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International Journal of Solids and Structures 43 (2006) 6100–6114

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A theory for multiple collisions of rigid solids and numerical simulation of granular flow

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Received 25 May 2005; received in revised form 25 July 2005
Available online 28 November 2005

Abstract

Granular materials such as soils are often modeled as a collection of discrete bodies. In this paper we apply a strategy for soil modeling which is based on a discretization of the media with rigid polygons: a mechanical description of instantaneous collisions, based on the principle of virtual work is presented. Finally, a numerical example based on the proposed theory is treated. A package of randomly generated particles has been arranged on a skid, simulating the behaviour of a landslide.

Keywords: Discrete model; Instantaneous collisions; Principle of virtual work; Numerical simulation of a landslide; Atomized efforts contact dynamics respecting Clausius–Duhem inequality (A-CD^2)

1. Introduction

Contact problems are of great interest in civil engineering as the interaction between particles involves strong nonlinearities: as a consequence, a numerical simulation of these problems turns out to be very complex. The classical continuum approach presents several drawbacks, especially when large strains and crack propagation take place: many authors have therefore proposed a discrete approach, based on the representation of the medium as a collection of discrete rigid bodies (Frémond, 1995; Emeriault and Cambou, 1996; Cundall and Strack, 1979; Nardin and Schrefler, 2004). In particular, methods based on molecular dynamics (MD), e.g., (Nardin and Schrefler, 2004; Zavarise et al., 1992; Oden and Lin, 1986; Lee and Oden, 1993), consider, in general, the continuum discretized by means of a collection of rigid disks suitably linked with contact elements where the definition of the contact model reproducing the behaviour of the media results from the overlapping of the disks. The mechanical answer is then governed by the contact law which transforms the error of the penalty contact formulation into a displacement field (Nardin and Schrefler, 2004). This approach

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.10.001

turns out to be particularly effective in solving quasi-static problems, less in dynamics problems such as fragmentations or granular flows, where gap functions can hardly be defined (Kane et al., 1999). Beside the MD methods, other approaches dealing with non-smooth contacts are available, including the present as well as Moreau (1994) or Jean (1992). These methods are usually known as contact dynamics (CD) methods.

In this paper, rigid bodies contacts are studied by means of the principle of virtual work. When rigid bodies get into collision, it is no longer possible to solve the classical smooth equation of motion (velocities are not derivable): the application of the principle of virtual work in association with appropriate constitutive laws relating internal stress and velocities, allows to obtain a set of equations of motion, valid both for smooth and for non-smooth evolutions. In particular, this approach allows to overcome the limits of the classical penalty method (e.g. it does not require the definition of gap functions) and respect the actual physical condition of non-interpenetration of the particles. Moreover, compared to Moreau (1994) or Jean (1992) the existence and the uniqueness of the solution as well as the respect of the Clausius–Duhem inequality can be proved. The presented approach will be therefore called atomized efforts contact dynamics respecting the Clausius–Duhem inequality (A-CD^2), as it describes multiple bodies contact dynamics (respecting the Clausius–Duhem inequality) by means of an "Atomization" of the efforts exerted during contact (see Section 3). As an illustration of this approach, the simulation of a dynamic evolution of a multi-particle system (a granular flow) will be finally given.

2. The mechanical and mathematical model

As a first approach to the problem of contact, the collision between a moving point and a rigid fixed body will be presented in the following. This simplified formulation reduces the DOFs of the system and allows to focus on the collision mechanism.

In the following, the equation of motion will be given in the time interval $[t_1, t_2]$.

During the short duration of the collision, very large stresses are developed at the contact point in order to keep the velocities compatible. We assume that collisions are instantaneous, thus forces have to be concentrated in time. This kind of efforts are identified as percussions $\vec{P}^{\text{int}}$ (Dimnet, 2002, 2004). Velocity is therefore discontinuous at the instant of the collision t_c and its left and right limits will be noted $-$ and $+$.

2.1. The principle of virtual work

Interior efforts (both forces $\vec{r}^{\text{int}}$ and percussions $\vec{P}^{\text{int}}$, which are forces concentrated in time) are defined by their work.

The principle of virtual work leads us to choose the following expression for the virtual work of the interior efforts (Dimnet, 2002, 2004; Dimnet et al., 2003):

$$W^{\text{int}}(t_1, t_2, t_c, \vec{V}) = -\int_{t_1}^{t_2} \vec{r}^{\text{int}}(\tau)\vec{V}\,\mathrm{d}\tau - \vec{P}^{\text{int}}(t_c)\frac{\vec{V}^-(t_c) + \vec{V}^+(t_c)}{2} \tag{1}$$

where $\vec{V}$ is a virtual velocity of the point and t_c is a virtual instant of collision. In particular, Eq. (1) allows to establish a duality between the internal percussion P^{int} and the quantity $\frac{\vec{V}^-(t_c)+\vec{V}^+(t_c)}{2}$, which can be interpreted as the rate of deformation at t_c.

The virtual work of the acceleration efforts is (Dimnet, 2002, 2004; Dimnet et al., 2003):

$$W^{\text{acc}}(t_1, t_2, t_c, \vec{V}) = \int_{t_1}^{t_2} m\frac{\mathrm{d}\vec{U}(\tau)}{\mathrm{d}\tau}\vec{V}(\tau)\,\mathrm{d}\tau + m(\vec{U}^+(t_c) - \vec{U}^-(t_c))\frac{\vec{V}^-(t_c) + \vec{V}^+(t_c)}{2} \tag{2}$$

where m is the mass of the point and $\vec{U}$ is the actual velocity.

Finally, the exterior percussion is given in the following way (Dimnet, 2002, 2004; Dimnet et al., 2003):

$$W^{\text{ext}}(t_1, t_2, t_c, \vec{V}) = \int_{t_1}^{t_2} \vec{r}^{\text{ext}}(\tau)\vec{V}(\tau)\,\mathrm{d}\tau + \vec{P}^{\text{ext}}(t_c)\frac{\vec{V}^-(t_c) + \vec{V}^+(t_c)}{2} \tag{3}$$

The principle of virtual work implies that for any velocity $\vec{V}$ and any time t_c, the following expression holds:

$$W^{acc}(t_1, t_2, t_c, \vec{V}) = W^{int}(t_1, t_2, t_c, \vec{V}) + W^{ext}(t_1, t_2, t_c, \vec{V}) \tag{4}$$

According to this principle, the equations of motion assume the following form on $[t_1, t_2]$:

$$m\frac{d\vec{U}}{d\tau} = -\vec{r}^{int} + \vec{r}^{ext} \quad \text{almost everywhere} \tag{5}$$

and

$$m(\vec{U}^+ - \vec{U}^-) = -\vec{P}^{int} + \vec{P}^{ext} \quad \text{everywhere} \tag{6}$$

The internal percussion $\vec{P}^{int}$ in Eq. (6) is generally unknown and depends on the deformation rate (as a consequence of the duality established by Eq. (1) between and $\vec{P}^{int}$ and $\frac{\vec{V}^-(t_c)+\vec{V}^+(t_c)}{2}$). It is therefore necessary to introduce an appropriate set of constitutive laws describing the behaviour and the interactions between the colliding bodies.

2.2. The constitutive laws

In the case of a contact problem, constitutive laws describe the interactions among particles during the collision and have to assure the non-interpenetration of the bodies.

Internal percussion is therefore divided in two parts, a dissipative percussion $\vec{P}^{d}$ and a reactive percussion $\vec{P}^{reac}$:

$$\vec{P}^{int} = \vec{P}^{d} + \vec{P}^{reac} \tag{7}$$

2.2.1. The dissipative percussion

The dissipative percussion describes the (dissipative) interactions among the colliding bodies. A general, associated dissipative interaction can be described introducing a pseudopotential of dissipation Φ^{d}, which is a convex, positive function, null in the origin (Moreau, 1966; Frémond, 1995; Dimnet and Frémond, 1999; Dimnet et al., 2001; Dimnet, 2002; Pfeiffer, 2001):

$$\vec{P}^{d} \in \partial\Phi^{d}\left(\frac{\vec{U}^+ + \vec{U}^-}{2}\right) \tag{8}$$

2.2.2. The reactive percussion

The term $\vec{P}^{reac}$ describes the reaction to the non-interpenetration condition, which implies $U_N^+ \geqslant 0$ (where $U_N^+ = \vec{U}^+ \cdot \vec{N}$, see Fig. 1). This percussion is equal to 0 if $U_N^+ > 0$, is active if $U_N^+ = 0$ and implies that the

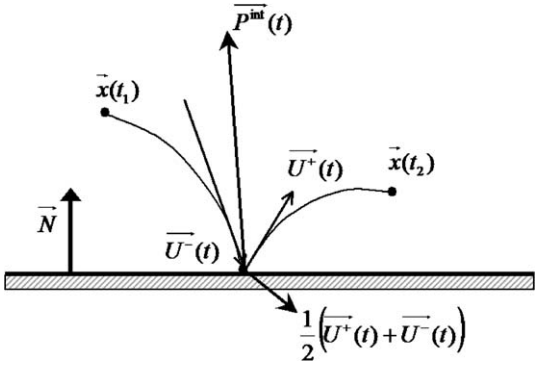

Fig. 1. Instantaneous collision between a point and a rigid and fixed plane.

condition $U_{\mathrm{N}}^{+} < 0$ cannot be verified. All these properties can be written by means of the indicator function (Moreau, 1966; Frémond, 1995) in the following way:

$$\vec{P}^{\mathrm{reac}} \in \partial I_K\left(\frac{\vec{U}^{+}+\vec{U}^{-}}{2}\cdot\vec{N}\right), \quad K=\left[\frac{U_{\mathrm{N}}^{-}}{2},+\infty\right] \tag{9}$$

In particular, as K is convex and contains the origin, the indicator functions I_K is a pseudopotential of dissipation (Dimnet and Frémond, 1999; Dimnet et al., 2001; Pfeiffer, 2001). The internal percussion can be therefore written in the following form:

$$\vec{P}^{\mathrm{int}} \in \partial\Phi\left(\frac{\vec{U}^{+}+\vec{U}^{-}}{2}\right) \quad \text{where } \Phi=\Phi^{\mathrm{d}}+I_K \tag{10}$$

i.e. the internal percussion derives from the pseudopotential of dissipation Φ.

2.3. Case of N colliding solids

The theory presented in the previous section, can be extended to a generalized form, valid for N colliding bodies. Contacts between solids are assumed to be punctual.

We can therefore consider N solids colliding at time t, defined by their mass m_i, a center of gravity G_i and an inertial tensor I_i. The kth contact between the solid i and the solid j takes place at points $A_{i,j,k}$. $S_{i,j}$ contains the contact points between the two solids: if no contact takes place, $S_{i,j}$ is not necessarily empty (i.e. if distant interactions are taken into account, $S_{i,j}$ is not empty). Percussion $\vec{P}^{\mathrm{int}}_{i,j,k}$ is applied at contact point $A_{i,j,k}$. $\vec{V}_i$ is the virtual velocity of the center of gravity G_i and $\vec{\omega}_i$ is the virtual rotational velocity. External percussions $\vec{P}^{\mathrm{ext}}_{i,l}$ are applied at points $B_{i,l}$ of solid i. S_i' contains the points $B_{i,l}$ on which external percussions are applied on the solid i.

If we define the vector $\widehat{V}=(\vec{V}_i,\vec{\omega}_i)$, we can write the relative velocities of the solids in contact at point $A_{i,j,k}$ in the following form:

$$\vec{D}_{i,j}(\widehat{V},A_{i,j,k})=\vec{V}_i+\vec{\omega}_i\wedge\overrightarrow{G_iA}_{i,j,k}-(\vec{V}_j+\vec{\omega}_j\wedge\overrightarrow{G_jA}_{i,j,k}) \tag{11}$$

and the velocities in contact points $B_{i,l}$ of solid i:

$$\vec{E}_i(\widehat{V},B_{i,l})=\vec{V}_i+\vec{\omega}_i\wedge\overrightarrow{G_iB}_{i,l} \tag{12}$$

In the following, we will focus our attention on the instant of the collision t. If we note $\vec{U}_i,\vec{\Omega}_i$, the actual velocities of the solid i and $\vec{V}_i,\vec{\omega}_i$ its virtual velocities, we can write the principle of virtual work in the following generalized form:

$$\begin{aligned}
&\forall\widehat{V},\sum_{i=1}^{N}\left\{m_i(\vec{U}_i^{+}-\vec{U}_i^{-})\cdot\left(\frac{\vec{V}_i^{+}+\vec{V}_i^{-}}{2}-\frac{\vec{U}_i^{+}+\vec{U}_i^{-}}{2}\right)+I_i(\vec{\Omega}_i^{+}-\vec{\Omega}_i^{-})\cdot\left(\frac{\vec{\omega}_i^{+}+\vec{\omega}_i^{-}}{2}-\frac{\vec{\Omega}_i^{+}+\vec{\Omega}_i^{-}}{2}\right)\right\}\\
&\quad+\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{A_{i,j,k}\in S_{i,j}}\vec{P}^{\mathrm{int}}_{i,j,k}\cdot\left\{\frac{1}{2}\left(\vec{D}_{i,j}(\widehat{V}^{+},A_{i,j,k})+\vec{D}_{i,j}(\widehat{V}^{-},A_{i,j,k})\right)\right.\\
&\quad\left.-\frac{1}{2}\left(\vec{D}_{i,j}(\widehat{U}^{+},A_{i,j,k})+\vec{D}_{i,j}(\widehat{U}^{-},A_{i,j,k})\right)\right\}-\sum_{i=1}^{N}\sum_{B_{i,l}\in S_i'}\vec{P}^{\mathrm{ext}}_{i,l}\cdot\left\{\frac{1}{2}\left(\vec{E}_i(\widehat{V}^{+},B_{i,l})+\vec{E}_i(\widehat{V}^{-},B_{i,l})\right)\right.\\
&\quad\left.-\frac{1}{2}\left(\vec{E}_i(\widehat{U}^{+},B_{i,l})+\vec{E}_i(\widehat{U}^{-},B_{i,l})\right)\right\}=0,
\end{aligned} \tag{13}$$

As in the previous case, it is necessary to introduce a set of constitutive laws describing the behaviour and the interactions among the colliding bodies.

2.4. The constitutive laws

Constitutive laws can be again defined in the following way:

$$\vec{P}^{\text{int}}_{i,j,k} \in \partial \Phi_{i,j,k}\left(\frac{\vec{D}_{i,j}(\widehat{U}^{+}, A_{i,j,k})) + \vec{D}_{i,j}(\widehat{U}^{-}, A_{i,j,k})}{2}\right) \tag{14}$$

$\Phi_{i,j,k}$ is a pseudopotential (or dissipation potential), sum of the pseudopotential describing the dissipation during the percussion and of the function $\widetilde{I_K} : \vec{D} \to I_K(\vec{D}\cdot\vec{N})$ assuring the non-interpenetration condition. The second term requires the existence of a normal, i.e. the boundary of at least one colliding solid has to be regular.

If we introduce Eq. (14) into Eq. (13) and apply the inequality of the sub-differential (see e.g. Frémond, 1995), we can write:

$$\begin{aligned}
&\forall \widehat{V}, \sum_{i=1}^{N}\left\{m_i(\vec{U}_i^{+} - \vec{U}_i^{-})\cdot\left(\frac{\vec{V}_i^{+} + \vec{V}_i^{-}}{2} - \frac{\vec{U}_i^{+} + \vec{U}_i^{-}}{2}\right) + I_i(\vec{\Omega}_i^{+} - \vec{\Omega}_i^{-})\cdot\left(\frac{\vec{\omega}_i^{+} + \vec{\omega}_i^{-}}{2} - \frac{\vec{\Omega}_i^{+} + \vec{\Omega}_i^{-}}{2}\right)\right\}\\
&\quad + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{A_{i,j,k}\in S_{i,j}}\left\{\Phi_{i,j,k}\left(\frac{1}{2}\left(\vec{D}_{i,j}(\widehat{V}^{+}, A_{i,j,k}) + \vec{D}_{i,j}(\widehat{V}^{-}, A_{i,j,k})\right)\right)\right.\\
&\quad \left. - \Phi_{i,j,k}\left(\frac{1}{2}\left(\vec{D}_{i,j}(\widehat{U}^{+}, A_{i,j,k}) + \vec{D}_{i,j}(\widehat{U}^{-}, A_{i,j,k})\right)\right)\right\}\\
&\quad - \sum_{i=1}^{N}\sum_{B_{i,l}\in S_i'}\vec{P}^{\text{ext}}_{i,l}\cdot\left\{\frac{1}{2}\left(\vec{E}_i(\widehat{V}^{+}, B_{i,l}) + \vec{E}_i(\widehat{V}^{-}, B_{i,l})\right) - \frac{1}{2}\left(\vec{E}_i(\widehat{U}^{+}, B_{i,l}) + \vec{E}_i(\widehat{U}^{-}, B_{i,l})\right)\right\} \geqslant 0.
\end{aligned} \tag{15}$$

If we note that every function

$$\widehat{V} \to \Phi_{i,j,k}(\vec{D}_{i,j}(\widehat{V}, A_{i,j,k})) = \Phi_{i,j,k}\left(\vec{V}_i + \vec{\omega}_i \wedge \overrightarrow{G_iA}_{i,j,k} - (\vec{V}_j + \vec{\omega}_j \wedge \overrightarrow{G_jA}_{i,j,k})\right) \tag{16}$$

is a pseudopotential of dissipation, the following function is a pseudopotential of dissipation, too:

$$\widehat{V} \to \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{A_{i,j,k}\in S_{i,j}}\Phi_{i,j,k}(\vec{D}_{i,j}(\widehat{V}, A_{i,j,k})) = \Phi(\widehat{V}) \tag{17}$$

The definition of the scalar product:

$$\langle \widehat{U}, \widehat{V}\rangle = \sum_{i=1}^{N}\left\{m_i\vec{U}_i\cdot\widehat{V}_i + I_i\vec{\Omega}_i\cdot\vec{\omega}_i\right\} \tag{18}$$

allows us to rewrite Eq. (15) in the following way:

$$\left\langle \widehat{U}^{+} - \widehat{U}^{-} - T^{\text{ext}}, \widehat{V} - \frac{\widehat{U}^{+} + \widehat{U}^{-}}{2}\right\rangle + \Phi(\widehat{V}) - \Phi\left(\frac{\widehat{U}^{+} + \widehat{U}^{-}}{2}\right) \geqslant 0 \tag{19}$$

for every $\widehat{V}$ belonging to R^{6N}, where T^{ext} is a vector of R^{6N} defined in the following way:

$$\langle T^{\text{ext}}, \widehat{V}\rangle = \sum_{k=1}^{N}\left\{\vec{R}_k\cdot\vec{V}_k + \vec{M}_k\cdot\vec{\omega}_k\right\} \tag{20}$$

where $\vec{R}_k$ is the resultant of all the exterior percussions applied on the solid k and $\vec{M}_k$ is the resultant of their angular moment applied with respect to the centre of gravity of the solid k.

Given that R^{6N} has a scalar product defined by $\langle .,.\rangle$ and given the definition of subgradient, the formulation (19) is equivalent to the following inclusion:

$$-(\widehat{U}^{+} - \widehat{U}^{-} - T^{\text{ext}}) \in \partial\Phi\left(\frac{\widehat{U}^{+} + \widehat{U}^{-}}{2}\right) \tag{21}$$

or, given $X = \frac{\widehat{U}^{+} + \widehat{U}^{-}}{2}$, to the following, more concise, form:

$$2\widehat{U}^{-} + T^{\text{ext}} \in 2X + \partial\Phi(X) \tag{22}$$

This inclusion is finally equivalent to the minimization problem

$$\text{Inf}\left\{\langle Y, Y\rangle + \Phi(Y) - \langle 2\widehat{U}^{-} + T^{\text{ext}}, Y\rangle | Y \in R^{6N}\right\} \tag{23}$$

as the argument X which gives the minimum verifies:

$$0(6N) \in 2X + \partial\Phi(X) - 2\widehat{U}^{-} - T^{\text{ext}} \tag{24}$$

i.e. the inclusion (22).

2.5. Coulomb's friction

In the previous section, constitutive laws have been derived from a pseudopotential of dissipation. However, many real behaviours are well modeled by means of non-associated constitutive laws. In particular, the behaviour of brittle materials such as rocks, concrete or ceramics is well represented by means of Coulomb's friction law. In this particular case, the tangential behaviour is described as follows:

$$|\vec{P}_{\text{T}}| \leqslant \mu|\vec{P}_{\text{N}}| \quad \text{where } \mu > 0 \text{ and} \tag{25}$$

$$\text{if } |\vec{P}_{\text{T}}| < \mu|\vec{P}_{\text{N}}| \text{ then } \vec{X}_{\text{T}} = 0 \tag{26}$$

$$\text{if } |\vec{P}_{\text{T}}| = \mu|\vec{P}_{\text{N}}| \text{ then } \exists\lambda \quad \text{such that } \vec{X}_{\text{T}} = \lambda\vec{P}_{\text{T}} \tag{27}$$

This behaviour does not derive from a pseudopotential of dissipation, however it can be shown (Dimnet, 2002) that the solution $\vec{U}^{+}$ of the problem exists and is unique if the following condition is respected:

$$\sum_{i,j,k} \mu_{i,j,k} < \frac{1}{16\sqrt{NC}M(1+M')^2} \tag{28}$$

where NC is the number of contacts and M,M' are positive real numbers (Dimnet, 2002).

Further details about the numerical treatment of Coulomb's friction law will be given in a forthcoming paper (Dal Pont and Dimnet, submitted for publication).

3. Numerical solution

In the previous section, a theory describing the behaviour of rigid bodies during instantaneous collisions has been proposed. In the following, the equations describing the evolution of the system will be integrated leading to a form suitable for a numerical solution. During its evolution, the system is subjected to internal and external actions.

The equations of motion for the velocity V (see also Section 2.3) have the following form:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -f^{\text{int}} + f^{\text{ext}} \quad \text{almost everywhere,} \tag{29}$$

$$V^{+} - V^{-} = -P^{\text{int}} + P^{\text{ext}} \quad \text{everywhere} \tag{30}$$

where f^{int} is a vector of R^{6N} representing the sum of the internal forces exerted on the system during the regular evolution of the system. In particular, at contact k the positions $6(k-1)+1,2,3$ represent the three linear momentums in the principal directions as well as $6(k-1)+4,5,6$ represent the three angular momentums. Similarly, f^{ext} represents the sum of the external forces exerted on the system during the regular evolution of the system (e.g. weight, see also Section 3.1) and $P^{\text{int}}/P^{\text{ext}}$ stand for the sum of the internal/external percussions during contacts.

If the cumulated efforts R^{int} and R^{ext} are taken into account:

$$R^{\text{int}}(t) = \int_0^t f^{\text{int}}(\tau)\,\mathrm{d}\tau + \int_0^t \sum_{t_i} P^{\text{int}}(t_i)\delta_{t_i}(\tau)\,\mathrm{d}\tau = \int_0^t f^{\text{int}}(\tau)\,\mathrm{d}\tau + \sum_{t_i} P^{\text{int}}(t_i)H(t-t_i), \tag{31}$$

$$R^{\text{ext}}(t) = \int_0^t f^{\text{ext}}(\tau)\,\mathrm{d}\tau + \int_0^t \sum_{t_j} P^{\text{ext}}(t_j)\delta_{tj}(\tau)\,\mathrm{d}\tau = \int_0^t f^{\text{ext}}(\tau)\,\mathrm{d}\tau + \sum_{t_j} P^{\text{ext}}(t_j)H(t-t_i) \tag{32}$$

where t_i and t_j are the instants when internal and external percussions are exerted, δ is the Dirac function and H is the Heaviside function. We can therefore write:

$$\mathrm{d}V(t) = -\mathrm{d}R^{\text{int}}(t) + \mathrm{d}R^{\text{ext}}(t) \tag{33}$$

The solution of this equation can be calculated introducing the following approximations:

- The time length $[0, T]$ is discretized in n regular steps $[t_k, t_{k+1}]$ of length $\Delta_n = \frac{T}{n}$. In this time step, active forces are "atomized", i.e. replaced by percussions exerted at the instant $\theta_k = t_k + \frac{1}{2}\Delta_n$.
- All the percussions exerted during the time gap $[t_k, t_{k+1}]$ are also exerted at the instant θ_k.
- Velocities are therefore discontinuous at the instants θ_k, when the percussions are exerted.

3.1. Atomization of a regular force

The "atomization" of a regular force f on the time interval $[t_k, t_{k+1}]$, consists in replacing it with the percussion P exerted at the instant θ_k. This result derives from the so-called Percussion Method (Dimnet, 2002). If f depends on a time-dependent quantity y, we can write:

$$\int_{t_k}^{t_{k+1}} f(y(\tau))\,\mathrm{d}\tau \simeq (t_{k+1}-t_k) f\left(\frac{y^+(\theta_k)+y^-(\theta_k)}{2}\right) \tag{34}$$

i.e. f can be replaced by the percussion:

$$P(t) = \Delta_n f\left(\frac{y^+(\theta_k)+y^-(\theta_k)}{2}\right)\delta_{\theta_k}(t) \tag{35}$$

which is the derivative of:

$$E_P(t) = \Delta_n f\left(\frac{y^+(\theta_k)+y^-(\theta_k)}{2}\right)H(t-\theta_k) \tag{36}$$

3.1.1. Constant force

A constant force f_0 can be atomized replacing in any interval a percussion of intensity $f_0\Delta_n$. If we consider, for example, the action of the weight $-\int_{t_k}^{t_{k+1}} g\,\mathrm{d}\tau$ at time θ_k, is approximated by

$$(0,0,-g\Delta_n,0,0,0,\ldots\ldots,0,0,-g\Delta_n,0,0,0,\ldots\ldots,0,0,-g\Delta_n,0,0,0) \tag{37}$$

3.1.2. Time dependent force

If the force $f(t)$ is exerted on the system at the time gap $[t_k, t_{k+1}]$, it can be replaced by a percussion exerted at the instant θ_k of intensity $\Delta_n f(\theta_k)$.

3.1.3. Position dependent force

If the force $f(Y(t))$ is exerted on the system at the time $[t_k, t_{k+1}]$, it can be replaced by a percussion exerted at the instant θ_k of intensity $\Delta_n f(Y(\theta_k))$. For example, that two point are elastically bounded and we note the elongation $x(t)$ at the instant t, the modulus of the elastic force exerted on every point is $k|x(t)|$. This force is therefore replaced by a percussion of intensity $\Delta_n k|x(\theta_k)|$.

3.1.4. Velocity dependent force

If the force $f(V)$ is exerted on the system at the time $[t_k, t_{k+1}]$, it can be replaced by a percussion exerted at the instant θ_k, where velocities are discontinuous. On $[t_k, \theta_k[$ the velocity of the elements of the system is $V^-(\theta_k)$ and on $]\theta_k, t_{k+1}]$ is $V^+(\theta_k)$. For this reason, the intensity of the atomized percussion is $f\left(\frac{V^-(\theta_k)+V^+(\theta_k)}{2}\right)\Delta_n$.

If the force $f(V)$ derives from a pseudopotential of dissipation, the internal percussion is an associated law. An example of this kind of behaviour is the viscous friction. If we consider that the contact in the point $A_{i,j,l}$ follows a viscous behaviour, we can write:

$$f(V) \in \partial\Phi_{i,j,k}(D_{\mathrm{T},i,j}(V, A_{i,j,l})) \tag{38}$$

$$\Phi_{i,j,k}(\vec{D}) = \frac{1}{2} v \vec{D}^2 \tag{39}$$

i.e. a function of the velocities, where $D_{\mathrm{T},i,j}(V, A_{i,j,l})$ gives the relative tangential velocity of the solids at point $A_{i,j,l}$. This force is replaced by a percussion of intensity:

$$P \in \Delta_n \partial\Phi_{i,j,k}\left(\frac{D_{\mathrm{T},i,j}(V^+(\theta_k), A_{i,j,l})) + D_{\mathrm{T},i,j}(V^-(\theta_k), A_{i,j,l})}{2}\right) \tag{40}$$

If we consider that at the instant t_k the solids i and j are in contact and that $D_{i,j}(V, A_{i,j,k})$ gives the relative velocity at $A_{i,j,l}$, noting $\vec{N}$ the normal vector, we can define the contact force $f(V)$ as a function of the normal right velocity:

$$f(V) = 0 \quad \text{if } D_{i,j}(V^+, A_{i,j,l}) \cdot \vec{N} > 0, \tag{41}$$

$$f(V) < 0 \quad \text{if } D_{i,j}(V^+, A_{i,j,l}) \cdot \vec{N} = 0 \tag{42}$$

which can be also written:

$$f(V) \in \partial I_{[0,+\infty[}\left(D_{i,j}(V^+, A_{i,j,l}) \cdot \vec{N}\right) \tag{43}$$

This force is replaced by the percussion:

$$P \in \Delta_n \partial I_{[0,+\infty[}\left(\frac{D_{i,j}(V^+, A_{i,j,l}) + D_{ij}(V^-, A_{i,j,l})}{2} \cdot \vec{N}\right) \tag{44}$$

given $D_{ij}(V^-, A_{i,j,l}) \cdot \vec{N} = 0$.

Finally, a friction force following Coulomb's law can also be taken into consideration. $\vec{V}_{\mathrm{T}}$ stands for the tangential friction velocity. The force $(f_{\mathrm{N}}, \vec{f}_{\mathrm{T}})$ has to verify:

$$\|\vec{f}_{\mathrm{T}}\| \leqslant \mu|f_{\mathrm{N}}| \quad \text{where } \mu > 0 \text{ and} \tag{45}$$

$$\text{if } \|\vec{f}_{\mathrm{T}}\| < \mu|f_{\mathrm{N}}| \text{ then } \vec{V}_{\mathrm{T}} = \vec{0} \tag{46}$$

$$\text{if } \|\vec{f}_{\mathrm{T}}\| = \mu|f_{\mathrm{N}}| \text{ then } \exists\lambda > 0 \quad \text{such that } \vec{V}_{\mathrm{T}} = -\lambda\vec{f}_{\mathrm{T}} \tag{47}$$

The expression of the atomized percussion $(P_{\mathrm{N}}, \vec{P}_{\mathrm{T}})$ which replaces the force according to Eqs. (34) and (35), has the same expression defined in Section 2.5:

$$\|\vec{P}_{\mathrm{T}}\| \leqslant \mu|P_{\mathrm{N}}| \quad \text{where } \mu > 0 \text{ and} \tag{48}$$

$$\text{if } \|\vec{P}_{\mathrm{T}}\| < \mu|P_{\mathrm{N}}| \text{ then } \vec{X}_{\mathrm{T}} = \vec{0}, \tag{49}$$

$$\text{if } \|\vec{P}_{\mathrm{T}}\| = \mu|P_{\mathrm{N}}| \text{ then } \exists\lambda > 0 \quad \text{such that } \vec{X}_{\mathrm{T}} = -\lambda\vec{P}_{\mathrm{T}}, \tag{50}$$

$$\text{where } \vec{X}_{\mathrm{T}} = \frac{1}{2}\left(\overrightarrow{V^+}_{\mathrm{T}} + \overrightarrow{V^-}_{\mathrm{T}}\right). \tag{51}$$

3.2. Solution algorithm

At each time θ_k, the problem to solve can be written in one of the three following forms:

$$\begin{aligned}
&\forall V, \langle U^+ - U^- - T^{\text{ext}}, V - \frac{U^+ + U^-}{2}\rangle + \Phi(V) - \Phi\left(\frac{U^+ + U^-}{2}\right) \geqslant 0 \\
&2U^- + T^{\text{ext}} \in 2X + \partial\Phi(X) \quad \text{where } X = \frac{U^- + U^+}{2} \\
&\inf_{Y\in R^{6n}}\{\langle Y, Y\rangle + \Phi(Y) - \langle 2U^- + T^{\text{ext}}, Y\rangle\}
\end{aligned} \tag{52}$$

In particular, if the internal percussions are derived from a pseudopotential of dissipation:

$$X \to \Phi_{i,j,k}(D_{i,j}(X, A_{i,j,k})) \tag{53}$$

the function:

$$X \to \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{A_{i,j,k}\in S_{i,j}} \Phi_{i,j,k}(D_{i,j}(X, A_{i,j,k})) = \Phi(X) \tag{54}$$

is a pseudopotential of dissipation that can be divided in two parts:

$$\Phi(X) = \Phi^{\text{d}}(X) + \Phi^{r}(X) \tag{55}$$

$\Phi^r(X)$ is the sum of all the indicator functions and is a function modelling the non-interpenetration conditions and renamed B_l ($l = 1, p$) (p is the total number of indicator functions):

$$\begin{aligned}
&\Phi^r(X) = \sum_{i=1}^{p} I\left[d_l\left(\frac{V^-}{2}\right), +\infty\right]^{(\varphi_l(X))} \\
&d_l(X) = D_{i,j}(X, B_l)\cdot\vec{N}_l, \quad l = 1, p
\end{aligned} \tag{56}$$

$\Phi^{\text{d}}(X)$ is considered differentiable.

The equation is therefore equivalent to:

$$\mathscr{F}(Y) = \langle Y, Y\rangle + \Phi(Y) - \langle 2V^- + P^e, Y\rangle, \quad \text{where } Y \in R^{6N} \tag{57}$$

which implies the following minimization problem, i.e. given 55, the reactive percussion is explicitly written providing the non-interpenetration condition:

$$\mathscr{F}(Y) = \langle Y, Y\rangle + \Phi^{\text{d}}(Y) - \langle 2V^- + P^e, Y\rangle \tag{58}$$

where $Y \in \Omega$, $\Omega = \left\{Y \in R^{6N} / \varphi_l(Y) = -d_l(Y) + d_l\left(\frac{V^-}{2}\right) \leqslant 0, \quad l = 1, p\right\}$.

The solution of this minimization problem is a saddle point of the application (Lagrangian) (Dimnet, 2002):

$$L(Y, \mu) \in \Omega \times R_+^p \to \mathscr{F}(Y) + \sum_{i=1}^{p} \mu_l \varphi_l(Y) \tag{59}$$

The domain Ω as well as the functions $\mathscr{F}$ and $\Phi^{\text{d}}(Y)$ are convex. This means that if X is a solution of the problem, it exists at least one $\lambda \in R_+^p$ such that (X, λ) is a saddle point of L. If (X, λ) is a saddle point of L then $X \in \Omega$ and X is a solution of the problem.

The numerical solution of such a problem can be found by means of an iterative method such as e.g. the classic Uzawa method (see e.g. Ciarlet, 1989).

4. Numerical simulations

To illustrate the theory, some numerical simulations based on the described numerical method will be presented. The objective of these simulations is to reproduce the behaviour of a rockslide in the perspective of a comparison with experimental results. At this stage, an experimental analysis of a landslide is in due course at the École Polytechnique Fédérale de Lausanne (EPFL) in Switzerland: the A-CD^2 method presented in the

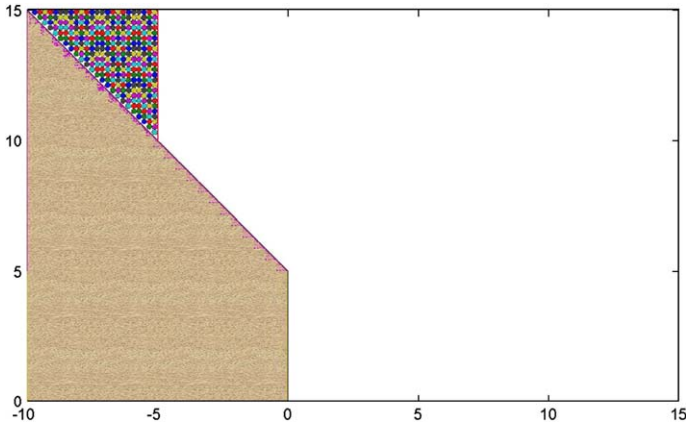

Fig. 2. Initial packing of the polygons ($r = 0.25$ m).

previous section turns out to be well adapted to reproduce such a dynamic case. Nevertheless, at this step of the study, a quantitative comparison between the presented results and the experiments is not yet possible due to the lack of material data and experimental results. The numerical analysis is still of interest for presenting the effectiveness of the proposed method dealing with dynamic problems as molecular dynamics methods are well adapted in reproducing quasi-statics problems but are less effective if applied to cases such as a granular flow (Kane et al., 1999).

In order to simulate the dispersion of material properties of an irregular assembly, a random generation of the initial configuration has been set. This means that the bodies have a random number of sides and a random initial rotation. The mass and the inertia modulus of every polygon is therefore calculated accordingly to its random number of sides. As particles are formed by polygons, contact detection turns out to be more complex than in the case of circular particles. The simulation handles both vertex–side contacts, vertex–vertex and

Table 1
Parameters used in the numerical simulation

Radius	0.25 m
Density	2500 kg m^{-3}
K_N	78 kg m^{-1}
K_T	78 kg m^{-1}
Δt	10^{-4} s

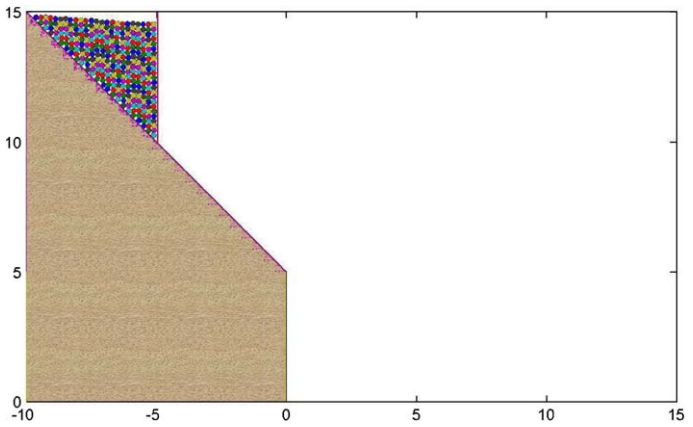

Fig. 3. Final packing of the polygons.

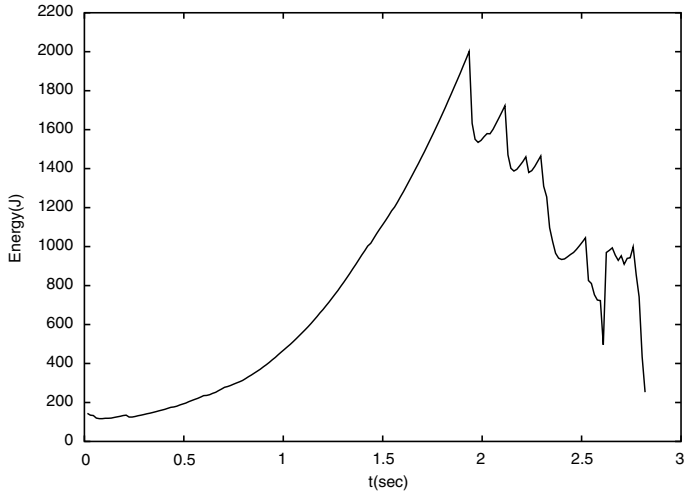

Fig. 4. Kinetic energy during packing.

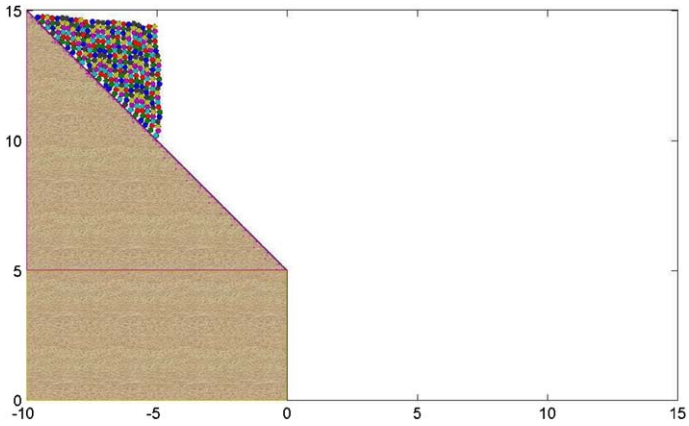

Fig. 5. Evolution of the system at $t = 3$ s.

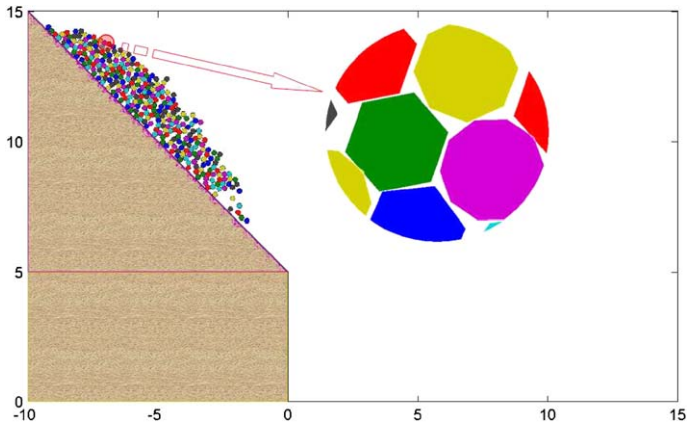

Fig. 6. Evolution of the system at $t = 5$ s.

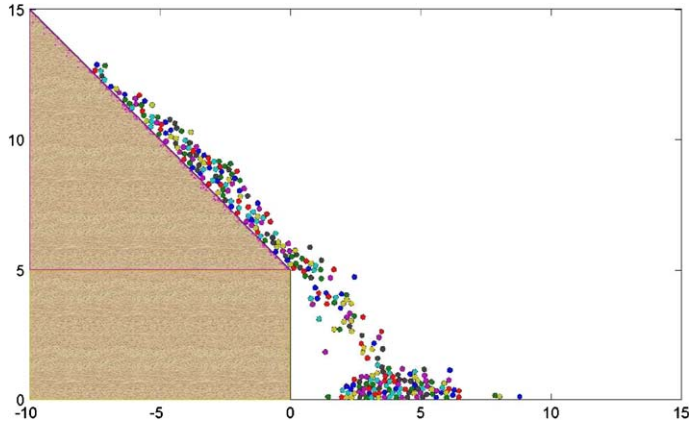

Fig. 7. Evolution of the system at $t = 8$ s.

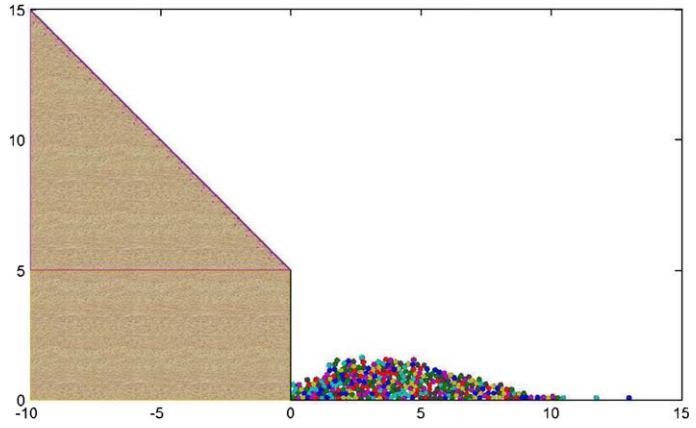

Fig. 8. Evolution of the system at $t = 12$ s.

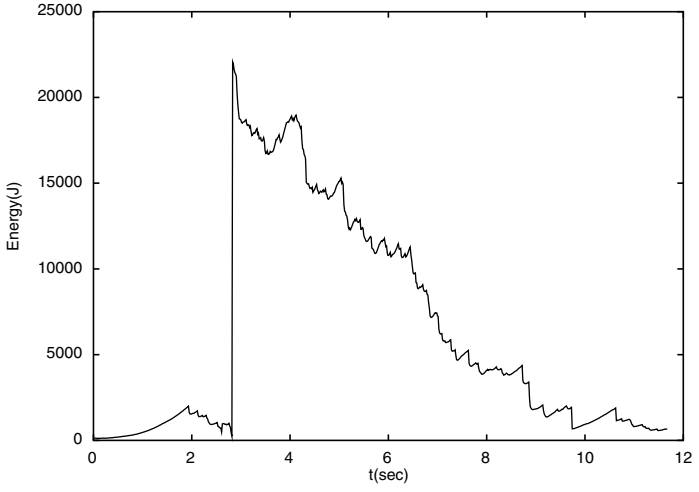

Fig. 9. Evolution of the kinetic energy of the system.

side–side contacts, by verifying if detected contact points belongs or not to the area delimited by the sides of the other polygon and vice-versa. In particular, the case of a vertex–vertex contact is dealt in (Dimnet and Frémond, 1999).

The behaviour of the landslide has been simulated by means of a packing of 300 rigid regular bodies (the radius r of the circumference inscribing the polygons is 0.25 m). The geometry of the slide is given in Fig. 2.

In this numerical example, the constitutive law describing the behaviour of the solids during the shock, is associated and quadratic, i.e. Coulomb's friction law is not taken into account. This leads to the following expression:

$$\Phi = \frac{1}{2}K_{\mathrm{T}}\left(\left(\vec{U}^{+}+\vec{U}^{-}\right)\cdot\vec{T}\right)^{2}+\frac{1}{2}K_{\mathrm{N}}\left(\left(\vec{U}^{+}+\vec{U}^{-}\right)\cdot\vec{N}\right)^{2}, \qquad K_{\mathrm{T}}\geqslant 0,\quad K_{\mathrm{N}}\geqslant 0 \tag{60}$$

where K_{T} and K_{N} represent the tangential and the normal elastic constants (see e.g. Frémond, 1995). This choice leads to the following form of the functional:

$$\text{Find } U^{+}\in C \text{ that minimises } J(V)=\frac{1}{2}a(V,V)-l(V) \tag{61}$$

The saddle point, solution of our problem, can be computed by means of the Uzawa method (Ciarlet, 1989).

The parameters describing the packing as well as the contact laws are summarised in Table 1.

In the first part of the analysis, particles are generated (as described above) as indicated in Fig. 2. The rigid bodies are then submitted to the action of gravity and the package arranges to a stable position. The final, stable configuration of the packing is computed observing the value of kinetic energy, i.e. if kinetic energy is below an established threshold, final stabilised position is considered to be achieved (see Figs. 3 and 4).

The barrier retaining the particles is then removed and the polygons can freely roll to the bottom of the slide (see Figs. 5–8). Fig. 9 gives the evolution of the kinetic energy during the whole evolution of the system.

In order to investigate the real behaviour of a landslide, it seems recommended to use the larger number of particles as possible. The same granular flow problem has been therefore solved using 1225 solids (see Fig. 10), showing (as expected) a more "fluid" behaviour, closer to the response of a real system.

The model can also be adapted to simulate problems characterised by different and more complex geometries. An example, taking into account the interactions of the particles with a wall placed at the bottom of the slide, is given in Fig. 11.

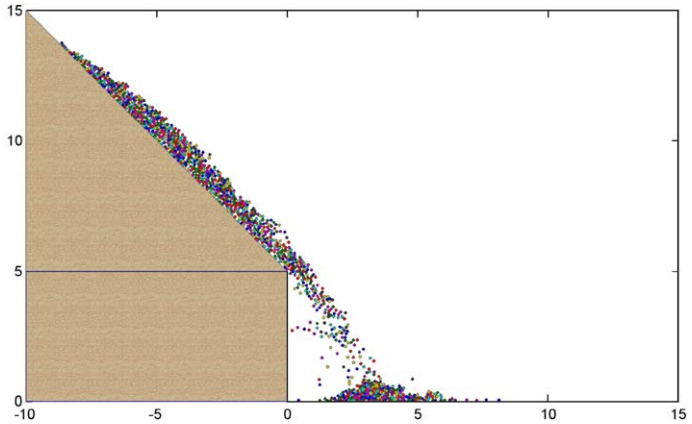

Fig. 10. System with 1225 solids ($r = 0.1$ m).

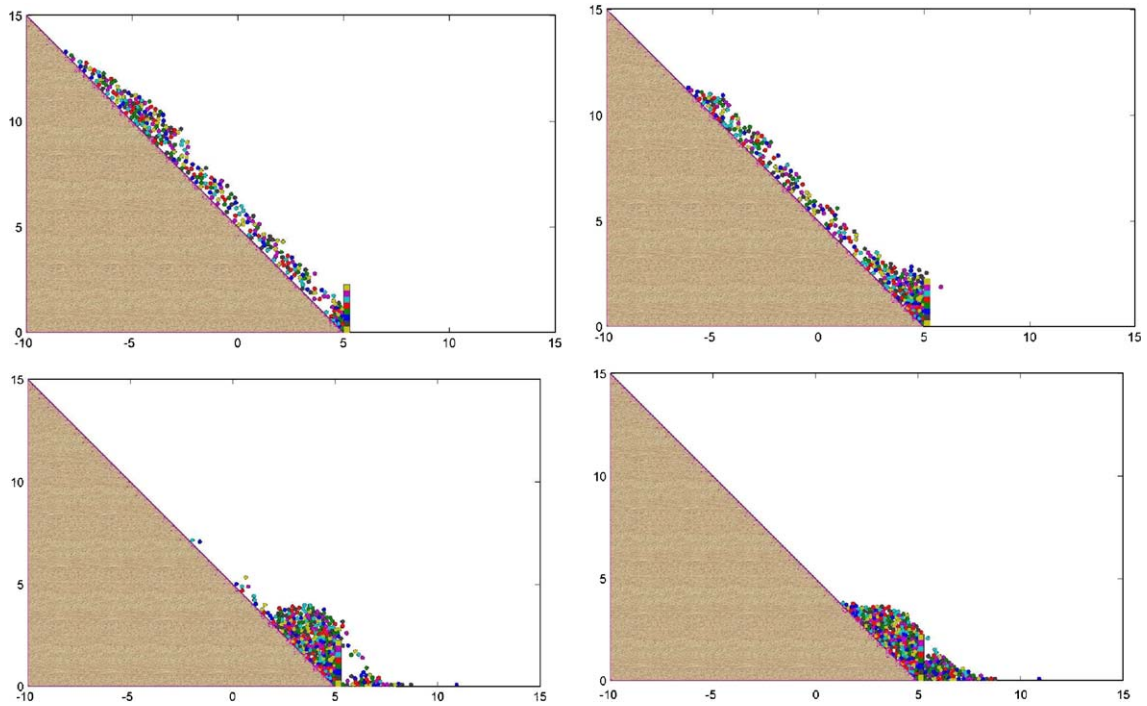

Fig. 11. Interaction of the solids with a wall.

5. Conclusion

This paper has presented a mathematical model based on the principle of virtual work for the description of the interactions among particles during an instantaneous contact. This approach shows interesting perspectives for technical and industrial applications. The presented model has been then applied in association with an associated quadratic constitutive law to describe the behaviour of an idealised landslide: these numerical results are to be compared with experimental applications. Many other applications are possible by introducing appropriate constitutive laws (e.g. by fitting experimental curves) which can also take into account non-associative models such as Coulomb's friction. The model can also be adapted to compute the evolution of an incompressible fluid or to describe the interactions between a rigid body and an incompressible fluid.

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