.1, is geometrical and quite different from Hopf's (or the Ruelle–Takens proof [10].) Let $I(y_1, y_2) = (y_1^2/2) + (y_2^2/2)$. Let $I_+(c, \tau, \epsilon)$ be the value of I at the point $(y_1, 0)$, where the trajectory of (7.13) starting at $(c, 0)$ hits $y_2 = 0$. $I_-(c, \tau, \epsilon)$ is defined by integrating backward. As in Lemma 7.2, we find the periodic orbits by solving the equations $I_+(c, \tau, \epsilon) - I_-(c, \tau, \epsilon) = 0$ for $\tau = \tau(c, \epsilon)$. Hence, we must show that

$$\left.\frac{\partial I_+}{\partial \tau}\right|_{\tau=\epsilon=0,\, c=1} \neq 0. \qquad \text{As before, } \left.\frac{\partial I_+}{\partial \tau}\right|_{\tau=\epsilon=0} = -\left.\frac{\partial I_-}{\partial \tau}\right|_{\tau=\epsilon=0}.$$

Now, $\partial I_+/\partial\tau = (\partial/\partial\tau) \oint (dI/dt)\, dt$, where the line integral is computed along the trajectory described above. Along this path, $dI/dt = \operatorname{Re} \lambda(\tau)[y_1^2 + y_2^2]$. (This requires that A_τ have the form as in (7.13).) By hypotheses, $(d/d\tau) \operatorname{Re} \lambda(\tau) \neq 0$, so $\partial I_+/\partial\tau \neq 0$.

We note that the Hopf theorem is about a one-parameter family of vector fields, in this proof artificially turned into a two-parameter system. On the other hand, the two-parameter family (7.4) of Theorem (7.1) cannot be collapsed into a one-parameter family because, when $\epsilon = 0$, (7.4) does not have the invariance under stretching ($x_i = \epsilon y_i$) that is displayed by $(7.13)_0$.

Also, we note that the transversality condition $(d/d\tau) \operatorname{Re} \lambda(\tau)|_{\tau=0} \neq 0$ alone implies the existence of a one-parameter family of periodic solutions for (7.13). A further hypothesis on the nonlinear terms is needed to ensure that there is, at most, one periodic solution for each value of τ. (See [9] for conditions ensuring that $(\partial^2\tau/\partial\epsilon^2)(1, 0) \neq 0$; the sign of this quantity determines the stability of the periodic orbit.) This closely parallels the situation in Theorems 7.1 and 7.2: Hypotheses 1–3 yield the existence of a one-parameter family of homoclinic orbits and a two-parameter family of periodic orbits, but further conditions on the nonlinear terms are needed to ensure that there is, at most, one periodic solution for each pair of parameter values μ, ν. For applications, it is sometimes useful to separate the criteria for existence of periodic solutions from those for stability (see, e.g., [23]).

Finally, we remark that the curve $\nu(1, \epsilon)$ (or equivalently $\tau(1, \epsilon)$), considered in Lemma 7.3, gives the values of ν at which (7.5) (resp. 7.4) has a Hopf bifurcation, when ϵ is fixed and ν (resp. τ) is varied.

8. Calculation of the Shock Trajectories by Iteration

In this section, we discuss how the shock trajectory may be found without explicitly calculating the center manifold. We start by discussing the case $k = 1$; our procedure is related to the one employed by Foy [6] for hyperbolic systems of conservation laws with viscosity.

Assume for definiteness, that the equations are (2.2). We may assume that

$$A = \begin{pmatrix} 0 & & \\ & B_+ & \\ & & B_- \end{pmatrix},$$

where B_+ (resp. B_-) is an $l_+ \times l_+$ (resp. $l_- \times l_-$) nonsingular matrix whose eigenvalues have positive (resp. negative) real part. Since $k = 1$, $l_+ + l_- = n - 1$. As in Section 4, let a be the first entry of the vector g and b the first entry of $Q(e, e)$, where $e = (1, 0,..., 0)$. Let $\epsilon = |(a/b)\,\mu\,|^{1/2}$. Let the scalar y and the vectors $U = (u_1, ..., u_{l_+})$ and $V = (v_1, ..., v_{l_-})$ be defined by $x_1 = \epsilon y_1$; $x_j = \epsilon^2 u_{j-1}$ for $2 \leqslant j \leqslant l_+ + 1$; $x_j = \epsilon^2 v_{j-l_+-1}$ for $l_+ + 1 \leqslant j \leqslant n$. Further, let $t = b^{-1}\epsilon^{-1}\bar{t}$. Then, (2.2) becomes

$$\dot{y} = 1 - y^2 + \epsilon r(y, U, V, \epsilon), \tag{8.1a}$$

$$\epsilon \dot{U} = B_+ U + g_+ + Q_+(e, e)\, y^2 + \epsilon r_+(y, U, V, \epsilon), \tag{8.1b}$$

$$\epsilon \dot{V} = B_- V + g_- + Q_-(e, e)\, y^2 + \epsilon r_-(y, U, V, \epsilon). \tag{8.1c}$$

In these equations, r, r_+ and r_- are $O(1)$, and g_+ and g_- are vectors of dimension l_+ and l_-, with $g = (a, g_+, g_-)$. Similarly, $Q = (b, Q_+, Q_-)$.

If $\epsilon = 0$, the solution to (8.1a) which is bounded for all t is tanh t (normalizing the t-origin so that $y(0) = 0$). We define $\eta(t)$ by $y(t) = \tanh t + \epsilon\eta(t)$. Also, let U_0 (resp. V_0) be the solution to $B_+ U + g_+ + Q_+(e, e) \tanh^2 t = 0$ (resp. $B_- v + g_- + Q_-(e, e) \tanh^2 t = 0$). ($U_0$ and V_0 exist since B_+ and B_- are invertible.) The vector $(\tanh t, U_0(t), V_0(t))$ is the lowest-order approximation to the trajectory we are seeking.

Equations (8.1) form the basis of the following iteration ($\eta_0 = 0$):

$$\dot{\eta}_{m+1} + 2 \tanh t \eta_{m+1} = -\epsilon \eta_m{}^2 + r(\tanh \tau, U_0, V_0, 0) + \epsilon\phi(\tau, \eta, U, V, \epsilon), \tag{8.2a}$$

where $r(\tanh \tau + \epsilon\eta, U, V, \epsilon) = r(\tanh \tau, U_0, V_0, 0) + \epsilon\phi(\tau, \eta, U, V, \epsilon)$. We choose the solution satisfying $\eta_{m+1}(0) = 0$:

$$\epsilon \dot{U}_{m+1} = B_+ U_{m+1} + g_+ + Q_+(e, e)[\tanh t]^2 + \epsilon\gamma^+(\eta_m, U_m, V_m, \epsilon, t), \tag{8.2b}$$

where $\gamma^+ = 2\eta + \epsilon\eta^2 + r_+(\tanh t + \epsilon\eta, U, V, \epsilon)$. Similarly,

$$\dot{V}\epsilon_{m+1} = B_- V_{m+1} + g_- + Q_-(e, e)[\tanh t]^2 + \epsilon\gamma^-. \tag{8.2c}$$

At each stage, we choose the unique solution to (8.2b) and (8.2c) that is bounded for all time. There is such a solution: Using the variation of constants formula, we may write the solution bounded as $t \to +\infty$ as

$$U_{m+1}(t) = -(1/\epsilon) \int_t^\infty e^{-B_+(t'-t)/\epsilon} \{g_+ + Q_+ \tanh^2 t' + \epsilon\gamma^+(\eta_m, U_m, V_m, \epsilon, t')\}\, dt'.$$

To see that this solution really is bounded as $t \to -\infty$, we let θ be defined by $t' = t + \epsilon\theta$. Then,

$$U_{m+1}(t) = -\int_0^\infty e^{-B_+\theta}\{g_+ + Q_+ \tanh^2(t+\epsilon\theta) + \epsilon\gamma^+(..., t+\epsilon\theta)\}\, d\theta. \tag{8.3a}$$

Similarly,

$$V_{m+1}(t) = \int_{-\infty}^0 e^{-B_-\theta}\{g_- + Q_- \tanh^2(t+\epsilon\theta) + \epsilon\gamma^-(..., t+\epsilon\theta)\}\, d\theta. \tag{8.3b}$$

Also, we note that the solution to (8.2a) may be expressed as

$$\eta_{m+1}(t) = \operatorname{sech}^2 t \int_0^t \cosh^2 t'[r(\tanh t', U_0, V_0, 0) + \epsilon(-\eta_m^2 + \phi)]\, dt'. \tag{8.3c}$$

For U, V bounded, $\eta_m(t)$ is bounded as $t \to \pm\infty$. Then, if U_m, V_m, and η_m are bounded, it follows that U_{m+1} and V_{m+1} are bounded.

Now, we show that there are uniform bounds for the U_m, V_m, and η_m. Let K_+ (resp. K_-) be an upper bound for

$$\left\| \int_0^\infty e^{-B_+\theta}\{g_+ + Q_+ \tanh(t+\epsilon\theta)\}\, d\theta \right\| \text{ and } \tfrac{1}{2}\| U_0 \| \text{ (resp.}$$

$$\left\| \int_{-\infty}^0 e^{-B_-\theta}\{g_- + Q_- \tanh(t+\epsilon\theta)\, d\theta \right\| \text{ and } \tfrac{1}{2}\| V_0 \|).$$

Let K be an upper bound for

$$\left| \operatorname{sech}^2 t \int_0^t \cosh^2 t' r(\tanh t', U_0, V_0, 0)\, dt' \right|.$$

Let ϵ be sufficiently small such that if $\| U \| < 2K_+$, $\| V \| < 2K_-$ and $| \eta | < 2K$, then

$$\left\| \epsilon \int_0^\infty e^{-B_+\theta} \gamma^+(\eta, U, V, \epsilon, t+\epsilon\theta)\, d\theta \right\| < K_+,$$

$$\left\| \epsilon \int_{-\infty}^0 e^{-B_-\theta} \gamma^-(\eta, U, V, \epsilon, t+\epsilon\theta)\, d\theta \right\| < K_-,$$

$$\left| \epsilon \operatorname{sech}^2 t \int_0^t \cosh^2 t'(-\eta^2 + \phi)\, dt \right| < K.$$

Then, since $\eta_0 = 0$, U_0, V_0 satisfy $| \eta_0 | < 2K$, $\| U_0 \| < 2K_+$, and $\| V_0 \| < 2K_-$, Eqs. (8.3) show that $2K$, $2K_+$ and $2K_-$ are uniform bounds for $| \eta_m |$, $\| U_m \|$, and $\| V_m \|$.

It is now easy to show that the iteration of (8.3) converges:

$$\eta_{m+1} - \eta_m = \epsilon \operatorname{sech}^2 t \int_0^t \cosh^2 t'(-\eta_m{}^2 + \eta_{m-1}^2 + \phi_m - \phi_{m-1})\, dt',$$

where $\phi_m = \phi(\eta_m, U_m, V_m, \epsilon, t')$. Since $\| \eta_m \|$, $\| U_m \|$, and $\| V_m \|$ are uniformly bounded, there are Lipshitz constants for ϕ. That is

$$| \phi_m - \phi_{m-1} | \leqslant L_\eta \| \eta_m - \eta_{m-1} \| + L_U \| U_m - U_{m-1} \| + L_V \| V_m - V_{m-1} \| .$$

Similarly,

$$U_{m+1} - U_m = -\epsilon \int_0^\infty e^{-B_+\theta} (\gamma_m{}^+ - \gamma_{m-1}^+)\, d\theta$$

and

$$V_{m+1} - V_m = \epsilon \int_{-\infty}^0 e^{-B_-\theta} (\gamma_m{}^- - \gamma_{m-1}^-)\, d\theta,$$

where $\gamma_m{}^\pm = \gamma^\pm(\eta_m, U_m, V_m, \epsilon, t)$. Again, there are Lipshitz constants for γ^+ and γ^-:

$$\begin{aligned} &\| \gamma_m{}^\pm - \gamma_{m-1}^\pm \| \\ &\qquad \leqslant L_\eta{}^\pm \| \eta_m - \eta_{m-1} \| + L_U{}^\pm \| U_m - U_{m-1} \| + L_V{}^\pm \| V_m - V_{m-1} \| . \end{aligned}$$

Now, for any vector $v(\theta)$,

$$\left\| \int_0^\infty e^{-B_+\theta}\, v\, d\theta \right\| \leqslant \left\| \int_0^\infty e^{-b_+\theta}\, d\theta \right\| \| v \| \leqslant (1/b_+) \| v \| ,$$

where b_+ is the smallest (positive) eigenvalue of B_+. Similarly,

$$\left\| \int_0^\infty e^{-B_-\theta}\, v\, d\theta \right\| \leqslant (1/b_-) \| v \| ,$$

where b_- is the absolute value of the numerically smallest eigenvalue of B_-. Also, $\cosh^2 t \leqslant e^{2t}$, $\operatorname{sech}^2 t \leqslant 4e^{-2t}$. Hence, $\operatorname{sech}^2 t \int_0^t \cosh^2 t' dt' \leqslant 2$.

Using the above estimates, we get

$$\begin{aligned} &\| \eta_{m+1} - \eta_m \| + \| U_{m+1} - U_m \| + \| V_{m+1} - V_m \| \\ &\qquad \leqslant \epsilon [(4M + 2L_\eta + (1/b_+) L_\eta{}^+ + (1/b_-) L_\eta{}^-) \| \eta_m - \eta_{m-1} \| \\ &\qquad\quad + (2L_U + (1/b_+) L_U{}^+ + (1/b_-) L_U{}^-) \| U_m - U_{m-1} \| \\ &\qquad\quad + (2L_V + (1/b_+) L_V{}^+ + (1/b_-) L_V{}^-) \| V_m - V_{m-1} \|] \\ &\qquad \leqslant \epsilon \text{ const } (\| \eta_m - \eta_{m-1} \| + \| U_m - U_{m-1} \| + \| V_m - V_{m-1} \|) \end{aligned}$$

Clearly by choosing ϵ sufficiently small, this is a contraction mapping.

Now, we turn to the case $k = 3$ where, after a suitable linear transformation, A can be assumed to have the diagonal block form $\operatorname{diag}(J, B_+, B_-)$ with B_+ and B_- as before and

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

After introducing ϵ and rescaling, we represent X by (Y, U, V), where Y is now three-dimensional and obtain the analog of (8.1):

$$\dot{Y} = JY + \tfrac{1}{2}(y_1^2 - 1)\,\mathbf{k} + \epsilon r(Y, U, V, \epsilon), \tag{8.4a}$$

$$\epsilon\dot{U} = B_+U + g_+ + Q_+(e, e)\,y_1^2 + \epsilon r_+\,, \tag{8.4b}$$

$$\epsilon\dot{V} = B_-V + g_- + Q_-(e, e)\,y_1^2 + \epsilon r_-\,. \tag{8.4c}$$

($\mathbf{k}$ is the third basis vector in the Y-space.) This system has (for small ϵ) a critical point with y_1 near 1 that we call (Y^+, U^+, V^+), and one with y_1 near -1 called (Y^-, U^-, V^-). In the case $k = 1$ the iterative scheme above made explicit use of the solution $\tanh t$ to the canonical problem. For $k = 3$ it appears to be preferable to use a somewhat different procedure in which the successive iterates are computed separately for $t > 0$ and $t < 0$, along with sequences of approximations to the critical points. We do not explicitly use solutions to the canonical problem, but the connecting trajectory for that case (discussed in Section 6 and below) is useful as a first approximation; starting with that trajectory, with $\epsilon = 0$, the iteration below converges in one step.

Now, suppose that we have the mth approximation for the connecting trajectory and for its endpoints. The $(m + 1)$th for the critical points is defined by

$$JY^{\pm}_{m+1} + \tfrac{1}{2}(y^{\pm}_{1,m+1} \mp 1)(y^{\pm}_{1,m} \pm 1)\,\mathbf{k} + \epsilon r(Y_m{}^{\pm}, U_m{}^{\pm}, V_m{}^{\pm}, \epsilon) = 0, \tag{8.5a}$$

$$B_+U^{\pm}_{m+1} + g_+ + Q_+(e, e)(y^{\pm}_{1,m+1})^2 + \epsilon r_+(Y_m{}^{\pm}, U_m{}^{\pm}, V_m{}^{\pm}, \epsilon) = 0, \tag{8.5b}$$

$$B_-U^{\pm}_{m+1} + g_- + Q_-(e, e)(y^{\pm}_{1,m+1})^2 + \epsilon r_-(Y_m{}^{\pm}, U_m{}^{\pm}, V_m{}^{\pm}, \epsilon) = 0. \tag{8.5c}$$

(It is easy to check, given Lipschitz constants for the functions r, that for small enough ϵ, these rules define sequences that converge to critical points.)

For the $(m+1)$th approximation to the connecting trajectory, with the normalization that $t = 0$ at the place where $y_1 = 0$, we first compute Y_{m+1} on $t \geqslant 0$ as a solution to the three-dimensional system

$$\dot{Y}_{m+1} - J_m^+(Y_{m+1} - Y_{m+1}^+)$$
$$= \epsilon r(Y_m, U_m, V_m, \epsilon) - \epsilon r(Y_m^\pm\ U_m^\pm, V_m^\pm, \epsilon) + \tfrac{1}{2}(y_{1,m} - y_{1,m}^+)^2\, \mathbf{k}, \tag{8.6}$$

where

$$J_m^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ y_{1,m}^+ & 0 & 0 \end{bmatrix}. \tag{8.7}$$

(It is easy to check that if all the above sequences converge, (8.6) becomes (8.4a).) Since $y_{1,m}^+$ is near 1, J_m^+ has one positive real eigenvalue $\lambda_+ = (y_{1,m}^+)^{1/3}$ (also near 1) with right eigenvector $E^+ = (1, \lambda_+, \lambda_+^2)$ and two conjugate complex ones $\omega\lambda_+$ and $\bar{\omega}\lambda_+$, where $\omega = (-1 + i(3^{1/2}))/2$. The right eigenvector F^+ corresponding to $\omega\lambda_+$ has components $(1, \lambda_+\omega, \lambda_+^2\omega^2)$, and the eigenvector corresponding to $\bar{\omega}\lambda_+$ is $\bar{F}^+$. The reciprocal basis of left eigenvectors is $E_*^+ = \frac{1}{3}(1, \lambda_+^{-1}, \lambda_+^{-2})$, $F_*^+ = \frac{1}{3}(1, (\omega\lambda_+)^{-1}, (\omega\lambda_+)^{-2})$ and $\bar{F}_*^+$. It is convenient to represent the solutions to (8.6) in the form

$$Y_{m+1} - Y_{m+1}^+ = C^+(t)\, E^+ + \operatorname{Re}(A^+(t) F^+). \tag{8.8}$$

Calling the right-hand side of (8.6) $f^+(t)$ (for brevity), one finds for C^+ and the complex function A^+ the equations

$$\dot{C}^+ - \lambda_+ C^+ = E_*^+ \cdot f^+(t), \tag{8.9}$$

$$\dot{A}^+ - \lambda_+\omega A^+ = 2F_*^+ \cdot f^+(t). \tag{8.10}$$

The only solution of (8.9) that approaches 0 as $t \to \infty$ is

$$C^+ = -e^{\lambda_+ t} \int_t^\infty e^{-\lambda_+ t'}\, E_*^+ \cdot f^+(t')\, dt'. \tag{8.11}$$

Assuming $\|f^+\| \leqslant \text{const}\ e^{-\alpha t}$, where α is some positive constant (it will be clear that this is the case, for small ϵ, for some value of α near $\frac{1}{2}$), all solutions of (8.10) tend to zero as $t \to \infty$ and are given by

$$A^+ = e^{\lambda_+\omega t}\left[A_0^+ + 2\int_0^t F_*^+ \cdot f^+(t')\, e^{-\lambda_+\omega t'}\, dt'\right] \tag{8.12}$$

for any value of $A_0{}^+$. (This is a two-parameter family parameterized by the real and imaginary parts of $A_0{}^+$.) Thus, the solutions of (8.6) that tend to Y^+_{m+1} as $t \to \infty$ have at $t = 0$ the value $Y_{m+1}(0) = Y^+_{m+1} + C^+(0)\, E^+ + \mathrm{Re}(A_0{}^+F^+)$. Of these, those with zero first component at $t = 0$ have the real part of $A_0{}^+$ determined by

$$\mathrm{Re}\, A_0{}^+ = -Y^+_{1,m+1} + \int_0^\infty e^{-\lambda_+ t} E_*{}^+ \cdot f^+(t)\, dt. \tag{8.13}$$

The imaginary part of $A_+{}^0$ remains arbitrary and we leave it so for the moment.

Next, we compute Y_{m+1} on $t < 0$ as a solution to the analogue of (8.6) obtained by replacing the $+$ superscripts with $-$ superscripts. Since J^- has eigenvalues close to the cube roots of -1, λ_+ is replaced by the negative number $\lambda_- = (Y^-_{1,m})^{1/3}$ and again we have, with the representation

$$Y_{m+1} - Y^-_{m+1} = C^-(t)\, E^- + \mathrm{Re}(A^-(t) F^-), \tag{8.14}$$

a unique function C^-, corresponding to solutions with $Y_{m+1} \to Y^-_{m+1}$ for $t \to -\infty$, given by

$$C^- = e^{\lambda_- t} \int_{-\infty}^t e^{-\lambda_- t'} E_*{}^- \cdot f^-(t')\, dt'. \tag{8.15}$$

Similarly, we have (for $t < 0$)

$$A^- = e^{\lambda_- \omega t} \left[A_0{}^- - 2 \int_t^0 F_*{}^- \cdot f^-(t')\, e^{-\lambda_- \omega t'}\, dt' \right], \tag{8.16}$$

and those solutions that also have first component zero for $t \to 0$ (from below) have the real part of $A_0{}^-$ determined by

$$\mathrm{Re}\, A_0{}^- = -y^-_{1,m+1} - \int_{-\infty}^0 e^{-\lambda_- t} E_*{}^- \cdot f^-(t)\, dt. \tag{8.17}$$

Finally, we determine the imaginary parts of $A_0{}^+$ and $A_0{}^-$ so that Y_{m+1} is continuous at $t = 0$ in the second and third components as well as in the first (where the limit from both sides is already zero). This requires

$$C^-(0)\, E^- + \mathrm{Re}(A_0{}^-F^-) = C^+(0)\, E^+ + \mathrm{Re}(A_0{}^+F^+). \tag{8.18}$$

Now, $\mathrm{Re}(A_0{}^\pm F^\pm) = \mathrm{Re}\, A_0{}^\pm \,\mathrm{Re}\, F^\pm - \mathrm{Im}\, A_0{}^\pm \,\mathrm{Im}\, F^\pm$, and $\mathrm{Im}\ F^\pm = (3^{1/2}/2)(0, \lambda_\pm, -\lambda_\pm{}^2)$. Since λ_+ and λ_- have opposite signs, the two vectors $\mathrm{Im}\, F^\pm$ are clearly linearly independent and span the subspace of vectors with zero first component. Thus, Im $A_0^{\pm}$ are uniquely determined and our rule for calculating Y_{m+1} from the data of the mth iterate is completely specified. To complete the iterative step we calculate U_{m+1} as the unique bounded solution of the equation (compare (8.4b))

$$\epsilon \dot{U}_{m+1} - B_+ U_{m+1} = g_+ + Q_+(e, e)\, y_{1,m+1}^2 + \epsilon r_+(Y_m, U_m, V_m, \epsilon) \tag{8.19}$$

and V_{m+1} from the analogous equation obtained from (8.4c). (That there are such solutions follows as in the $k = 1$ case.) If it is the case that Y_m, U_m and V_m approach Y_m^+, U_m^+ and V_m^+ with a bound of the form const $e^{-\alpha t}$ as $t \to \infty$, it is clear that $\| f^+ \|$ is similarly bounded (assuming Lipschitz continuity of the functions r); (8.11) then shows that $| C^+ | \leqslant$ const $e^{-\alpha t}$. Since λ_+ is near 1 for sufficiently small ϵ, (8.12) then shows that $| A^+ | \leqslant$ const $e^{-\lambda_+ t/2}$ + const $e^{-\alpha t}$. This is less than const $e^{-\alpha t}$ if α is positive and somewhat $< \frac{1}{2}$, say, $\frac{1}{4}$. Then, we see that $\| Y_{m+1} - Y_{m+1}^+ \| <$ const. $e^{-\alpha t}$ and it is clear from (8.19) that U_{m+1} and V_{m+1} also approach their limits with $e^{-\alpha t}$ bounds. Similar considerations apply for $t < 0$, so if we start our iteration procedure initially with such bounds, they will continue to hold.

It seems that the convergence of these sequences to solutions to our original problem could be proved, for sufficiently small ϵ, in a manner similar to the proof sketched above for the $k = 1$ case, *provided* that it could be done for $\epsilon = 0$. However, the latter appears to us to be more difficult than one might at first expect and it is for this reason that we adopted the approach of Section 6 in proving the existence of the solution to the canonical problem. The iteration does, however, converge, and reasonably rapidly. We first computed the solution to the canonical problem essentially by this method, later checking it by other calculations related to the method of Section 6. Successive iterates agreed in the sixth decimal place after seven iterations, starting with $y_1 = \operatorname{sgn}(t)$, $y_2 = y_3 = 0$. Numerical values are given in Table I. As mentioned above, this solution of the canonical problem would be useful as a starting point in applying the iteration to the general problem for small ϵ. For this purpose, high accuracy is of no consequence and one may use the following simple approximations to this function and its derivatives: Let $c(t)$ and $s(t)$ be the real and imaginary parts of $e^{\omega t}$; then, for $t \geqslant 0$, we have

$$y \cong 1 - 0.9073c + 0.3705s - 0.0927c^2 - 0.0379cs - 0.1368s^2,$$

$$y' \cong 0.7767c + 0.6027s + 0.0652c^2 - 0.0584cs + 0.1679s^2,$$

$$y'' \cong 0.1249c - 0.9793s - 0.1249c^2 + 0.2946cs - 0.1174s^2.$$

TABLE I — Values of the Odd Bounded Solution of $y''' = \frac{1}{2}(y^2 - 1)$

t	y	y'	y''
0.0	0.00000	0.84193	0.00000
0.2	0.16772	0.83198	−0.09906
0.4	0.33150	0.80268	−0.19258
0.6	0.48761	0.75565	−0.27554
0.8	0.63274	0.69344	−0.34387
1.0	0.76419	0.61927	−0.39476
1.2	0.87990	0.53679	−0.42685
1.4	0.97860	0.44978	−0.44015
1.6	1.05975	0.36190	−0.43592
1.8	1.12352	0.27643	−0.41641
2.0	1.17068	0.19616	−0.38450
2.2	1.20248	0.12325	−0.34341
2.4	1.22057	0.05920	−0.29637
2.6	1.22681	0.00489	−0.24640
2.8	1.22320	−0.03935	−0.19616
3.0	1.21173	−0.07370	−0.14780
3.2	1.19434	−0.09871	−0.10297
3.4	1.17282	−0.11520	−0.06281
3.6	1.14876	−0.12419	−0.02803
3.8	1.12357	−0.12679	0.00107
4.0	1.09839	−0.12414	0.02449
4.2	1.07418	−0.11736	0.04248
4.4	1.05166	−0.10748	0.05543
4.6	1.03133	−0.09549	0.06386
4.8	1.01355	−0.08220	0.06836
5.0	0.99849	−0.06837	0.06952

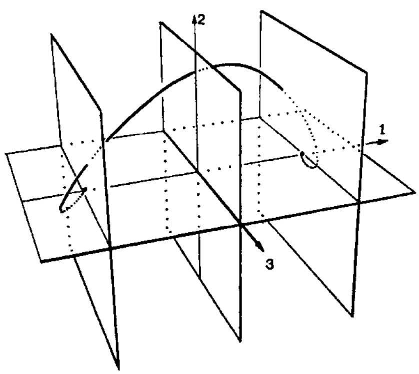

FIG. 3. The trajectory of $y''' = \frac{1}{2}(y^2 - 1)$ which joins the critical points.

The absolute error in these formulas is less than 0.001 for y, 0.0024 for y', and 0.004 for y'' on $t \geqslant 0$. (For $t \leqslant 0$ one should use the fact that y and y'' are odd, y' is even to compute the values.) Fig. 3 is an attempt at a perspective drawing of the connecting trajectory for this canonical example.

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