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# Some basic theorems in elastostatics of micropolar materials with voids

M. Marin

*Faculty of Mathematics, University ofBrasov, Str. Iuliu Maniu, No. 50, 2200 Brasov, Romania*

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**Abstract**

This work consists of basic equations and conditions of theory of micropolar bodies with voids. The initial-boundary value problem, in this context, is formulated. Moreover, some basic results of equilibrium are presented.

*Keywords:* Micropolar; Voids; Reciprocal theorem; Uniqueness; Minimum principle

## 1. Introduction

The origin of the theories of bodies with voids goes back to Goodman and Cowin [1]. The authors have introduced an additional degree of kinematic freedom to develop a continuum theory for flowing granular materials. Nunziato and Cowin [5] have established the theory of elastic materials with voids. Iesan [3] has established the equations of thermoelasticity of materials with voids.

In [6], Rusu applies the theory of semigroups of operators to obtain the existence and uniqueness of the solutions for the initial-boundary value problems in thermoelasticity of materials with voids. The objective of our paper is to extend the previous results to the case of the micropolar bodies with voids.

The theories of bodies with voids do not represent a material length scale, but are quite sufficient for a large number of solid mechanics applications. Our work is dedicated to the behavior of porous solids in which the matrix material is elastic and the interstices are voids of material. The intended applications of this theory are to geological materials, like rocks and soils and to manufactured porous materials.

First we establish the basic equations of the initial-boundary value problem in the context of linear of micropolar bodies with voids and we prove some basic theorems in elastostatics.


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## 2. Notations, formulas and problem formulation

For convenience, the notation and terminology choosen are almost identical to those of [3, 4]. Following Goodman and Cowin [1] we assume that the mass density has the form:

$$\rho = v\gamma, \tag{2.1}$$

where $\gamma$ is the density of the matrix material and $v$ is the volume fraction field ($0 < v \leqslant 1$).

The function $v$ represents a measure of the volume change of the bulk material which results from void compaction or distention. In the reference configuration we also have

$$\rho_0 = v_0\gamma_0. \tag{2.2}$$

Let $B$ be an open region of Euclidian three-dimensional space and bounded by the piece-wise smooth surface $\partial B$. We consider that the body, at time $t = 0$, occupies the region $B_0$ of Euclidian three-dimensional space and is bounded by the piece-wise smooth surface $\partial B_0$.

We use a fixed system of rectangular Cartesian axes and adopt the Cartesian tensor notation. The points in $B$ are denoted by $x_i$ and and in $B_0$ by $X_A$. The variable $t$ is time and $t \in [0, t_0)$. We shall employ the usual summation over repeated subscripts while subscripts preceded by a comma denote the partial differentiation with respect to the spatial argument. We also use a superposed dot to denote partial differentiation with respect to $t$.

Latin indices are understood to range over the integers (1, 2, 3) while the Greek indices have the range (1, 2).

The behavior of a micropolar body with voids is characterized by the following kinematic variables:

$$x_i = x_i(X, t), \quad \varphi_i = \varphi_i(X, t), \quad v = v(X, t), \qquad (X, t) \in \bar{B}_0 x[0, t_0). \tag{2.3}$$

We postulate an energy balance at time $t$ in the form:

$$\int_P \rho_0(\dot{e} + \dot{x}_i\ddot{x}_i + Y_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k\dot{v}\ddot{v})\,\mathrm{d}V = \int_P \rho_0(f_i\dot{v}_i + g_i\dot{\varphi}_i + L\dot{v})\,\mathrm{d}V + \int_{\partial P} \rho_0(T_i\dot{v}_i + M_i\dot{\varphi}_i + h\dot{v})\,\mathrm{d}A, \tag{2.4}$$

where $P$ is the corresponding region, at time $t = 0$, of an arbitrary region $\mathscr{P}$ of the continuum and $\partial P$ is corresponding $\partial P$.

In (2.4) we have used the following notations: $e$ — the internal energy per unit mass; $f_i$, $g_i$ — the body force and body couple per unit mass; $T_i$, $M_i$ — the stress vector and couple stress vector associated with the surface $\partial\mathscr{P}$, but measured per unit area of the surface $\partial P$; $L$ — the intrinsic equilibrated body force per unit mass; $h$ — the equilibrated stress associated with the surface $\partial\mathscr{P}$, but measured per unit area of the surface $\partial P$; $k$ — the equilibrated inertia; $Y_{ij}$ — the coefficients of inertia.

The physical significance of the functions $L$, $h$ and $k$ are presented in [1, 5].

Following the procedure of Green and Rivlin [2], to obtain the basic equations, we use the balance of energy and the invariance requirements under superposed rigid body motions.

First, by using a constant superposed rigid body translation velocity, we deduce

$$T_i = T_{Ki}N_i, \qquad T_{Ki,K} + \rho_0 f_i = \rho_0 \ddot{x}_i,$$
$$M_i = M_{Ki}N_K, \qquad M_{Ki,K} + \varepsilon_{ijs}x_{j,L}T_{Ls} + \rho_0 g_i = \rho_0 Y_{ij}\ddot{\varphi}_j, \tag{2.5}$$

where $N_A$ is the unit outward normal vector to the surface $\partial P$, $T_{Ki}$ is the first Piola–Kirchhoff stress tensor and $M_{Ki}$ is the couple stress tensor associated with surfaces in the deformed body which were originally the coordinate planes perpendicular to the axes $X_K$ through the point $(X_K)$, measured per unit area of these planes.

If we apply Eq. (2.4) to a region which in reference state was a tetrahedron bounded by coordinate plane and by a plane whose unit normal is $N_K$, we obtain

$$h = H_A N_A, \tag{2.6}$$

$H_A$ being the equilibrated stress associated with the surface $\partial\mathscr{P}$ which were originally coordinate planes perpendicular to the axes $X_K$ through the point $(X_A)$, measured per unit area of these planes.

With the aid of above relations, (2.5) and (2.6), we obtain the local form of the energy balance:

$$\rho_0 \dot{e} = H_K \dot{v}_{,K} - g\dot{v} + T_{Ki}(\dot{x}_{i,K} + \varepsilon_{ijs}x_{j,K}\dot{\varphi}_s) + M_{Ki}\dot{\varphi}_{i,K}, \tag{2.7}$$

where

$$g = \rho_0 \not{k}\ddot{v} + \tfrac{1}{2}\rho_0 \dot{\not{k}}\dot{v} - H_{K,K} - \rho_0 L. \tag{2.8}$$

We note that

$$x_{i,K} = u_{i,K} + \delta_{iK}, \tag{2.9}$$

where $u_i$ are the components of the displacement vector.

With the notations

$$T_{Ki} = x_{i,L}T_{KL}, \quad M_{Ki} = x_{i,L}M_{KL},$$
$$E_{K1} = \dot{x}_{i,L}(\dot{u}_{i,K} + \varepsilon_{ijs}x_{j,K}\dot{\varphi}_s), \quad \Lambda_{KL} = x_{i,K}\dot{\varphi}_{i,K}, \tag{2.10}$$

we can write (2.7) in the form

$$\rho_0 \dot{e} = H_K \dot{v}_{,K} - g\dot{v} + T_{KL}E_{KL} + M_{KL}\Lambda_{KL}. \tag{2.11}$$

In the context of the linear theory, we have

$$u_K = \delta_{iK}u_i, \quad \varphi_K = \delta_{iK}\varphi_i, \quad E_{KL} = \dot{u}_{L,K} + \varepsilon_{LKM}\dot{\varphi}_M, \quad \Lambda_{KL} = \dot{\varphi}_{L,K}, \tag{2.12}$$

where $\varepsilon_{LKM} = \delta_{iL}\delta_{iK}\delta_{rM}\varepsilon_{ijr}$.

By using the notations

$$\varepsilon_{KL} = u_{L,K} + \varepsilon_{LKM}\varphi_M, \qquad \gamma_{KL} = \varphi_{L,K}, \tag{2.13}$$

the energy balance (2.11) can be written in the form

$$\rho_0 \dot{e} = T_{KL}\dot{\varepsilon}_{KL} + M_{KL}\dot{\gamma}_{KL} + H_K \dot{v}_{,K} - g\dot{v}. \tag{2.14}$$

A micropolar elastic material with voids which does not conduct heat is defined by the following constitutive equations:

$$\begin{aligned} e &= \hat{e}(\varepsilon_{KL}, \gamma_{KL}, v, v_{,A}, \dot{v}, X_B), \\ T_{KL} &= \hat{T}_{KL}(\varepsilon_{KL}, \gamma_{KL}, v, v_{,A}, \dot{v}, X_B), \\ M_{KL} &= \hat{M}_{KL}(\varepsilon_{KL}, \gamma_{KL}, v, v_{,A}, \dot{v}, X_B), \\ H_K &= \hat{H}_K(\varepsilon_{KL}, \gamma_{KL}, v, v_{,A}, \dot{v}, X_B), \\ g &= \hat{g}(\varepsilon_{KL}, \gamma_{KL}, v, v_{,A}, \dot{v}, X_B). \end{aligned} \tag{2.15}$$

From the entropy production inequality (see [5]), we have

$$T_{KL} = \rho_0 \frac{\partial e}{\partial E_{KL}}, \qquad M_{KL} = \rho_0 \frac{\partial e}{\partial \gamma_{KL}}, \qquad H_K = \rho_0 \frac{\partial e}{\partial v_{,K}}, \qquad g = -\rho_0 \frac{\partial e}{\partial v} + F, \tag{2.16}$$

where $F$ is called the dissipation function. Because of linear theory, we may write

$$T_{Ki} = \delta_{iL} T_{KL}, \quad M_{Ki} = \delta_{iL} M_{KL}.$$

If we define

$$\begin{aligned} &f_{Ki} = \delta_{iK} f_i, \quad g_K = \delta_{iK} g_i, \quad Y_{KL} = Y_{ij} \delta_{iL} \delta_{jM} \\ &t_{ij} = \delta_{iK} \delta_{jL} T_{KL}, \quad m_{ij} = \delta_{iK} \delta_{jL} M_{KL}, \quad h_i = \delta_{iK} H_K, \end{aligned} \tag{2.17}$$

then all the above relations can be written using small indices. So, the geometrical equations (2.13) become

$$\varepsilon_{ij} = u_{j,i} + \varepsilon_{ijk} \varphi_k, \quad \gamma_{ij} = \varphi_{j,i}. \tag{2.18}$$

Let us introduce the notation:

$$\sigma = v - v_0,$$

$v_0$ is the volume fraction field for the reference configuration. If we assume that the body is free of initial stress and couple stress and has zero intrinsic equilibrated body force, then we can write

$$\begin{aligned} \rho_0 e = {} & \tfrac{1}{2} A_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + B_{ijmn} \varepsilon_{ij} \gamma_{mn} + \tfrac{1}{2} C_{ijmn} \gamma_{ij} \gamma_{mn} + \tfrac{1}{2} \xi \sigma^2 \\ & + \tfrac{1}{2} A_{ij} \sigma_{,i} \sigma_{,j} + B_{ij} \varepsilon_{ij} \sigma + C_{ij} \gamma_{ij} \sigma + D_{ijk} \varepsilon_{ij} \sigma_{,k} + E_{ijk} \gamma_{ij} \sigma_{,k} + d_i \sigma \sigma_{,i}, \end{aligned} \tag{2.19}$$

where the constitutive coefficients are prescribed functions of $X_s$ and they obey the following symmetries:

$$A_{ijmn} = A_{mnij}, \qquad C_{ijmn} = C_{mnij}, \qquad A_{ij} = A_{ji}. \tag{2.20}$$

From (2.16) and (2.19),

$$\begin{aligned} t_{ij} &= A_{ijmn} \varepsilon_{mn} + B_{ijmn} \gamma_{mn} + B_{ij} \sigma + D_{ijk} \sigma_{,k}, \\ m_{ij} &= B_{mnij} \varepsilon_{mn} + C_{ijmn} \gamma_{mn} + C_{ij} \sigma + E_{ijk} \sigma_{,k}, \\ h_i &= D_{mni} \varepsilon_{mn} + E_{mni} \gamma_{mn} + d_i \sigma + A_{ij} \sigma_{,j}. \end{aligned} \tag{2.21}$$

In the context of linear theory, the dissipation function $F$ must be a linear function in the independent constitutive variables. Following [5], we have

$$F = -\tau\dot{\sigma}, \tag{2.22}$$

where $\tau$ is a positive constitutive coefficient.

Then from (2.16) and (2.19) we obtain

$$g = -\tau\dot{\sigma} - B_{ij}\varepsilon_{ij} - C_{ij}\gamma_{ij} - d_i\sigma_{,i} - \xi\sigma. \tag{2.23}$$

Nunziato and Cowin [5] postulate that $\ell$ can be written in the form

$$\ell = \alpha + \hat{\alpha}\sigma, \tag{2.24}$$

where $\alpha$ and $\hat{\alpha}$ are prescribed functions of $X_i$ such that Eq. (2.8) reduces to

$$h_{i,i} + g + \rho_0 L = \rho_0\alpha\ddot{\sigma}. \tag{2.25}$$

Using small indices, the equations of motion $(2.5)_2$ and $(2.5)_4$ can be written in the following form:

$$\begin{aligned} &t_{ij,j} + \rho_0 f_i = \rho_0\ddot{u}_i,\\ &m_{ij,j} + \varepsilon_{ijk}t_{jk} + \rho_0 g_i = I_{ij}\ddot{\varphi}_j, \end{aligned} \tag{2.26}$$

where $I_{ij} = \rho_0 Y_{ij}$.

Finally, we consider the general boundary conditions

$$\begin{aligned} &u_i = \tilde{u}_i \quad \text{on } \overline{\partial B_1}\times[0,t_0), && t_i \equiv t_{ji}n_j = \tilde{t}_i \quad \text{on } \partial B_2\times[0,t_0),\\ &\varphi_i = \tilde{\varphi}_i \quad \text{on } \overline{\partial B_2}\times[0,t_0), && m_i \equiv m_{ji}n_j = \tilde{m}_i \quad \text{on } \partial B_4\times[0,t_0),\\ &\sigma = \tilde{\sigma} \quad \text{on } \overline{\partial B_5}\times[0,t_0), && h \equiv h_i n_i = \tilde{h} \quad \text{on } \partial B_6\times[0,t_0), \end{aligned} \tag{2.27}$$

where $n_i = \delta_{iK}N_K$ and $\overline{\partial B_i}$, $\partial B_i$ are subsets of the surfaces $\partial B$, such that

$$\overline{\partial B_1}\cup\partial B_2 = \overline{\partial B_3}\cup\partial B_4 = \overline{\partial B_5}\cup\partial B_6 = \partial B, \quad \partial B_1\cap\partial B_2 = \partial B_3\cap\partial B_4 = \partial B_5\cap\partial B_6 = \emptyset,$$

and the following standard initial conditions hold:

$$\begin{aligned} &u_i(X,0) = u_{0i}(X), \quad \dot{u}_i(X,0) = \dot{u}_{0i}(X), \quad \varphi_i(X,0) = \varphi_{0i}(X),\\ &\dot{\varphi}_i(X,0) = \dot{\varphi}_{0i}(X), \quad \sigma(X,0) = \sigma_0(X), \quad \dot{\sigma}(X,0) = \dot{\sigma}_0(X), \quad X\in\bar{B}. \end{aligned} \tag{2.28}$$

In (2.27) and (2.28) $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}_i$, $\tilde{m}_i$, $\tilde{\sigma}$, $\tilde{h}$, $u_{0i}$, $\dot{u}_{0i}$, $\varphi_{0i}$, $\dot{\varphi}_{0i}$, $\sigma_0$ and $\dot{\sigma}_0$ are prescribed functions.

In conclusion, the initial-boundary value problem of the linear theory of micropolar bodies with voids, consists of

- equations of motion: (2.26),
- balance of the equilibrated forces: (2.25),
- constitutive equations: (2.21) and (2.23),
- geometrical equations: (2.18),
- boundary conditions: (2.27),
- initial conditions: (2.28).

## 3. Basic theorems in elastostatics

Now we deduce some basic theorems in the time-independent behavior of linearly micropolar elastic materials with voids. In this context, the basic equations become

Equations of equilibrium:

$$\begin{aligned} &t_{ij,j} + \rho_0 f_i = 0, \\ &m_{ij,j} + \varepsilon_{ijk} t_{jk} + \rho_0 g_i = 0, \\ &h_{i,i} + g + \rho_0 L = 0; \end{aligned} \tag{3.1}$$

Constitutive equations:

$$\begin{aligned} &t_{ij} = A_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + B_{ij}\sigma + D_{ijk}\sigma_{,k}, \\ &m_{ij} = B_{mnij}\varepsilon_{mn} + C_{ijmn}\gamma_{mn} + C_{ij}\sigma + E_{ijk}\sigma_{,k}, \\ &h_i = D_{mni}\varepsilon_{mn} + E_{mni}\gamma_{mn} + d_i\sigma + A_{ij}\sigma_{,j}, \\ &g = -B_{ij}\varepsilon_{ij} - C_{ij}\gamma_{ij} - d_i\sigma_{,i} - \xi\sigma; \end{aligned} \tag{3.2}$$

Geometrical equations:

$$\varepsilon_{ij} = u_{j,i} + \varepsilon_{ijk}\varphi_k, \quad \gamma_{ij} = \varphi_{j,i}; \tag{3.3}$$

Boundary conditions:

$$\begin{aligned} &u_i = \tilde{u}_i \quad \text{on } \overline{\partial B_1}, \qquad t_{ij}n_j = \tilde{t}_i \quad \text{on } \partial B_2, \\ &\varphi_i = \tilde{\varphi}_i \quad \text{on } \overline{\partial B_3}, \qquad m_{ij}n_j = \tilde{m}_i \quad \text{on } \partial B_4, \\ &\sigma = \tilde{\sigma} \quad \text{on } \overline{\partial B_5}, \qquad h_i n_i = \tilde{h} \quad \text{on } \partial B_6. \end{aligned} \tag{3.4}$$

To avoid repeated regularity assumptions, we postulate

(i) $\rho_0$ is continuous and strictly positive on $\bar{B}$;

(ii) $A_{ijmn}, B_{ijmn}, C_{ijmn}, B_{ij}, D_{ijk}, C_{ij}, E_{ijk}, A_{ij}, d_i$ and $\xi$ are of class $C^2$ on $B$ and satisfy the symmetry relations (2.20);

(iii) $f, g$ and $L$ are continuous on $\bar{B}$.

An ordered array of functions $(u_i, \varphi_i, \sigma)$ is a *statically admissible* vector field on $\bar{B}$ provided $u_i, \varphi_i, \sigma \in C^1(\bar{B}) \cap C^2(B)$.

An ordered array of functions $(t_{ij}, m_{ij}, h_i, g)$ with the properties:

$-t_{ij}, m_{ij}, \sigma \in C^0(\bar{B}) \cap C^1(B)$;

$-t_{ij,j}, m_{ij,j}, h_{i,i}, g \in C^0(\bar{B})$

is an *admissible system of internal forces* on $\bar{B}$. By an *admissible state* on $\bar{B}$ we mean an ordered array $s = \{u_i, \varphi_i, \sigma, \varepsilon_{ij}, \gamma_{ij}, t_{ij}, m_{ij}, h_i, g\}$ with the properties:

$-(u_i, \varphi_i, \sigma)$ is a statistically admissible vector field on $\bar{B}$;

$-(t_{ij}, m_{ij}, h_i, g)$ is an admissible system of internal forces;

$-\varepsilon_{ij} = \varepsilon_{ji} \in C^0(\bar{B}) \cap C^1(B)$.

By an *external data system* on $\bar{B}$ we mean an ordered array $I, I = (f_i, g_i, L, \tilde{u}_i, \tilde{\varphi}_i, \tilde{\sigma}, \tilde{t}_i, \tilde{m}_i, \tilde{h})$ with the properties:
$-f_i, g_i, L \in C^0(\bar{B})$;
$-\tilde{u}_i \in C^0(\overline{\partial B_1}), \tilde{\varphi}_i \in C^0(\overline{\partial B_3}), \tilde{\sigma} \in C^0(\overline{\partial B_5})$;
$-\tilde{t}_i$ are piecewise regular on $\partial B_2$, $\tilde{m}_i$ are piecewise regular on $\partial B_4$, $\tilde{h}$ are piecewise regular on $\partial B_6$.

We say that $s = \{u_i, \varphi_i, \sigma, \varepsilon_{ij}, \gamma_{ij}, t_{ij}, m_{ij}, h_i, g\}$ is an *elastic state* on $\bar{B}$ corresponding to the forces $(f_i, g_i, L)$ if $s$ is an admissible state that satisfies the Eqs. (3.1)–(3.3).

### 3.1. Reciprocal theorem

We suppose that the body is subjected to two external data systems:

$$I^{(\alpha)} = \{f_i^{(\alpha)}, g_i^{(\alpha)}, L^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{\varphi}_i^{(\alpha)}, \tilde{\sigma}^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{m}_i^{(\alpha)}, \tilde{h}^{(\alpha)}\} \quad (\alpha = 1, 2)$$

and let

$$s^{(\alpha)} = \{u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \sigma^{(\alpha)}, \varepsilon_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}, m_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)}\} \quad (\alpha = 1, 2)$$

be two elastic states which correspond to the external data systems $I^{(\alpha)}$, respectively.

**Theorem 3.1.** *If* $s^{(\alpha)}$ $(\alpha = 1, 2)$ *are the elastic states corresponding to the different external data systems* $I^{(\alpha)}$ $(\alpha = 1, 2)$, *then we have the following Betti-type relation*:

$$\int_B \rho_0(f_i^{(1)}u_i^{(2)} + g_i^{(1)}\varphi_i^{(2)} + L^{(1)}\sigma^{(2)})\,dv + \int_{\partial B} \rho_0(t_i^{(1)}u_i^{(2)} + m_i^{(1)}\varphi_i^{(2)} + h^{(1)}\sigma^{(2)})\,da$$
$$= \int_B \rho_0(f_i^{(2)}u_i^{(1)} + g_i^{(2)}\varphi_i^{(1)} + L^{(2)}\sigma^{(1)})\,dv + \int_{\partial B} \rho_0(t_i^{(2)}u_i^{(1)} + m_i^{(2)}\varphi_i^{(1)} + h^{(2)}\sigma^{(1)})\,da, \tag{3.5}$$

*where* $t_i^{(\alpha)} = t_{ij}^{(\alpha)}n_j$, $m_i^{(\alpha)} = m_{ij}^{(\alpha)}n_j$, $h^{(\alpha)} = h_i^{(\alpha)}n_i$.

**Proof.** We consider the expression $U_{\alpha\beta}$, defined by

$$2U_{\alpha\beta} = t_{ij}^{(\alpha)}\varepsilon_{ij}^{(\beta)} + m_{ij}^{(\alpha)}\gamma_{ij}^{(\beta)} + h_i^{(\alpha)}\sigma_{,i}^{(\beta)} - g^{(\alpha)}\sigma^{(\beta)}. \tag{3.6}$$

With the aid of the constitutive equations (3.2) we get

$$2U_{\alpha\beta} = A_{ijmn}\varepsilon_{mn}^{(\alpha)}\varepsilon_{ij}^{(\beta)} + B_{ijmn}(\gamma_{mn}^{(\alpha)}\varepsilon_{ij}^{(\beta)} + \varepsilon_{ij}^{(\alpha)}\gamma_{mn}^{(\beta)}) + C_{ijmn}\gamma_{mn}^{(\alpha)}\gamma_{ij}^{(\beta)} + B_{ij}(\sigma^{(\alpha)}\varepsilon_{ij}^{(\beta)} + \varepsilon_{ij}^{(\alpha)}\sigma^{(\beta)})$$
$$+ C_{ij}(\sigma^{(\alpha)}\gamma_{ij}^{(\beta)} + \gamma_{ij}^{(\alpha)}\sigma^{(\beta)}) + D_{ijk}(\sigma_{,k}^{(\alpha)}\varepsilon_{ij}^{(\beta)} + \varepsilon_{ij}^{(\alpha)}\sigma_{,k}^{(\beta)})$$
$$+ E_{ijk}(\sigma_{,k}^{(\alpha)}\gamma_{ij}^{(\beta)} + \gamma_{ij}^{(\alpha)}\sigma_{,k}^{(\beta)}) + d_i(\sigma^{(\alpha)}\sigma_{,i}^{(\beta)} + \sigma_{,i}^{(\alpha)}\sigma^{(\beta)})$$
$$+ A_{ij}\sigma_{,j}^{(\alpha)}\sigma_{,j}^{(\beta)} + \xi\sigma^{(\alpha)}\sigma^{(\beta)}. \tag{3.7}$$

It is easy to see that from (3.7) we have

$$U_{12} = U_{21}. \tag{3.8}$$

On the other part, in view of relations (3.3), we deduce

$$
\begin{aligned}
2U_{\alpha\beta} &= t_{ji}^{(\alpha)}(u_{i,j}^{(\beta)} + \varepsilon_{ijk}\varphi_k^{(\beta)}) + m_{ji}^{(\alpha)}\varphi_{i,j}^{(\beta)} + h_i^{(\alpha)}\sigma_{,i}^{(\beta)} - g^{(\alpha)}\sigma^{(\beta)} \\
&= (t_{ji}^{(\alpha)}u_i^{(\beta)})_{,j} - t_{ij,j}^{(\alpha)}u_i^{(\beta)} + (m_{ji}^{(\alpha)}\varphi_i^{(\beta)})_{,j} - m_{ji,j}^{(\alpha)}\varphi_i^{(\beta)} \\
&\quad + \varepsilon_{ijk}t_{ji}^{(\alpha)}\varphi_k^{(\beta)} + (h_i^{(\alpha)}\sigma^{(\beta)})_{,i} - h_{i,i}^{(\alpha)}\sigma^{(\beta)} - g^{(\alpha)}\sigma^{(\beta)}.
\end{aligned}
$$

Next, by using the equations of equilibrium (3.1), from the above expression we arrive at

$$
2U_{\alpha\beta} = (t_{ji}^{(\alpha)}u_i^{(\beta)} + m_{ji}^{(\alpha)}\varphi_i^{(\beta)} + h_i^{(\alpha)}\sigma^{(\beta)})_{,i} + \rho_0 f_i^{(\alpha)}u_i^{(\beta)} + \rho_0 g_i^{(\alpha)}\varphi_i^{(\beta)} + \rho_0 L^{(\alpha)}\sigma^{(\beta)}. \tag{3.9}
$$

By integrating over $B$, in (3.9), with aid of the divergence theorem, we obtain

$$
2\int_B U_{\alpha\beta}\,\mathrm{d}v = \int_{\partial B}(t_i^{(\alpha)}u_i^{(\beta)} + m_i^{(\alpha)}\varphi_i^{(\beta)} + h^{(\alpha)}\sigma^{(\beta)})\,\mathrm{d}a + \int_B \rho_0(f_i^{(\alpha)}u_i^{(\beta)} + g_i^{(\alpha)}\varphi_i^{(\beta)} + L^{(\alpha)}\sigma^{(\beta)})\,\mathrm{d}v. \tag{3.10}
$$

From (3.8) and (3.10) we arrive at the desired result (3.5). □

### 3.2. Uniqueness results

Let

$$
I = \{f_i, g_i, L, \tilde{u}_i, \tilde{\varphi}_i, \tilde{\sigma}, \tilde{t}_i, \tilde{m}_i, \tilde{h}\}
$$

be an external data system, and

$$
s = \{u_i, \varphi_i, \varepsilon_{ij}, \gamma_{ij}, t_{ij}, m_{ij}, h_i, g\}
$$

the corresponding elastic state.

**Theorem 3.2.** *The following relation hold:*

$$
2\int_B \rho_0 e\,\mathrm{d}v = \int_{\partial B}(t_i u_i + m_i\varphi_i + h\sigma)\mathrm{d}a + \int_B \rho_0(f_i u_i + g_i\varphi_i + L\sigma)\,\mathrm{d}v. \tag{3.11}
$$

**Proof.** By using the constitutive equations (3.2), we get

$$
\begin{aligned}
&t_{ij}\varepsilon_{ij} + m_{ij}\gamma_{ij} + h_i\sigma_{,i} - g\sigma \\
&\quad = A_{ijmn}\varepsilon_{mn}\varepsilon_{ij} + 2B_{ijmn}\varepsilon_{ij}\gamma_{mn} + C_{ijmn}\gamma_{ij}\gamma_{mn} + 2B_{ij}\varepsilon_{ij}\sigma + 2C_{ij}\gamma_{ij}\sigma \\
&\qquad + 2D_{ijk}\varepsilon_{ij}\sigma_{,k} + 2E_{ijk}\varepsilon_{ij}\sigma_{,k} + A_{ij}\sigma_{,i}\sigma_{,j} + 2d_i\sigma\sigma_{,i} + \xi\sigma^2 = 2\rho_0 e.
\end{aligned} \tag{3.12}
$$

On the other part, in view of geometrical equations (3.3) and, next, by using the equations of equilibrium, we are led to

$$
\begin{aligned}
&t_{ij}\varepsilon_{ij} + m_{ij}\gamma_{ij} + h_i\sigma_{,i} - g\sigma \\
&\quad = (t_{ji}u_i + m_{ji}\varphi_i + h_i\sigma)_{,i} + \rho_0(f_i u_i + g_i\varphi_i + L\sigma).
\end{aligned} \tag{3.13}
$$

From (3.12) and (3.13) we deduce

$$2\rho_0 e = (t_{ji}u_i + m_{ji}\varphi_i + h_i\sigma)_{,i} + \rho_0 (f_i u_i + g_i\varphi_i + L\sigma). \tag{3.14}$$

By integrating over $B$ in (3.14) and using the divergence theorem we arrive at desired result (3.11). □

**Theorem 3.3.** *If the internal energy is a positive-definite quadratic form in the variables $\varepsilon_{ij}$, $\gamma_{ij}$, $\sigma$ and $\sigma_{,k}$, for any $X \in B$, then any two solutions of the our mixed problem are equal, modulo a rigid displacement.*

*Moreover, if $\partial B_1$ is non-empty, then the mixed problem has at most one solution on $B$.*

**Proof.** Let $(u_i^{(1)}, \varphi_i^{(1)}, \sigma^{(1)})$, $(u_i^{(1)} + v, \varphi_i^{(1)} + \psi, \sigma^{(1)} + \theta)$ be two solutions of the mixed problem. According to the linearity of our problem, we deduce that $(v_i, \psi_i, \theta)$ is also solution, but it corresponds to $f_i = 0$, $g_i = 0$ and $L = 0$.

Moreover, we have

$$t_i v_i + m_i\psi_i + h\theta = 0 \quad \text{on } \partial B.$$

With these considerations, the relation (3.10) reduces to

$$\int_B \rho_0 e \, dv = 0,$$

where $e$ is the internal energy corresponding to $(v_i, \psi_i, \theta)$. Since $e$ is positive-definite quadratic form, we deduce

$$\varepsilon_{ij} = 0, \qquad \gamma_{ij} = 0, \qquad \sigma = 0,$$

where $\varepsilon_{ij}$ and $\gamma_{ij}$ correspond to the differences of two solutions.

Therefore,

$$v_{j,i} + \varepsilon_{jik}\psi_k = 0, \qquad \psi_{j,i} = 0, \quad \sigma = 0,$$

and then

$$v_{i,j} + v_{j,i} = 0, \qquad \psi_i = \tfrac{1}{2}\varepsilon_{ijk}v_{k,j}, \quad \psi_{i,j} = 0.$$

Finally, we obtain

$$v_i = \alpha_i + \varepsilon_{ijk}\beta_j x_k, \quad \psi_i = \beta_i,$$

where $\alpha_i$ and $\beta_i$ are arbitrary constants. Moreover, if $\partial B_1$ is non-empty, then $\alpha_i = \beta_i = 0$, so that the proof of Theorem 3.3 is complete. □

### *3.3. Minimum principle*

An admissible state that satisfies the constitutive equations (3.2), the geometrical equations (3.3) and the following boundary conditions

$$u_i = \tilde{u}_i \quad \text{on } \overline{\partial B_1}, \qquad \varphi_i = \tilde{\varphi}_i \quad \text{on } \overline{\partial B_3}, \qquad \sigma = \tilde{\sigma} \text{ on } \overline{\partial B_5},$$

is called a *kinematically admissible state*. We denote with $K$ the following set

$$K = \{A\colon A \text{ is a kinematically admissible state}\}.$$

Let $F$ be the functional on $K$, defined by

$$F(A) = \int_B \rho_0 e\,\mathrm{d}v - \int_B \rho_0(f_i u_i + g_i \varphi_i + L\sigma)\,\mathrm{d}v - \int_{\partial B_2} \tilde{t}_i u_i\,\mathrm{d}a - \int_{\partial B_4} \tilde{m}_i \varphi_i\,\mathrm{d}a - \int_{\partial B_6} \tilde{h}\sigma\,\mathrm{d}a. \quad (3.15)$$

**Theorem 3.4.** *Assume that the internal energy is a positive-definite quadratic form. If $S$ is a solution of our mixed problem, then*

$$F(S) \leqslant F(A), \quad (3.16)$$

*for every $A \in K$, and equality holds only if $A = S$, modulo a rigid displacement.*

**Proof.** First, we observe that

$$\begin{aligned} F(A) = \tfrac{1}{2}\int_B &(A_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2B_{ijmn}\varepsilon_{ij}\gamma_{mn} + C_{ijmn}\gamma_{ij}\gamma_{mn} + \xi\sigma^2 \\ &+ A_{ij}\sigma_{,i}\sigma_{,j} + 2B_{ij}\varepsilon_{ij}\sigma + 2C_{ij}\gamma_{ij}\sigma + 2D_{ijk}\varepsilon_{ij}\sigma_{,k} \\ &+ 2E_{ijk}\gamma_{ij}\sigma_{,k} + 2d_i\sigma\sigma_{,i})\,\mathrm{d}v - \int_B \rho_0(f_i u_i + g_i\varphi_i + L\sigma)\,\mathrm{d}v \\ &- \int_{\partial B_2} \tilde{t}_i u_i\,\mathrm{d}a - \int_{\partial B_4} \tilde{m}_i\varphi_i\,\mathrm{d}a - \int_{\partial B_6} \tilde{h}\sigma\,\mathrm{d}a. \end{aligned} \quad (3.17)$$

Let $S = \{u_i, \varphi_i, \sigma, \varepsilon_{ij}, \gamma_{ij}, t_{ij}, m_{ij}, h_i, g\}$ be a solution of the mixed problem and $A = \{u_i^*, \varphi_i^*, \sigma^*, \varepsilon_{ij}^*, r_{ij}^*, t_{ij}^*, m_{ij}^*, h_i^*, g^*\} \in K$.

We denote with $A'$ the following kinematically admissible state:

$$A' = \{v_i, \psi_i, \theta, e_{ij}, \kappa_{ij}, \tau_{ij}, \mu_{ij}, q_i, \eta\},$$

where we have used the notations:

$$v_i = u_i^* - u_i, \quad \psi_i = \varphi_i^* - \varphi_i, \quad \theta = \sigma^* - \sigma, \quad e_{ij} = \varepsilon_{ij}^*\varepsilon_{ij}, \quad \kappa_{ij} = \gamma_{ij}^* - \gamma_{ij},$$

$$\tau_{ij} = t_{ij}^* - t_{ij}, \quad \mu_{ij} = m_{ij}^* - m_{ij}, \quad q_i = h_i^* - h_i, \quad \eta = g^* - g.$$

We deduce that $A'$ is a kinematically admissible state such that

$$\begin{aligned} &e_{ij} = v_{j,i} + \varepsilon_{ijk}\psi_k, \quad \kappa_{ij} = \psi_{j,i}, \\ &\tau_{ij} = A_{ijmn}e_{mn} + B_{ijmn}\kappa_{mn} + B_{ij}\theta + D_{ijk}\theta_{,k}, \\ &\mu_{ij} = B_{mnij}e_{mn} + C_{ijmn}\kappa_{mn} + C_{ij}\theta + E_{ijk}\theta_{,k}, \\ &q_i = D_{mni}e_{mn} + E_{mni}\kappa_{mn} + d_i\theta + A_{ij}\theta_{,j}, \\ &\eta = B_{ij}e_{ij} - C_{ij}\kappa_{ij} - d_i\theta_{,i} - \xi\theta. \\ &v_i = 0 \quad \text{on } \partial B_1, \qquad \psi_i = 0 \quad \text{on } \partial B_3, \qquad \sigma = 0 \quad \text{on } \partial B_5. \end{aligned} \quad (3.18)$$

Also, we have

$$\tau_{ij}n_j = \tilde{t}_1 \quad \text{on } \partial B_2, \qquad \mu_{ij}n_j = \tilde{m}_i \quad \text{on } \partial B_4, \qquad q_i n_i = \tilde{h} \quad \text{on } \partial B_6. \tag{3.19}$$

Then, from (3.17) and (3.18), we obtain

$$F(A) = F(S) + \int_B \rho_0 e' \,\mathrm{d}v + \int_B (t_{ij}e_{ij} + m_{ij}\kappa_{ij} + h_i\theta_{,i} - g\theta)\,\mathrm{d}v$$
$$- \int_B \rho_0(f_i v_i + g_i\psi_i + L\theta)\,\mathrm{d}v - \int_{\partial B_2} \tilde{t}_i v_i \,\mathrm{d}a - \int_{\partial B_4} \tilde{m}_i\psi_i\,\mathrm{d}a - \int_{\partial B_6} \tilde{h}\theta\,\mathrm{d}a, \tag{3.20}$$

where $e'$ is the internal energy which corresponds to $A'$.

In view of $(3.18)_1$ and with aid of equations of equilibrium (3.1), we have

$$t_{ij}e_{ij} + m_{ij}\kappa_{ij} + h_i\theta_{,i} - g\theta$$
$$= \rho_0(f_i v_i + g_i\psi_i + L\theta) + (t_{ij}v_j + m_{ij}\psi_j + h_i\theta)_{,i}. \tag{3.21}$$

By integrating over $B$ in (3.21), with aid of the divergence theorem and $(3.18)_6$, we have

$$\int_B (t_{ij}e_{ij} + m_{ij}\kappa_{ij} + h_i\theta_{,i} - g\theta)\,\mathrm{d}v$$
$$= \int_B \rho_0(f_i v_i + g_i\psi_i + L\theta)\,\mathrm{d}v + \int_{\partial B_2} t_{ij}n_j v_i\,\mathrm{d}a + \int_{\partial B_4} m_{ij}n_j\psi_i\,\mathrm{d}a + \int_{\partial B_6} q_i n_i\theta\,\mathrm{d}a. \tag{3.22}$$

From (3.20) and (3.22), we obtain

$$F(A) = F(S) + \int_B \rho_0 e'\,\mathrm{d}v + \int_{\partial B_2} (t_{ij}n_j - \tilde{t}_i)\,v_i\,\mathrm{d}a$$
$$+ \int_{\partial B_4} (m_{ij}n_j - \tilde{m}_i)\psi_i\,\mathrm{d}a + \int_{\partial B_6} (q_i n_i - \tilde{h})\theta\,\mathrm{d}a. \tag{3.23}$$

In view of the boundary conditions, from (3.23)

$$F(A) = F(S) + \int_B \rho_0 e'\,\mathrm{d}v.$$

But the internal energy $e'$ is a positive-definite quadratic form and thus we arrive at the desired results (3.16).

Further, equality holds if and only if $e' = 0$, i.e.,

$$\varepsilon_{ij}^* = \varepsilon_{ij} \quad \text{and} \quad \gamma_{ij}^* = \gamma_{ij},$$

from which we deduce the conclusion of Theorem 3.4. □

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