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New real-variable characterizations of Musielak–Orlicz Hardy spaces

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ABSTRACT

Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ be such that $\varphi(x, \cdot)$ is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight. The Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the grand maximal function f^* belongs to the Musielak–Orlicz space $L^\varphi(\mathbb{R}^n)$. Luong Dang Ky established its atomic characterization. In this paper, the authors establish some new real-variable characterizations of $H^\varphi(\mathbb{R}^n)$ in terms of the vertical or the non-tangential maximal functions, or the Littlewood–Paley g-function or g_λ^*-function, via first establishing a Musielak–Orlicz Fefferman–Stein vector-valued inequality. Moreover, the range of λ in the g_λ^*-function characterization of $H^\varphi(\mathbb{R}^n)$ coincides with the known best results, when $H^\varphi(\mathbb{R}^n)$ is the classical Hardy space $H^p(\mathbb{R}^n)$, with $p \in (0, 1]$, or its weighted variant.

1. Introduction

As the generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum–Orlicz in [1] and Orlicz in [2]. Since then, the theory of the Orlicz spaces themselves has been well developed and these spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [3–5]. Moreover, Orlicz–Hardy spaces are also suitable substitutes of Orlicz spaces in dealing with many problems of analysis; see, for example, [6–12]. Recall that Orlicz–Hardy spaces and their dual spaces were first studied by Strömberg [7] and Janson [6] on $\mathbb{R}^n$ and Viviani [8] on spaces of homogeneous type in the sense of Coifman and Weiss [13].

Recently, Ky [14] introduced a new *Musielak–Orlicz Hardy space*, $H^\varphi(\mathbb{R}^n)$, via the grand maximal function, which generalizes both the Orlicz–Hardy space of Strömberg [7] and Janson [6] and the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ with $\omega \in A_\infty(\mathbb{R}^n)$ studied by García-Cuerva [15] and Strömberg and Torchinsky [16], here and in what follows, $A_q(\mathbb{R}^n)$ with $q \in [1, \infty]$ denotes the *class of Muckenhoupt's weights* (see, for example, [17,15] for their definitions and properties) and we always assume that φ is a *growth function*, which means that $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is a Musielak–Orlicz function such that $\varphi(x, \cdot)$ is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight. Musielak–Orlicz functions are the natural generalization of Orlicz functions that may vary in the spatial variables; see, for example, [18,19,14,20]. Recall that the motivation to study function spaces of Musielak–Orlicz type comes from applications to elasticity, fluid dynamics, image processing, nonlinear partial differential equations and the calculus of variation; see, for example, [21–23,18,19,14].

In [14], Ky established the atomic characterization of $H^\varphi(\mathbb{R}^n)$ and, moreover, Ky [14] further introduced the BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$, which was proved to be the dual space of $H^\varphi(\mathbb{R}^n)$; as an interesting application, Ky proved that the class of pointwise multipliers for $\mathrm{BMO}(\mathbb{R}^n)$, characterized by Nakai and Yabuta [24], is the dual space of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$, where

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$H^{\log}(\mathbb{R}^n)$ is the Musielak–Orlicz Hardy space related to the growth function

$$\varphi(x,t) := \frac{t}{\log(e+|x|)+\log(e+t)}$$

for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. Furthermore, the Lusin area function and the molecular characterizations of $H^{\varphi}(\mathbb{R}^n)$ were obtained in [25]. As an application of the Lusin area function characterization of $H^{\varphi}(\mathbb{R}^n)$, the φ-Carleson measure characterization of $\mathrm{BMO}_{\varphi}(\mathbb{R}^n)$ was also given in [25]. It is worth noticing that some special Musielak–Orlicz Hardy spaces appear naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [22,23]), and the endpoint estimates for the div–curl lemma and the commutators of singular integral operators (see [26,22,27]). Moreover, the local Musielak–Orlicz Hardy space, $h^{\varphi}(\mathbb{R}^n)$, and its dual space, $\mathrm{bmo}_{\varphi}(\mathbb{R}^n)$, were studied in [28] and some applications of $h^{\varphi}(\mathbb{R}^n)$ and $\mathrm{bmo}_{\varphi}(\mathbb{R}^n)$, to pointwise multipliers of BMO-type spaces and to the boundedness of local Riesz transforms and pseudo-differential operators on $h^{\varphi}(\mathbb{R}^n)$, were also obtained in [28].

In this paper, we establish some new real-variable characterizations of $H^{\varphi}(\mathbb{R}^n)$ in terms of the vertical or the non-tangential maximal functions, and in terms of the Littlewood–Paley g-function or g_{λ}^*-function, via first establishing a Musielak–Orlicz Fefferman–Stein vector-valued inequality. Moreover, the range of λ in the g_{λ}^*-function characterization of $H^{\varphi}(\mathbb{R}^n)$ coincides with the known best results, when $H^{\varphi}(\mathbb{R}^n)$ is the classical Hardy space $H^p(\mathbb{R}^n)$, with $p \in (0,1]$, or its weighted variant.

To be precise, this paper is organized as follows.

In Section 2, we recall some notions concerning growth functions and some of their properties established in [14]. Then via some skillful applications of these properties on growth functions, such as their equivalent property that

$$\varphi(x,t) \sim \int_0^t \frac{\varphi(x,s)}{s}\,ds \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty)$$

(see Lemma 2.4(ii) below), we establish an interpolation theorem of Musielak–Orlicz type (see Theorem 2.7) and also a vector-valued version (see Theorem 2.9). As a corollary, we immediately obtain a Musielak–Orlicz Fefferman–Stein vector-valued inequality (see Theorem 2.10), which plays a key role in establishing the g-function characterization of $H^{\varphi}(\mathbb{R}^n)$ in Section 4 and might also be very useful in some other applications including the further study of function spaces of Musielak–Orlicz type, for example, Besov-type and Triebel–Lizorkin-type spaces.

Section 3 is devoted to establishing some maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$ in terms of the vertical and the non-tangential maximal functions (see Theorem 3.7), via first obtaining some key inequalities (see Theorem 3.6) involving the grand, the vertical and the tangential Peetre-type maximal functions.

In Section 4, by using the Lusin area function characterization of $H_{\varphi}(\mathbb{R}^n)$ established in [25], we obtain the g-function characterization of $H^{\varphi}(\mathbb{R}^n)$ (see Theorem 4.4). To do so, except using the Musielak–Orlicz Fefferman–Stein vector-valued inequality established in Theorem 2.10 of this paper, we also need to invoke the discrete Calderón reproducing formula obtained by Lu and Zhu [29, Theorem 2.1] and some key estimates from [29, Lemmas 2.1 and 2.2] (see also the estimates (4.2) and (4.3)). Moreover, by borrowing some ideas from Folland and Stein [30] and Aguilera and Segovia [31], we further obtain the Littlewood–Paley g_{λ}^*-function characterization of $H^{\varphi}(\mathbb{R}^n)$ for all $\lambda \in (2q/p,\infty)$ (see Theorem 4.8). We point out that even when $\varphi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, or $\varphi(x,t) := w(x)t^p$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, with $p \in (0,1]$, $q \in [1,\infty)$ and $w \in A_q(\mathbb{R}^n)$, the range of λ is the known best possible; see, respectively, [30, p. 221, Corollary 7.4] and [31, Theorem 2]. In this sense, the range of λ in Theorem 4.8 might also be the best possible.

We remark that the Littlewood–Paley function characterizations of $H^{\varphi}(\mathbb{R}^n)$ have local variants, which will be studied in a forthcoming paper; see [28] for the definition of local Musielak–Orlicz Hardy spaces $h^{\varphi}(\mathbb{R}^n)$.

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a *positive constant* depending on the indicated parameters $\gamma, \beta, \ldots$. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The *symbol* $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$ and by χ_E its *characteristic function*. For any cube $Q \subset \mathbb{R}^n$, we use $\ell(Q)$ to denote its side length. We also set $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any given function g on $\mathbb{R}^n$, if $\int_{\mathbb{R}^n} g(x)\,dx \neq 0$, we let $L_g := -1$; otherwise, we let $L_g \in \mathbb{Z}_+$ be the *maximal integer* such that g has vanishing moments up to order L_g, namely, $\int_{\mathbb{R}^n} g(x)x^{\alpha}\,dx = 0$ for all multi-indices α with $|\alpha| \le L_g$.

2. Preliminaries

In Section 2.1, we first recall some notions concerning growth functions and some of their properties established in [14]. Then in Section 2.2 we establish an interpolation theorem of Musielak–Orlicz type and also a vector-valued version. As a corollary, we obtain a Musielak–Orlicz type Fefferman–Stein vector-valued inequality. In Section 2.3, we recall the notion of Musielak–Orlicz Hardy spaces and some of their known properties.

2.1. Growth functions

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for $t \in (0,\infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$ (see, for example, [20,32,33]). The function Φ is said to be of *upper type p* (resp. *lower type p*) for some $p \in [0,\infty)$, if there exists a positive constant C such that for all $t \in [1,\infty)$ (resp. $t \in [0, 1]$) and $s \in [0,\infty)$, $\Phi(st) \le Ct^p\Phi(s)$.

For a given function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of *uniformly upper type p* (resp. *uniformly lower type p*) for some $p \in [0,\infty)$ if there exists a positive constant C such that for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in [1,\infty)$ (resp. $s \in [0, 1]$), $\varphi(x, st) \le Cs^p\varphi(x, t)$. Let

$$i(\varphi) := \sup\{p \in (0,\infty) : \varphi \text{ is of uniformly lower type } p\}. \tag{2.1}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$; see below for some examples.

Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfy that $x \mapsto \varphi(x, t)$ is measurable for all $t \in [0,\infty)$. Following [14], $\varphi(\cdot, t)$ is said to be *uniformly locally integrable* if, for all compact sets K in $\mathbb{R}^n$,

$$\int_K \sup_{t\in(0,\infty)} \left\{ |\varphi(x,t)| \left[\int_K |\varphi(y,t)|\,dy \right]^{-1} \right\} dx < \infty.$$

Definition 2.1. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be uniformly locally integrable. The function $\varphi(\cdot, t)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{A}_q(\varphi) := \sup_{t\in[0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,dx \left\{ \int_B [\varphi(y,t)]^{-q'/q}\,dy \right\}^{q/q'} < \infty, \tag{2.2}$$

where $1/q + 1/q' = 1$, or

$$\mathbb{A}_1(\varphi) := \sup_{t\in[0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,dx \left(\operatorname{esssup}_{y\in B}[\varphi(y,t)]^{-1}\right) < \infty.$$

Here the first supremums are taken over all $t \in [0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

Recall that $\mathbb{A}_q(\mathbb{R}^n)$ with $q \in [1,\infty)$ in Definition 2.1 was introduced by Ky [14]. We have the following properties for $\mathbb{A}_q(\mathbb{R}^n)$ with $q \in [1,\infty)$, whose proofs are similar to those in [15,34].

Lemma 2.2. (i) *$\mathbb{A}_1(\mathbb{R}^n) \subset \mathbb{A}_p(\mathbb{R}^n) \subset \mathbb{A}_q(\mathbb{R}^n)$ for $1 \le p \le q < \infty$.*
(ii) *If $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in (1,\infty)$, then there exists $q \in (1, p)$ such that $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$.*

Let $\mathbb{A}_\infty(\mathbb{R}^n) := \cup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n)$ and define the *critical index*, $q(\varphi)$, of $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ by

$$q(\varphi) := \inf\left\{q \in [1,\infty) : \varphi \in \mathbb{A}_q(\mathbb{R}^n)\right\}. \tag{2.3}$$

By Lemma 2.2(ii), we see that if $q(\varphi) \in (1,\infty)$, then $\varphi \notin \mathbb{A}_{q(\varphi)}(\mathbb{R}^n)$. Moreover, there exists $\varphi \notin \mathbb{A}_1(\mathbb{R}^n)$ such that $q(\varphi) = 1$ (see, for example, [35]).

Now we introduce the notion of growth functions.

Definition 2.3. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a *Musielak–Orlicz function*, namely,
  (i)$_1$ the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
  (i)$_2$ the function $\varphi(\cdot, t)$ is a measurable function for all $t \in [0,\infty)$.
(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.
(iii) φ is of positive uniformly lower type p for some $p \in (0, 1]$ and of uniformly upper type 1.

Clearly, $\varphi(x, t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty(\mathbb{R}^n)$ and Φ is an Orlicz function of lower type p for some $p \in (0, 1]$ and of upper type 1. It is known that, for $p \in (0, 1]$, if $\Phi(t) := t^p$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type p and of upper type p; for $p \in [\frac{1}{2}, 1]$, if $\Phi(t) := t^p/\ln(e + t)$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type q for $q \in (0, p)$ and of upper type p; for $p \in (0, \frac{1}{2}]$, if $\Phi(t) := t^p \ln(e + t)$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type p and of upper type q for $q \in (p, 1]$. Recall that if an Orlicz function is of upper type $p \in (0, 1)$, then it is also of upper type 1. Another typical and useful growth function is

$$\varphi(x,t) := \frac{t^\alpha}{[\ln(e+|x|)]^\beta + [\ln(e+t)]^\gamma}$$

for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, with any $\alpha \in (0, 1]$, $\beta \in [0,\infty)$ and $\gamma \in [0, 2\alpha(1 + \ln 2)]$; more precisely, $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$, φ is of uniformly upper type α and $i(\varphi) = \alpha$ which is not attainable (see [14]).

Throughout the paper, we *always assume that* φ *is a growth function* as in Definition 2.3. Let us now introduce the Musielak–Orlicz space.

The *Musielak–Orlicz space* $L^{\varphi}(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx < \infty$ with *Luxembourg norm*

$$\|f\|_{L^{\varphi}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

In what follows, for any measurable subset E of $\mathbb{R}^n$, we denote $\int_E \varphi(x, t)\, dx$ by the *symbol* $\varphi(E, t)$ for any $t \in [0, \infty)$.

The following Lemmas 2.4–2.6 on the properties of growth functions are, respectively, [14, Lemmas 4.1, 4.2 and 4.3].

Lemma 2.4. (i) *Let* φ *be a growth function. Then* φ *is uniformly* σ*-quasi-subadditive on* $\mathbb{R}^n \times [0, \infty)$, *namely, there exists a positive constant* C *such that for all* $(x, t_j) \in \mathbb{R}^n \times [0, \infty)$ *with* $j \in \mathbb{N}$,

$$\varphi\left(x, \sum_{j=1}^{\infty} t_j\right) \le C \sum_{j=1}^{\infty} \varphi(x, t_j).$$

(ii) *Let* φ *be a growth function and*

$$\widetilde{\varphi}(x, t) := \int_0^t \frac{\varphi(x, s)}{s}\, ds \quad \textit{for all } (x, t) \in \mathbb{R}^n \times [0, \infty).$$

Then $\widetilde{\varphi}$ *is a growth function, which is equivalent to* φ*; moreover,* $\widetilde{\varphi}(x, \cdot)$ *is continuous and strictly increasing.*

Lemma 2.5. *Let* φ *be a growth function. Then*

(i) *For all* $f \in L^{\varphi}(\mathbb{R}^n) \setminus \{0\}$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\|f\|_{L^{\varphi}(\mathbb{R}^n)}}\right) = 1.$$

(ii) $\lim_{k\to\infty} \|f_k\|_{L^{\varphi}(\mathbb{R}^n)} = 0$ *if and only if*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi(x, |f_k(x)|) dx = 0.$$

Lemma 2.6. *For a given positive constant* $\widetilde{C}$, *there exists a positive constant* C *such that the following hold:*

(i) *The inequality*

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le \widetilde{C} \quad \textit{for } \lambda \in (0, \infty)$$

implies that $\|f\|_{L^{\varphi}(\mathbb{R}^n)} \le C\lambda$.

(ii) *The inequality*

$$\sum_j \varphi\left(Q_j, \frac{t_j}{\lambda}\right) \le \widetilde{C} \quad \textit{for } \lambda \in (0, \infty)$$

implies that

$$\inf\left\{\alpha > 0 : \sum_j \varphi\left(Q_j, \frac{t_j}{\alpha}\right) \le 1\right\} \le C\lambda,$$

where $\{t_j\}_j$ *is a sequence of positive constants and* $\{Q_j\}_j$ *a sequence of cubes.*

2.2. The Musielak–Orlicz Fefferman–Stein vector-valued inequality

In this subsection, we establish an interpolation theorem of operators, in the spirit of the Marcinkiewicz interpolation theorem, associated with a growth function, which may have independent interest. In what follows, for any nonnegative locally integrable function w on $\mathbb{R}^n$ and $p \in (0, \infty)$, the *space* $L^p_w(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right\}^{1/p} < \infty.$$

Theorem 2.7. *Let $p_1, p_2 \in (0, \infty)$, $p_1 < p_2$ and φ be a Musielak–Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+. If $0 < p_1 < p_\varphi^- \le p_\varphi^+ < p_2 < \infty$ and T is a sublinear operator defined on $L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ satisfying that for $i \in \{1, 2\}$, all $\alpha \in (0, \infty)$ and $t \in (0, \infty)$,*

$$\varphi(\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}, t) \le C_i \alpha^{-p_i} \int_{\mathbb{R}^n} |f(x)|^{p_i} \varphi(x, t)\, dx, \tag{2.4}$$

where C_i is a positive constant independent of f, t and α. Then T is bounded on $L^\varphi(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that for all $f \in L^\varphi(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi(x, |Tf(x)|)\, dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx.$$

Proof. First observe that for all $t \in (0, \infty)$,

$$\int_{\mathbb{R}^n} |f(x)|^p \varphi(x, t)\, dx < \infty \quad \text{if and only if} \quad \int_{\mathbb{R}^n} |f(x)|^p \varphi(x, 1)\, dx < \infty.$$

Thus, the spaces $L^p_{\varphi(\cdot,t)}(\mathbb{R}^n)$ and $L^p_{\varphi(\cdot,1)}(\mathbb{R}^n)$ coincide as sets. Now we show that $L^\varphi(\mathbb{R}^n) \subset L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$.

For any given $t \in (0, \infty)$, we decompose $f \in L^\varphi(\mathbb{R}^n)$ as

$$f = f\chi_{\{x\in\mathbb{R}^n: |f(x)|>t\}} + f\chi_{\{x\in\mathbb{R}^n: |f(x)|\le t\}} =: f^t + f_t.$$

Then by the fact that φ is of uniformly lower type p_φ^- and $p_1 < p_\varphi^-$, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} |f^t(x)|^{p_1} \varphi(x, 1)\, dx &\lesssim \int_{\{x\in\mathbb{R}^n: |f(x)|>t\}} |f(x)|^{p_1} \left[\frac{t}{|f(x)|}\right]^{p_\varphi^-} \varphi\left(x, \frac{|f(x)|}{t}\right) dx \\ &\lesssim t^{p_1} \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{t}\right) dx < \infty,\end{aligned}$$

namely, $f^t \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n)$. Similarly we have $f_t \in L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ and hence Tf is well defined.

By the fact that T is sublinear and Lemma 2.4(ii), we further see that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x, |Tf(x)|)\, dx &\sim \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n: |Tf(x)|>t\}} \varphi(x, t)\, dx\, dt \\ &\lesssim \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n: |Tf^t(x)|>t/2\}} \varphi(x, t)\, dx\, dt + \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n: |Tf_t(x)|>t/2\}} \cdots =: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

On I_1, since T is of weak type (p_1, p_1) (namely, (2.4) with $i = 1$), φ is of uniformly lower type p_φ^- and $p_1 < p_\varphi^-$, we conclude that

$$\begin{aligned}\mathrm{I}_1 &\lesssim \int_0^\infty \frac{1}{t} \left(\frac{t}{2}\right)^{-p_1} \int_{\mathbb{R}^n} |f^t(x)|^{p_1} \varphi(x, t)\, dx\, dt \\ &\sim \int_0^\infty \frac{1}{t^{1+p_1}} \int_{\{x\in\mathbb{R}^n: |f(x)|>t\}} |f(x)|^{p_1} \varphi(x, t)\, dx\, dt \\ &\sim \int_0^\infty \frac{1}{t^{1+p_1}} \int_{\{x\in\mathbb{R}^n: |f(x)|>t\}} \varphi(x, t) \left[\int_t^{|f(x)|} p_1 s^{p_1-1}\, ds + t^{p_1}\right] dx\, dt \\ &\sim \int_0^\infty s^{p_1-1} \int_{\{x\in\mathbb{R}^n: |f(x)|>s\}} \int_0^s \frac{\varphi(x, t)}{t^{1+p_1}}\, dt\, dx\, ds + \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n: |f(x)|>t\}} \varphi(x, t)\, dx\, dt \\ &\lesssim \int_0^\infty s^{p_1-1} \int_{\{x\in\mathbb{R}^n: |f(x)|>s\}} \varphi(x, s) s^{-p_\varphi^-} \int_0^s \frac{1}{t^{1+p_1-p_\varphi^-}}\, dt\, dx\, ds + \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx \\ &\sim \int_0^\infty \frac{1}{s} \int_{\{x\in\mathbb{R}^n: |f(x)|>s\}} \varphi(x, s) dx\, ds + \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx \sim \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx.\end{aligned}$$

Also, from the weak type (p_2, p_2) of T (namely, (2.4) with $i = 2$), the uniformly upper type p_φ^+ property of φ and $p_\varphi^+ < p_2$, we deduce that

$$\mathrm{I}_2 \lesssim \int_0^\infty \frac{1}{t} \left(\frac{t}{2}\right)^{-p_2} \int_{\mathbb{R}^n} |f_t(x)|^{p_2} \varphi(x, t)\, dx\, dt$$

$$
\begin{aligned}
&\sim \int_0^\infty \frac{1}{t^{1+p_2}} \int_{\{x\in\mathbb{R}^n:|f(x)|\le t\}} |f(x)|^{p_2}\varphi(x,t)\,dx\,dt \\
&\sim \int_0^\infty \frac{1}{t^{1+p_2}} \int_{\{x\in\mathbb{R}^n:|f(x)|\le t\}} \varphi(x,t)\int_0^{|f(x)|} p_2 s^{p_2-1}\,ds\,dx\,dt \\
&\sim \int_0^\infty s^{p_2-1} \int_{\{x\in\mathbb{R}^n:|f(x)|> s\}} \int_s^\infty \frac{\varphi(x,t)}{t^{1+p_2}}\,dt\,dx\,ds \\
&\lesssim \int_0^\infty s^{p_2-1} \int_{\{x\in\mathbb{R}^n:|f(x)|> s\}} \varphi(x,s)s^{-p_\varphi^+}\int_s^\infty \frac{1}{t^{1+p_2-p_\varphi^+}}\,dt\,dx\,ds \\
&\sim \int_0^\infty \frac{1}{s} \int_{\{x\in\mathbb{R}^n:|f(x)|> s\}} \varphi(x,s)dx\,ds \sim \int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx.
\end{aligned}
$$

Thus, T is bounded on $L^\varphi(\mathbb{R}^n)$, which completes the proof of Theorem 2.7. □

Recall that for any locally integrable function f and $x\in\mathbb{R}^n$, the *Hardy–Littlewood maximal function* $Mf(x)$ is defined by

$$
Mf(x) := \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|\,dy,
$$

where the supremum is taken over all balls B containing x. Let $q(\varphi)$ be as in (2.3). As a simple corollary of Theorem 2.7, together with the fact that for any $p\in(q(\varphi),\infty)$ if $q(\varphi)\in(1,\infty)$ or if $q(\varphi)=1$ and $\varphi\notin\mathbb{A}_1(\mathbb{R}^n)$, or for any $p\in[1,\infty)$ if $q(\varphi)=1$ and $\varphi\in\mathbb{A}_1(\mathbb{R}^n)$, there exists a positive constant $C_{(p,\varphi)}$ such that for all $f\in L^p_{\varphi(\cdot,t)}(\mathbb{R}^n)$ and $t\in(0,\infty)$,

$$
\varphi(\{x\in\mathbb{R}^n : |Mf(x)|>\alpha\},t) \le C_{(p,\varphi)}\alpha^{-p}\int_{\mathbb{R}^n}|f(x)|^p\varphi(x,t)\,dx,
$$

we immediately obtain the following boundedness of M on $L^\varphi(\mathbb{R}^n)$. We omit the details.

Corollary 2.8. *Let φ be a Musielak–Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+ satisfying $q(\varphi)<p_\varphi^-\le p_\varphi^+<\infty$, where $q(\varphi)$ is as in (2.3). Then the Hardy–Littlewood Maximal function M is bounded on $L^\varphi(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that for all $f\in L^\varphi(\mathbb{R}^n)$,*

$$
\int_{\mathbb{R}^n}\varphi(x,Mf(x))\,dx \le C\int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx.
$$

The *space* $L^\varphi(\ell^r,\mathbb{R}^n)$ is defined to be the set of all $\{f_j\}_{j\in\mathbb{Z}}$ satisfying $[\sum_j |f_j|^r]^{1/r}\in L^\varphi(\mathbb{R}^n)$ and let

$$
\|\{f_j\}_j\|_{L^\varphi(\ell^r,\mathbb{R}^n)} := \left\|\left[\sum_j |f_j|^r\right]^{1/r}\right\|_{L^\varphi(\mathbb{R}^n)}.
$$

We have the following vector-valued interpolation theorem of Musielak–Orlicz type.

Theorem 2.9. *Let p_1, p_2 and φ be as in Theorem 2.7 and $r\in[1,\infty]$. Assume that T is a sublinear operator defined on $L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n)+L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ satisfying that for $i\in\{1,2\}$ and all $\{f_j\}_j\in L^{p_i}_{\varphi(\cdot,1)}(\ell^r,\mathbb{R}^n)$, $\alpha\in(0,\infty)$ and $t\in(0,\infty)$,*

$$
\varphi\left(\left\{x\in\mathbb{R}^n : \left[\sum_j |Tf_j(x)|^r\right]^{\frac1r}>\alpha\right\},t\right) \le C_i\alpha^{-p_i}\int_{\mathbb{R}^n}\left[\sum_j |f_j(x)|^r\right]^{\frac{p_i}{r}}\varphi(x,t)\,dx, \tag{2.5}
$$

where C_i is a positive constant independent of $\{f_j\}_j$, t and α. Then there exists a positive constant C such that for all $\{f_j\}_j\in L^\varphi(\ell^r,\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n}\varphi\left(x,\left[\sum_j |Tf_j(x)|^r\right]^{1/r}\right)dx \le C\int_{\mathbb{R}^n}\varphi\left(x,\left[\sum_j |f_j(x)|^r\right]^{1/r}\right)dx.
$$

Proof. For all $\{f_j\}_j\in L^\varphi(\ell^r,\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, let

$$
n_j(x) := \frac{f_j(x)}{[\sum_j |f_j(x)|^r]^{1/r}} \quad \text{if } \left[\sum_j |f_j(x)|^r\right]^{1/r}\ne 0,
$$

and $n_j(x) = 0$ otherwise. Then $[\sum_j |n_j(x)|^r]^{1/r} = 1$ for all $x \in \mathbb{R}^n$. Consider the operator

$$A(g) := \left[\sum_j |T(gn_j)|^r\right]^{1/r},$$

where $g \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$. Then, for all $g_1,\ g_2 \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, by the sublinear property of T and Minkowski's inequality, we see that

$$\begin{aligned} A(g_1+g_2)(x) &= \left[\sum_j |T((g_1+g_2)n_j)(x)|^r\right]^{1/r} \\ &\le \left\{\sum_j \left[|T(g_1n_j)(x)| + |T(g_2n_j)(x)|\right]^r\right\}^{1/r} \\ &\le \left[\sum_j |T(g_1n_j)(x)|^r\right]^{1/r} + \left[\sum_j |T(g_2n_j)(x)|^r\right]^{1/r} \\ &= A(g_1)(x) + A(g_2)(x). \end{aligned}$$

Thus, A is sublinear. Moreover, by (2.5), we further conclude that for all $i \in \{1, 2\}, \alpha \in (0,\infty),\ t \in (0,\infty)$ and $g \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$,

$$\begin{aligned} \varphi(\{x \in \mathbb{R}^n : |A(g)(x)| > \alpha\}, t) &= \varphi\left(\left\{x \in \mathbb{R}^n : \left[\sum_j |T(gn_j)(x)|^r\right]^{1/r} > \alpha\right\}, t\right) \\ &\lesssim \alpha^{-p_i} \int_{\mathbb{R}^n} \left[\sum_j |gn_j(x)|^r\right]^{p_i/r} \varphi(x,t)\,dx \\ &\lesssim \alpha^{-p_i} \int_{\mathbb{R}^n} |g(x)|^{p_i} \varphi(x,t)\,dx, \end{aligned}$$

which implies that A satisfies (2.4). Thus, if setting $g = [\sum_j |f_j|^r]^{1/r}$, from Theorem 2.7, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_j |Tf_j(x)|^r\right]^{1/r}\right) dx &= \int_{\mathbb{R}^n} \varphi(x, |Ag(x)|)\,dx \lesssim \int_{\mathbb{R}^n} \varphi(x, |g(x)|)\,dx \\ &\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_j |f_j(x)|^r\right]^{1/r}\right) dx, \end{aligned}$$

which completes the proof of Theorem 2.9. □

By using Theorem 2.9 and [36, Theorem 3.1(a)], we immediately obtain the following Musielak–Orlicz Fefferman–Stein vector-valued inequality, which, when $\varphi(x,t) := t^p$ for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$ with $p \in (1,\infty)$, was obtained by Fefferman and Stein in [37, Theorem 1] and, when $\varphi(x,t) := w(x)t^p$ for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$ with $p \in (1,\infty)$, $q \in (1,\infty)$ and $w \in A_q(\mathbb{R}^n)$, by Andersen and John in [36, Theorem 3.1]. We point out that to apply Theorem 2.9, we need $r \in (1,\infty]$.

Theorem 2.10. *Let $r \in (1,\infty]$, φ be a Musielak–Orlicz function with uniformly lower type p_φ^- and upper type p_φ^+, $q \in (1,\infty)$ and $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$. If $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, then there exists a positive constant C such that, for all $\{f_j\}_{j\in\mathbb{Z}} \in L^\varphi(\ell^r, \mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{j\in\mathbb{Z}} \left[M(f_j)(x)\right]^r\right\}^{1/r}\right) dx \le C \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_{j\in\mathbb{Z}} |f_j(x)|^r\right]^{1/r}\right) dx.$$

2.3. Musielak–Orlicz Hardy spaces

In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). For $m \in \mathbb{N}$, define

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \sup_{\beta\in\mathbb{Z}^n_+, |\beta|\le m+1} (1+|x|)^{(m+2)(n+1)} |\partial_x^\beta \psi(x)| \le 1 \right\}.$$

Then for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *nontangential grand maximal function*, f_m^*, of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi\in\mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t,\ t\in(0,\infty)} |f * \psi_t(y)|, \tag{2.6}$$

where for all $t \in (0, \infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor, \tag{2.7}$$

where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.3) and (2.1), we *denote* $f^*_{m(\varphi)}$ *simply by* f^*.

Now we recall the definition of the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ introduced by Ky [14] as follows.

Definition 2.11. Let φ be a growth function. The *Musielak–Orlicz Hardy space* $H^\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$ with the *quasi-norm*

$$\|f\|_{H^\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}.$$

When $\varphi(x, t) = w(x)\Phi(t)$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, with w being a Muckenhoupt weight and Φ an Orlicz function, $H^\varphi(\mathbb{R}^n)$ is just the weighted Hardy–Orlicz space which includes the classical Hardy–Orlicz spaces of Janson [6] ($w = 1$ in this context) and the classical weighted Hardy spaces of García-Cuerva [15] and Strömberg and Torchinsky [16] ($\Phi(t) := t^p$ for all $t \in (0, \infty)$ in this context).

In order to introduce the atomic Musielak–Orlicz Hardy space, Ky [14] introduced the following local Musielak–Orlicz space.

Definition 2.12. For any cube Q in $\mathbb{R}^n$, the *space* $L^q_\varphi(Q)$ for $q \in [1, \infty]$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ supported in Q such that

$$\|f\|_{L^q_\varphi(Q)} := \begin{cases} \sup_{t\in(0,\infty)} \left[\frac{1}{\varphi(Q,t)} \int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)\,dx\right]^{1/q} < \infty, & q \in [1,\infty); \\ \|f\|_{L^\infty(\mathbb{R}^n)} < \infty, & q = \infty. \end{cases}$$

Now, we recall the atomic Musielak–Orlicz Hardy spaces introduced by Ky [14] as follows.

Definition 2.13. A triplet (φ, q, s) is said to be *admissible*, if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{N}$ satisfies $s \ge m(\varphi)$. A measurable function a is called a (φ, q, s)*-atom* if it satisfies the following three conditions:

(i) $a \in L^q_\varphi(Q)$ for some cube Q;
(ii) $\|a\|_{L^q_\varphi(Q)} \le \|\chi_Q\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for any $|\alpha| \le s$.

The *atomic Musielak–Orlicz Hardy space* $H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be represented as a sum of multiples of (φ, q, s)-atoms, that is, $f = \sum_j b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where, for each j, b_j is a multiple of some (φ, q, s)-atom supported in some cube Q_j, with the property $\sum_j \varphi(Q_j, \|b_j\|_{L^q_\varphi(Q_j)}) < \infty$. For any given sequence of multiples of (φ, q, s)-atoms, $\{b_j\}_j$, let

$$\Lambda_q(\{b_j\}_j) := \inf\left\{\lambda > 0 : \sum_j \varphi\left(Q_j, \frac{\|b_j\|_{L^q_\varphi(Q_j)}}{\lambda}\right) \le 1\right\}$$

and then define

$$\|f\|_{H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)} := \inf\left\{\Lambda_q(\{b_j\}_j) : f = \sum_j b_j \text{ in } \mathcal{S}'(\mathbb{R}^n)\right\},$$

where the infimum is taken over all decompositions of f as above.

The following Proposition 2.14 is just [14, Theorem 3.1].

Proposition 2.14. *Let (φ, q, s) be admissible. Then $H^\varphi(\mathbb{R}^n) = H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ with equivalent norms.*

3. Maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$

In this section, we establish some maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$. First, we recall the notions of the vertical, the tangential and the nontangential maximal functions. In what follows, let the *space* $\mathcal{D}(\mathbb{R}^n)$ be the space of all $C^{\infty}(\mathbb{R}^n)$ functions with compact support, endowed with the inductive limit topology, and $\mathcal{D}'(\mathbb{R}^n)$ its *topological dual space*, endowed with the weak-$*$ topology.

Definition 3.1. Let

$$\psi_0 \in \mathcal{D}(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} \psi_0(x)\,dx \neq 0. \tag{3.1}$$

For $j \in \mathbb{Z}$, $A,\ B \in [0, \infty)$ and $y \in \mathbb{R}^n$, let $m_{j,\,A,\,B}(y) := (1 + 2^j|y|)^A 2^{B|y|}$. The *vertical maximal function* $\psi_0^+(f)$ of f associated to ψ_0 is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_0^+(f)(x) := \sup_{j\in\mathbb{Z}} |(\psi_0)_j * f(x)|, \tag{3.2}$$

the *tangential Peetre-type maximal function* $\psi_{0,\,A,\,B}^{**}(f)$ of f associated to ψ_0 is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_{0,\,A,\,B}^{**}(f)(x) := \sup_{j\in\mathbb{Z},\ y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,\,A,\,B}(y)} \tag{3.3}$$

and the *nontangential maximal function* $(\psi_0)_{\triangledown}^*(f)$ of f associated to ψ_0 is defined by setting, for all $x \in \mathbb{R}^n$,

$$(\psi_0)_{\triangledown}^*(f)(x) := \sup_{|x-y|<t} |(\psi_0)_t * f(y)|,$$

here and in what follows, for all $x \in \mathbb{R}^n$, $(\psi_0)_j(x) := 2^{jn}\psi_0(2^j x)$ for all $j \in \mathbb{Z}$ and $(\psi_0)_t(x) := \frac{1}{t^n}\psi_0(\frac{x}{t})$ for all $t \in (0, \infty)$.

Obviously, for all $x \in \mathbb{R}^n$, we have $\psi_0^+(f)(x) \le (\psi_0)_{\triangledown}^*(f)(x) \lesssim \psi_{0,\,A,\,B}^{**}(f)(x)$.

In order to establish the vertical or the nontangential maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$, we first establish some inequalities in the norm of $L^{\varphi}(\mathbb{R}^n)$ involving the maximal functions $\psi_{0,\,A,\,B}^{**}(f)$, $\psi_0^+(f)$ and f^*. We begin with some technical lemmas and the following Lemma 3.2 is just [38, Theorem 1.6].

Lemma 3.2. *Let ψ_0 be as in (3.1) and $\psi(x) := \psi_0(x) - \frac{1}{2^n}\psi_0(\frac{x}{2})$ for all $x \in \mathbb{R}^n$. Then for any given integer $L \in \mathbb{N}$, there exist $\eta_0,\ \eta \in \mathcal{D}(\mathbb{R}^n)$ such that $L_\eta \ge L$ and, for all $f \in \mathcal{D}'(\mathbb{R}^n)$,*

$$f = \eta_0 * \psi_0 * f + \sum_{j\in\mathbb{N}} \eta_j * \psi_j * f \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

For $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $B \in [0, \infty)$ and $x \in \mathbb{R}^n$, let

$$K_B f(x) := \int_{\mathbb{R}^n} |f(y)| 2^{-B|x-y|}\,dy,$$

here and in what follows, $L^1_{\rm loc}(\mathbb{R}^n)$ denotes the *space of all locally integrable functions on* $\mathbb{R}^n$.

Lemma 3.3. *Let $p \in (1, \infty)$, $q \in (1, \infty]$ and $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$. Then there exist positive constants C and $B_0 := B_0(\varphi, n)$ such that for all $t \in (0, \infty)$, $B \ge B_0/p$ and $\{f^j\}_j \in L^p_{\varphi(\cdot,t)}(\ell^q, \mathbb{R}^n)$,*

$$\left\|\{K_B(f^j)\}_j\right\|_{L^p_{\varphi(\cdot,t)}(\ell^q,\mathbb{R}^n)} \le C \left\|\{f^j\}_j\right\|_{L^p_{\varphi(\cdot,t)}(\ell^q,\mathbb{R}^n)}.$$

Lemma 3.3 is just [38, Lemma 2.11].

Lemma 3.4. *Let ψ_0 be as in (3.1) and $r \in (0, \infty)$. Then for any $A, B \in [0, \infty)$, there exists a positive constant C, depending only on n, r, ψ_0, A and B, such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,*

$$|(\psi_0)_j * f(x)|^r \le C \sum_{k=j}^{\infty} 2^{(j-k)Ar} 2^{kn} \int_{\mathbb{R}^n} \frac{|(\psi_0)_k * f(x-y)|^r}{m_{j,\,Ar,\,Br}(y)}\,dy.$$

Proof. When $j \in \mathbb{N}$, Lemma 3.4 is just [38, Lemma 2.9]. We now show the conclusion of Lemma 3.4 for all $j \in \mathbb{Z}$.

By Lemma 3.2, there exist $\eta_0, \eta \in \mathcal{D}(\mathbb{R}^n)$ such that $L_\eta \geq A$ and, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f = \eta_0 * \psi_0 * f + \sum_{k\in\mathbb{N}} \eta_k * \psi_k * f.$$

We dilate this identity with $2^j, j \in \mathbb{Z}$, namely, for $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\langle f_j, \phi\rangle = \langle f, 2^{-jn}\phi(2^{-j}\cdot)\rangle$. By an elementary calculation, we see that, for all $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f_j = (\eta_0)_j * (\psi_0)_j * f_j + \sum_{k\in\mathbb{N}} \eta_{k+j} * \psi_{k+j} * f_j.$$

We rewrite the above equality to conclude that, for all $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f = (\eta_0)_j * (\psi_0)_j * f + \sum_{k\in\mathbb{N}} \eta_{k+j} * \psi_{k+j} * f. \tag{3.4}$$

Then, replacing [38, (2.12)] by (3.4), similar to the proof of [38, Lemma 2.9], we obtain the conclusion of Lemma 3.4, which completes the proof of Lemma 3.4. □

The proof of the following lemma is quite similar to that of [38, Lemma 2.10], and we omit the details.

Lemma 3.5. *Let ψ_0 be as in (3.1) and $r \in (0,\infty)$. Then there exists a positive constant A_0, depending only on the support of ψ_0, such that for any $A \in (\max\{A_0, \frac{n}{r}\}, \infty)$ and $B \in [0,\infty)$, there exists a positive constant C, depending only on n, r, ψ_0, A and B, such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,*

$$\left[(\psi_0)^*_{j,\,A,\,B}(f)(x)\right]^r \leq C\sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)}\left\{M(|(\psi_0)_k * f|^r)(x) + K_{Br}(|(\psi_0)_k * f|^r)(x)\right\},$$

where

$$(\psi_0)^*_{j,\,A,\,B}(f)(x) := \sup_{y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,\,A,\,B}(y)} \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 3.6. *Let φ be a growth function as in Definition 2.3, $R \in (0,\infty)$, ψ_0 as in (3.1), $\psi_0^+(f)$, $\psi^{**}_{0,\,A,\,B}(f)$, and f^* be respectively as in (3.2), (3.3) and (2.6) with $m = m(\varphi)$. Let $A_1 := \max\{A_0,\ nq(\varphi)/i(\varphi)\}$, $B_1 := B_0/i(\varphi)$ and integer $N_0 := \lfloor 2A_1\rfloor + 1$, where A_0 and B_0 are respectively as in Lemmas 3.5 and 3.3. Then for any $A \in (A_1,\infty)$, $B \in (B_1,\infty)$ and integer $N \geq N_0$, there exists a positive constant C, depending only on A, B, N, R, ψ_0, φ and n, such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\left\|\psi^{**}_{0,\,A,\,B}(f)\right\|_{L^\varphi(\mathbb{R}^n)} \leq C\left\|\psi_0^+(f)\right\|_{L^\varphi(\mathbb{R}^n)} \tag{3.5}$$

and

$$\left\|f^*\right\|_{L^\varphi(\mathbb{R}^n)} \leq C\left\|\psi_0^+(f)\right\|_{L^\varphi(\mathbb{R}^n)}. \tag{3.6}$$

Proof. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. First, we prove (3.5). Let $A \in (A_1,\infty)$ and $B \in (B_1,\infty)$. By $A_1 = \max\{A_0,\ nq(\varphi)/i(\varphi)\}$ and $B_1 = B_0/i(\varphi)$, we know that there exists $r_0 \in (0, \frac{i(\varphi)}{q(\varphi)})$ such that $A > \frac{n}{r_0}$ and $Br_0 > \frac{B_0}{q(\varphi)}$, where A_0 and B_0 are respectively as in Lemmas 3.5 and 3.3. Thus, by Lemma 3.5, for all $x \in \mathbb{R}^n$, we know that

$$\left[(\psi_0)^*_{j,\,A,\,B}(f)(x)\right]^{r_0} \lesssim \sum_{k=j}^{\infty} 2^{(j-k)(Ar_0-n)}\left\{M\left(|(\psi_0)_k * f|^{r_0}\right)(x) + K_{Br_0}\left(|(\psi_0)_k * f|^{r_0}\right)(x)\right\}, \tag{3.7}$$

where M is the Hardy–Littlewood maximal function. Let $\psi_0^+(f)$ and $\psi^{**}_{0,\,A,\,B}(f)$ be respectively as in (3.2) and (3.3). We notice that for any $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $|(\psi_0)_k * f(x)| \leq \psi_0^+(f)(x)$, which, together with (3.7), implies that for all $x \in \mathbb{R}^n$,

$$\left[\psi^{**}_{0,\,A,\,B}(f)(x)\right]^{r_0} \lesssim M([\psi_0^+(f)]^{r_0})(x) + K_{Br_0}([\psi_0^+(f)]^{r_0})(x). \tag{3.8}$$

By $r_0 < \frac{i(\varphi)}{q(\varphi)}$, we see that there exist $q \in (q(\varphi),\infty)$ and $p_0 \in (0, i(\varphi))$ such that $r_0 q < p_0$, $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ and φ is of uniformly lower type p_0. Thus, $\widetilde{\varphi}(x,t) := \varphi(x, t^{1/r_0})$ is of uniformly lower type p_0/r_0. Then from (3.8), Lemmas 2.4(i) and 3.3, Theorem 2.9 and Corollary 2.8, together with the fact that $p_0/r_0 > q > q(\varphi)$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(x, \psi^{**}_{0,\,A,\,B}(f)(x)\right) dx &\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left\{M\left([\psi_0^+(f)]^{r_0}\right)(x)\right\}^{1/r_0}\right) dx \\ &\quad + \int_{\mathbb{R}^n} \varphi\left(x, \left\{K_{Br_0}\left([\psi_0^+(f)]^{r_0}\right)(x)\right\}^{1/r_0}\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \psi_0^+(f)(x)\right) dx.\end{aligned}$$

Replacing f by f/λ with $\lambda \in (0, \infty)$ in the above inequality and noticing that $\psi^{**}_{0,A,B}(f/\lambda) = \psi^{**}_{0,A,B}(f)/\lambda$ and $\psi^{+}_0(f/\lambda) = \psi^{+}_0(f)/\lambda$, we see that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi^{**}_{0,A,B}(f)(x)}{\lambda}\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi^{+}_0(f)(x)}{\lambda}\right) dx, \tag{3.9}$$

which, together with the arbitrariness of $\lambda \in (0, \infty)$, implies (3.5).

Now, we prove (3.6). By $N_0 = \lfloor 2A_1 \rfloor + 1$, we know that there exists $A \in (A_1, \infty)$ such that $2A < N_0$. In the remainder of this proof, we fix $A \in (A_1, \infty)$ satisfying $2A < N_0$ and $B \in (B_1, \infty)$. Let integer $N \geq N_0$. For any $\gamma \in \mathcal{S}_N(\mathbb{R}^n)$, $t \in (0, 1)$ and $j \in \mathbb{Z}_+$, from Lemma 3.2 and (3.4), it follows that

$$\gamma_t * f = \gamma_t * (\eta_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \gamma_t * \eta_k * \psi_k * f, \tag{3.10}$$

where $\eta_0,\ \eta \in \mathcal{D}(\mathbb{R}^n)$ with $L_\eta \geq N$ and ψ is as in Lemma 3.2.

For any given $t \in (0, 1)$ and $x \in \mathbb{R}^n$, let $2^{-j_0-1} \leq t < 2^{-j_0}$ for some $j_0 \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$ satisfy $|z - x| < t$. Then, by (3.10), we conclude that

$$\begin{aligned} |\gamma_t * f(z)| &\leq \left|\gamma_t * (\eta_0)_{j_0} * (\psi_0)_{j_0} * f(z)\right| + \sum_{k=j_0+1}^{\infty} |\gamma_t * \eta_k * \psi_k * f(z)| \\ &\leq \int_{\mathbb{R}^n} \left|\gamma_t * (\eta_0)_{j_0}(y)\right| \left|(\psi_0)_{j_0} * f(z-y)\right| dy + \sum_{k=j_0+1}^{\infty} \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)|\, |\psi_k * f(z-y)|\, dy \\ &=: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{3.11}$$

To estimate I_1, from

$$\psi^{**}_{0,A,B}(f)(x) = \sup_{j\in\mathbb{Z}_+,\, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B}(y)} = \sup_{j\in\mathbb{Z}_+,\, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(z-y)|}{m_{j,A,B}(y+x-z)},$$

we infer that $\left|(\psi_0)_{j_0} * f(z-y)\right| \leq \psi^{**}_{0,A,B}(f)(x) m_{j_0,A,B}(y+x-z)$, which, together with the facts that $m_{j_0,A,B}(y+x-z) \leq m_{j_0,A,B}(x-z) m_{j_0,A,B}(y)$ and $m_{j_0,A,B}(x-z) \lesssim 2^A$, implies that $|(\psi_0)_{j_0} * f(z-y)| \lesssim 2^A \psi^{**}_{0,A,B}(f)(x) m_{j_0,A,B}(y)$. Thus, we have

$$\mathrm{I}_1 \lesssim 2^A \left\{\int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)| m_{j_0,A,B}(y)\, dy\right\} \psi^{**}_{0,A,B}(f)(x).$$

To estimate I_2, by the definition of ψ, we see that, for any $k \in \mathbb{N}$,

$$|\psi_k * f(z-y)| \leq |(\psi_0)_k * f(z-y)| + |(\psi_0)_{k-1} * f(z-y)|.$$

By the definition of $\psi^{**}_{0,A,B}(f)$ and the facts that for any $k \in \mathbb{N}$,

$$m_{k,A,B}(y+x-z) \leq m_{k,A,B}(x-z) m_{k,A,B}(y)$$

and $m_{k,A,B}(x-z) \lesssim 2^{(k-j_0)A}$, we conclude that

$$|(\psi_0)_k * f(z-y)| \leq \psi^{**}_{0,A,B}(f)(x) m_{k,A,B}(y+x-z) \lesssim 2^{(k-j_0)A} m_{k,A,B}(y) \psi^{**}_{0,A,B}(f)(x).$$

Similarly, we also have $|(\psi_0)_{k-1} * f(z-y)| \lesssim 2^{(k-j_0)A} m_{k,A,B}(y) \psi^{**}_{0,A,B}(f)(x)$. Thus,

$$\mathrm{I}_2 \lesssim \sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} \left\{\int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)| m_{k,A,B}(y)\, dy\right\} \psi^{**}_{0,A,B}(f)(x).$$

From (3.11) and the above estimates of I_1 and I_2, it follows that

$$\begin{aligned} |\gamma_t * f(z)| &\lesssim \left\{\int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)| m_{j_0,A,B}(y)\, dy + \sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)| m_{k,A,B}(y)\, dy\right\} \\ &\quad \times \psi^{**}_{0,A,B}(f)(x). \end{aligned} \tag{3.12}$$

Assume that $\operatorname{supp}(\eta_0) \subset B(0, R_0)$. Then $\operatorname{supp}((\eta_0)_j) \subset B(0, 2^{-j} R_0)$ for all $j \in \mathbb{Z}_+$. Moreover, by $\operatorname{supp}(\gamma) \subset B(0, R)$ and $2^{-j_0-1} \leq t < 2^{-j_0}$, we see that $\operatorname{supp}(\gamma_t) \subset B(0, 2^{-j_0} R)$. From this, we further deduce that $\operatorname{supp}(\gamma_t * (\eta_0)_{j_0}) \subset B(0, 2^{-j_0}(R_0 + R))$ and

$$|\gamma_t * (\eta_0)_{j_0}(y)| \lesssim \int_{\mathbb{R}^n} |\gamma_t(s)|\,|(\psi_0)_{j_0}(y-s)|\,ds \lesssim 2^{j_0 n} \int_{\mathbb{R}^n} |\gamma_t(s)|\,ds \sim 2^{j_0 n},$$

which implies that

$$\int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)| m_{j_0,\,A,\,B}(y)\,dy \lesssim 2^{j_0 n} \int_{B(0,2^{-j_0}(R_0+R))} (1+2^{j_0}|y|)^A 2^{B|y|}\,dy \lesssim 1. \tag{3.13}$$

Moreover, since η has vanishing moments up to order N, it was proved in [38, (2.13)] that $\|\gamma_t * \eta_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{(j_0-k)N} 2^{j_0 n}$ for all $k \in \mathbb{N}$ with $k \geq j_0+1$, which, together with the facts that $N > 2A$ and $\operatorname{supp}(\gamma_t * \eta_k) \subset B(0, 2^{-j_0}R_0 + 2^{-k}R)$, implies that

$$\begin{aligned}
&\sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)| m_{k,\,A,\,B}(y)\,dy \\
&\quad \lesssim \sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} 2^{(j_0-k)N} 2^{j_0 n} (2^{-j_0}R_0 + 2^{-k}R)^n \left[1 + 2^k(2^{-j_0}R_0 + 2^{-k}R)\right]^A 2^{(2^{-j_0}R_0 + 2^{-k}R)B} \\
&\quad \lesssim \sum_{k=j_0+1}^{\infty} 2^{(j_0-k)(N-2A)} \lesssim 1.
\end{aligned} \tag{3.14}$$

Thus, from (3.12)–(3.14), we deduce that $|\gamma_t * f(z)| \lesssim \psi_{0,\,A,\,B}^{**}(f)(x)$. Then, by the arbitrariness of $t \in (0,1)$ and $z \in B(x,t)$, we know that $f^*(x) \lesssim \psi_{0,\,A,\,B}^{**}(f)(x)$, which, together with (3.9), implies that, for any $\lambda \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, f^*(x)/\lambda\right)\,dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \psi_0^+(f)(x)/\lambda\right)\,dx.$$

From this, we infer that (3.6) holds, which completes the proof of Theorem 3.6. □

From Theorem 3.6, we immediately deduce the following vertical and the nontangential maximal function characterizations of $H^\varphi(\mathbb{R}^n)$. We omit the details.

Theorem 3.7. *Let φ be a growth function as in Definition 2.3, and ψ_0, ψ_0^+ and $(\psi_0)_\triangledown^*$ as in Definition 3.1. Then the followings are equivalent:*

(i) $f \in H^\varphi(\mathbb{R}^n)$;
(ii) $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\psi_0^+(f) \in L^\varphi(\mathbb{R}^n)$;
(iii) $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $(\psi_0)_\triangledown^*(f) \in L^\varphi(\mathbb{R}^n)$.

Moreover, for all $f \in H^\varphi(\mathbb{R}^n)$, $\|f\|_{H^\varphi(\mathbb{R}^n)} \sim \|\psi_0^+(f)\|_{L^\varphi(\mathbb{R}^n)} \sim \left\|(\psi_0)_\triangledown^*(f)\right\|_{L^\varphi(\mathbb{R}^n)}$, where the implicit constants are independent of f.

4. The Littlewood–Paley g-function and g_λ^*-function characterizations of $H^\varphi(\mathbb{R}^n)$

In this section, we establish the Littlewood–Paley g-function and g_λ^*-function characterizations of $H^\varphi(\mathbb{R}^n)$.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function, $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$,

$$\int_{\mathbb{R}^n} \phi(x) x^\gamma\,dx = 0 \tag{4.1}$$

for all $|\gamma| \leq m(\varphi)$, where $m(\varphi)$ is as in (2.7) and, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\int_0^\infty |\hat{\phi}(\xi t)|^2 \frac{dt}{t} = 1.$$

Recall that for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *$g$-function*, the *Lusin area integral* and the *g_λ^*-function*, with $\lambda \in (1,\infty)$, of f are defined, respectively, by setting, for all $x \in \mathbb{R}^n$,

$$g(f)(x) := \left[\int_0^\infty |f * \phi_t(y)|^2 \frac{dt}{t}\right]^{1/2},$$

$$S(f)(x) := \left[\int_0^\infty \int_{\{y\in\mathbb{R}^n:|y-x|<t\}} |f * \phi_t(y)|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\lambda^*(f)(x) := \left[\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |f * \phi_t(y)|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2}.$$

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is called to *vanish weakly at infinity*, if for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$; see, for example, [30, p. 50]. We have the following useful property of $H^\varphi(\mathbb{R}^n)$, which is just [25, Lemma 4.12].

Proposition 4.1. *Let φ be a growth function. If $f \in H^\varphi(\mathbb{R}^n)$, then f vanishes weakly at infinity.*

The following Proposition 4.2 is just [25, Theorem 4.11].

Proposition 4.2. *Let φ be a growth function. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $S(f) \in L^\varphi(\mathbb{R}^n)$ and, moreover,*

$$\frac{1}{C}\|S(f)\|_{L^\varphi(\mathbb{R}^n)} \le \|f\|_{H^\varphi(\mathbb{R}^n)} \le C\|S(f)\|_{L^\varphi(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

Similar to the proof of [14, Lemma 5.1], we easily obtain the following boundedness of the Littlewood–Paley g-function from $H^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$. We omit the details.

Proposition 4.3. *Let φ be a growth function. If $f \in H^\varphi(\mathbb{R}^n)$, then $g(f) \in L^\varphi(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that for all $f \in H^\varphi(\mathbb{R}^n)$,*

$$\|g(f)\|_{L^\varphi(\mathbb{R}^n)} \le C\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

We have the following Littlewood–Paley g-function characterization of $H^\varphi(\mathbb{R}^n)$.

Theorem 4.4. *Let φ be as in Definition 2.3. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in L^\varphi(\mathbb{R}^n)$ and, moreover,*

$$\frac{1}{C}\|g(f)\|_{L^\varphi(\mathbb{R}^n)} \le \|f\|_{H^\varphi(\mathbb{R}^n)} \le C\|g(f)\|_{L^\varphi(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

Proof. By Propositions 4.1–4.3, it suffices to prove that if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $g(f) \in L^\varphi(\mathbb{R}^n)$, then $\|S(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^\varphi(\mathbb{R}^n)}$.

For $j \in \mathbb{Z}$, let

$$\mathcal{D}_j := \{I \subset \mathbb{R}^n : I \text{ is a dyadic cube and } \ell(I) = 2^{-j}\}.$$

From [29, Theorem 2.1, Lemmas 2.1 and 2.2], we deduce that for any fixed $L \in [0, m(\varphi)+1)$, $K \in \mathbb{N}$ and $r \in (\frac{n}{n+K}, 1]$, there exists $N \in \mathbb{N}$ large enough such that, for all $j \in \mathbb{Z}$, $f \in \mathcal{S}'(\mathbb{R}^n)$, $u, u^* \in I$ and $x_{\widetilde{I}} \in \widetilde{I}$,

$$\begin{aligned}|\phi_j * f(u)| &\lesssim \sum_{\widetilde{j}\in\mathbb{Z}} \sum_{\widetilde{I}\in\mathcal{D}_{\widetilde{j}+N}} \frac{2^{-|j-\widetilde{j}|L}2^{-(\min\{j,\widetilde{j}\})K}|\widetilde{I}|}{(2^{-\min\{j,\widetilde{j}\}}+|u-x_I|)^{n+K}}|(\phi_{\widetilde{j}} * f)(x_{\widetilde{I}})| \\ &\lesssim \sum_{\widetilde{j}\in\mathbb{Z}} 2^{-|j-\widetilde{j}|L+n\left(1-\frac{1}{r}\right)[\min(\widetilde{j},j)-\widetilde{j}]} \left[M\left(\left[\sum_{\widetilde{I}\in\mathcal{D}_{\widetilde{j}+N}} |\phi_{\widetilde{j}} * f(x_{\widetilde{I}})|\chi_{\widetilde{I}}\right]^r\right)(u^*)\right]^{1/r}. \end{aligned} \tag{4.2}$$

We point out that L in [29, Lemma 2.1] must be strictly less than $M+1$, which cannot be arbitrary, as claimed in [29, Lemma 2.1]. This is why we need to restrict $L \in [0, m(\varphi)+1)$.

Let $i(\varphi)$, $q(\varphi)$ and $m(\varphi)$ be, respectively, as in (2.1), (2.3) and (2.7). By $m(\varphi)+1 = \lfloor n[q(\varphi)/i(\varphi)-1]\rfloor + 1$ and the definitions of $q(\varphi)$ and $i(\varphi)$, we know that there exist $q_0 \in (q(\varphi), \infty)$ and $p_0 \in (0, i(\varphi))$ such that $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$, φ is uniformly lower type p_0 and $L := n(q_0/p_0 - 1) < m(\varphi)+1$. Then $\frac{n}{n+L} = \frac{p_0}{q_0} < \frac{p_0}{q(\varphi)}$. Choosing $r \in (\frac{n}{n+L}, \frac{p_0}{q(\varphi)})$, by (4.2), we further conclude that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} S(f)(x) &\sim \left[\sum_{j\in\mathbb{Z}} 2^{jn}\int_{B(x,2^{-j})} |\phi_j * f(y)|^2\,dy\right]^{1/2} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{jn}\int_{B(x,2^{-j})} \left\{\sum_{\widetilde{j}\in\mathbb{Z}} 2^{-|j-\widetilde{j}|L+n\left(1-\frac{1}{r}\right)[\max(\widetilde{j},j)-j]} \left[M\left(\left[\sum_{\widetilde{I}\in\mathcal{D}_{\widetilde{j}+N}} |\phi_{\widetilde{j}} * f(x_{\widetilde{I}})|\chi_{\widetilde{I}}\right]^r\right)(x)\right]^{1/r}\right\}^2 dy\right)^{1/2} \\ &\sim \left\{\sum_{\widetilde{j}\in\mathbb{Z}} \left[M\left(\left[\sum_{\widetilde{I}\in\mathcal{D}_{\widetilde{j}+N}} |\phi_{\widetilde{j}} * f(x_{\widetilde{I}})|\chi_{\widetilde{I}}\right]^r\right)(x)\right]^{2/r}\right\}^{1/2}. \end{aligned} \tag{4.3}$$

Choose K large enough such that $\frac{n}{n+K} < \frac{p_0}{q(\varphi)}$ and $r \in (\max\{\frac{n}{n+L}, \frac{n}{n+K}\}, \frac{p_0}{q(\varphi)})$. Let $\widetilde{\varphi}(x,t) := \varphi(x, t^{1/r})$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. From the fact that φ is of uniformly upper type 1 and lower type p_0, it follows that $\widetilde{\varphi}$ is of uniformly upper type $1/r$ and lower type p_0/r. Then, by Theorem 2.10, together with $1/r > p_0/r > q(\varphi)$, we conclude that

$$\int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{j\in\mathbb{Z}} \left[M(f_j^r)(x)\right]^{2/r}\right\}^{1/2}\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_{j\in\mathbb{Z}} |f_j(x)|^2\right]^{1/2}\right) dx,$$

which, together with (4.3), further implies that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x, S(f)(x))\,dx &\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{\widetilde{j}\in\mathbb{Z}}\left[M\left(\left[\sum_{\widetilde{I}\in\mathcal{D}_{\widetilde{j}+N}} |\phi_{\widetilde{j}} * f(x_{\widetilde{I}})|\chi_{\widetilde{I}}\right]^r\right)(x)\right]^{2/r}\right\}^{1/2}\right) dx \\ &\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{\widetilde{j}\in\mathbb{Z}}\sum_{\widetilde{I}\in\mathcal{D}_{\widetilde{j}+N}} \left[|\phi_{\widetilde{j}} * f(x_{\widetilde{I}})|\chi_{\widetilde{I}}(x)\right]^2\right\}^{1/2}\right) dx \sim \int_{\mathbb{R}^n} \varphi\left(x, g(f)(x)\right)\,dx,\end{aligned}$$

where, in the last step, we used the arbitrariness of $x_{\widetilde{I}} \in \widetilde{I}$. This finishes the proof of Theorem 4.4. □

It is easy to see that $S(f)(x) \le g_\lambda^*(f)(x)$ for all $x \in \mathbb{R}^n$, which, together with Proposition 4.2, immediately implies the following conclusion.

Proposition 4.5. *Let φ be as in Definition 2.3 and $\lambda \in (1,\infty)$. If $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishes weakly at infinity and $g_\lambda^*(f) \in L^\varphi(\mathbb{R}^n)$, then $f \in H^\varphi(\mathbb{R}^n)$ and, moreover,*

$$\|f\|_{H^\varphi(\mathbb{R}^n)} \le C\|g_\lambda^*(f)\|_{L^\varphi(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

Next we consider the boundedness of g_λ^* on $L^\varphi(\mathbb{R}^n)$. To this end, we need to introduce the following variant of the Lusin area function S. For all $\alpha \in (0,\infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S_\alpha(f)(x) := \left[\int_0^\infty \int_{\{y\in\mathbb{R}^n:|y-x|<\alpha t\}} |f * \phi_t(y)|^2 (\alpha t)^{-n}\frac{dy\,dt}{t}\right]^{1/2}.$$

The following technical lemma plays a key role to obtain the g_λ^*-function characterization of $H^\varphi(\mathbb{R}^n)$, whose proof was motivated by Folland and Stein [30, p. 218, Theorem 7.1] and Aguilera and Segovia [31, Theorem 1].

Lemma 4.6. *Let $q \in [1,\infty)$, φ be as in Definition 2.3 and $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $\alpha \in [1,\infty)$, $t \in [0,\infty)$ and measurable functions f,*

$$\int_{\mathbb{R}^n} \varphi(x, S_\alpha(f)(x))\,dx \le C\alpha^{n(q-p/2)} \int_{\mathbb{R}^n} \varphi(x, S(f)(x))\,dx.$$

Proof. For all $\lambda \in (0,\infty)$, let $A_\lambda := \{x \in \mathbb{R}^n : S(f)(x) > \lambda\alpha^{n/2}\}$ and

$$U := \{x \in \mathbb{R}^n : M(\chi_{A_\lambda})(x) > (4\alpha)^{-n}\},$$

where M is the Hardy–Littlewood maximal function. Since $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, we see that

$$\begin{aligned}\varphi(U,\lambda) &= \varphi\left(\{x \in \mathbb{R}^n : M(\chi_{A_\lambda})(x) > (4\alpha)^{-n}\}, \lambda\right) \\ &\lesssim (4\alpha)^{nq}\|\chi_{A_\lambda}\|^q_{L^q_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)} \sim \alpha^{nq}\varphi(A_\lambda,\lambda)\end{aligned} \tag{4.4}$$

and, by [31, Lemma 2], we know that

$$\alpha^{n(1-q)} \int_{U^\complement} [S_\alpha(f)(x)]^2\varphi(x,\lambda)\,dx \lesssim \int_{A_\lambda^\complement} [S(f)(x)]^2\varphi(x,\lambda)\,dx. \tag{4.5}$$

Thus, from (4.4) and (4.5), it follows that

$$\begin{aligned}\varphi\left(\left\{x \in \mathbb{R}^n : S_\alpha(f)(x) > \lambda\right\}, \lambda\right) &\le \varphi(U,\lambda) + \varphi\left(U^\complement \cap \{x \in \mathbb{R}^n : S_\alpha(f)(x) > \lambda\}, \lambda\right) \\ &\lesssim \alpha^{nq}\varphi(A_\lambda,\lambda) + \lambda^{-2}\int_{U^\complement} [S_\alpha(f)(x)]^2\varphi(x,\lambda)\,dx\end{aligned}$$

$$\lesssim \alpha^{nq}\varphi(A_\lambda,\lambda)+\alpha^{n(q-1)}\lambda^{-2}\int_{A_\lambda^\complement}[S(f)(x)]^2\varphi(x,\lambda)\,dx$$
$$\sim \alpha^{nq}\varphi(A_\lambda,\lambda)+\alpha^{n(q-1)}\lambda^{-2}\int_0^{\lambda\alpha^{n/2}} t\varphi(\left\{x\in\mathbb{R}^n:\ S(f)(x)>t\right\},\lambda)\,dt,$$

which, together with the assumption that $\alpha\in[1,\infty)$, Lemma 2.4(ii), the uniformly lower type p and upper type 1 properties of φ, further implies that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\varphi(x,S_\alpha(f)(x))\,dx\\
&=\int_0^\infty \frac{1}{\lambda}\varphi\left(\left\{x\in\mathbb{R}^n:\ S_\alpha(f)(x)>\lambda\right\},\lambda\right)\,d\lambda\\
&\lesssim \alpha^{nq}\int_0^\infty \frac{1}{\lambda}\varphi(A_\lambda,\lambda)\,d\lambda+\alpha^{n(q-1)}\int_0^\infty\lambda^{-3}\int_0^{\lambda\alpha^{n/2}} t\varphi(\{x\in\mathbb{R}^n:\ S(f)(x)>t\},\lambda)\,dt\,d\lambda\\
&\lesssim \alpha^{n(q-p/2)}\int_0^\infty\frac{1}{\lambda}\varphi(\{x\in\mathbb{R}^n:\ S(f)(x)>\lambda\},\lambda)\,d\lambda\\
&\quad+\alpha^{n(q-1)}\left\{\int_0^\infty\lambda^{-3}\int_0^\lambda \lambda\varphi(\{x\in\mathbb{R}^n:\ S(f)(x)>t\},t)\,dt\,d\lambda\right.\\
&\quad\left.+\int_0^\infty\lambda^{-3}\int_\lambda^{\lambda\alpha^{n/2}}(\lambda/t)^p t\varphi(\{x\in\mathbb{R}^n:S(f)(x)>t\},t)\,dt\,d\lambda\right\}\\
&\lesssim \alpha^{n(q-p/2)}\int_{\mathbb{R}^n}\varphi(x,S(f)(x))\,dx+\alpha^{n(q-1)}\left\{\int_0^\infty\frac{1}{t}\lambda\varphi(\{x\in\mathbb{R}^n:\ S(f)(x)>t\},t)\,dt\right.\\
&\quad\left.+\int_0^\infty\frac{1}{t}\left[\alpha^{(2-p)n/2}-1\right]\varphi(\{x\in\mathbb{R}^n:\ S(f)(x)>t\},t)\,dt\right\}\\
&\lesssim \alpha^{n(q-p/2)}\int_{\mathbb{R}^n}\varphi(x,S(f)(x))\,dx.
\end{aligned}$$

This finishes the proof of Lemma 4.6. □

Using Lemma 4.6, we obtain the following boundedness of g_λ^* from $H^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.

Proposition 4.7. *Let φ be as in Definition 2.3, $q\in[1,\infty)$, $\varphi\in\mathbb{A}_q(\mathbb{R}^n)$, and $\lambda\in(2q/p,\infty)$. Then, there exists a positive constant $C_{(\varphi,q)}$ such that, for all $f\in H^\varphi(\mathbb{R}^n)$,*

$$\|g_\lambda^*(f)\|_{L^\varphi(\mathbb{R}^n)}\le C_{(\varphi,q)}\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

Proof. For all $f\in H^\varphi(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}
\left[g_\lambda^*(f)(x)\right]^2&=\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|f*\phi_t(y)|^2\frac{dy\,dt}{t^{n+1}}+\sum_{k=1}^\infty\int_0^\infty\int_{2^{k-1}t\le|x-y|<2^kt}\cdots\\
&\lesssim[Sf(x)]^2+\sum_{k=1}^\infty 2^{-kn(\lambda-1)}\left[S_{2^k}f(x)\right]^2.
\end{aligned}\tag{4.6}$$

Then from (4.6), Lemmas 2.4(i) and 4.6, and $\lambda\in(2q/p,\infty)$, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^n}\varphi(x,g_\lambda^*(f)(x))\,dx&\lesssim\sum_{k=0}^\infty\int_{\mathbb{R}^n}\varphi\left(x,2^{-kn(\lambda-1)/2}S_{2^k}(f)(x)\right)\,dx\\
&\lesssim\sum_{k=0}^\infty 2^{-knp(\lambda-1)/2}2^{kn(q-p/2)}\int_{\mathbb{R}^n}\varphi\left(x,S(f)(x)\right)\,dx\\
&\lesssim\int_{\mathbb{R}^n}\varphi(x,S(f)(x))\,dx.
\end{aligned}$$

By Lemma 2.5(i), we see that

$$\int_{\mathbb{R}^n}\varphi\left(x,\frac{g_\lambda^*(f)(x)}{\|f\|_{H^\varphi(\mathbb{R}^n)}}\right)\,dx\lesssim\int_{\mathbb{R}^n}\varphi\left(x,\frac{S(f)(x)}{\|f\|_{H^\varphi(\mathbb{R}^n)}}\right)\,dx$$

$$\sim \int_{\mathbb{R}^n} \varphi\left(x, \frac{S(f)(x)}{\|S(f)\|_{L^\varphi(\mathbb{R}^n)}}\right) dx \sim 1,$$

which, together with Lemma 2.6(i), then completes the proof of Proposition 4.7. □

By Propositions 4.1, 4.5 and 4.7, we have the following g_λ^*-function characterization of $H^\varphi(\mathbb{R}^n)$. We omit the details.

Theorem 4.8. *Let φ be as in Definition 2.3, $q \in [1, \infty)$, $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ and $\lambda \in (2q/p, \infty)$. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g_\lambda^*(f) \in L^\varphi(\mathbb{R}^n)$ and, moreover,*

$$\frac{1}{C}\|g_\lambda^*(f)\|_{L^\varphi(\mathbb{R}^n)} \le \|f\|_{H^\varphi(\mathbb{R}^n)} \le C\|g_\lambda^*(f)\|_{L^\varphi(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

We point out that the range of λ in Theorem 4.8 is the known best possible, even when $\varphi(x, t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, or $\varphi(x, t) := w(x)t^p$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, with $q \in [1, \infty)$ and $w \in A_q(\mathbb{R}^n)$; see, respectively, [30, p. 221, Corollary 7.4] and [31, Theorem 2].

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