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Relative unitary commutator calculus, and applications ☆

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ABSTRACT

This note revisits localisation and patching method in the setting of generalised unitary groups. Introducing certain subgroups of relative elementary unitary groups, we develop relative versions of the conjugation calculus and the commutator calculus in unitary groups, which are both more general, and substantially easier than the ones available in the literature. For the general linear group such relative commutator calculus has been recently developed by the first and the third authors. As an application we prove the mixed commutator formula,

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] = \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big],$$

for two form ideals (I, Γ) and (J, Δ) of a form ring (A, Λ). This answers two problems posed in a paper by Alexei Stepanov and the second author.

千里之行，始于足下。
—老子

A journey of a thousand miles begins with a single step.
–Lao Tze

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1. Introduction

One of the most powerful ideas in the study of classical groups over rings is localisation. It allows to reduce many important problems over various classes of rings subject to commutativity conditions, to similar problems for semi-local rings. Localisation comes in a number of versions. The two most familiar ones are **localisation and patching**, proposed by Daniel Quillen [38] and Andrei Suslin [43], and **localisation–completion**, proposed by Anthony Bak [3].

Originally, the above papers addressed the case of the general linear group $\mathrm{GL}(n, A)$. Soon thereafter, Suslin himself, Vyacheslav Kopeiko, Marat Tulenbaev, Leonid Vaserstein, Li Fuan, Eiichi Abe, and others proposed working versions of localisation and patching for other classical groups, such as symplectic and orthogonal ones, as well as unitary groups, under some additional simplifying assumptions, see, for example, [26,45,27,28] and further references in [47,9,41,6,23].

In the most general setting of quadratic modules, similar development took more time. In fact, the first full scale treatment of localisation–completion was proposed only in the Bielefeld thesis by the first author [19,20]. Quite remarkably, the first exhaustive treatment of localisation and patching came only afterwards, in the St.-Petersburg thesis by Victor Petrov [34–36] and was *strongly* influenced by [19,20].

As a matter of fact, both methods rely on a large body of common calculations, and technical facts, known as **conjugation calculus** and **commutator calculus**. Oftentimes these calculations are even referred to as the **yoga of conjugation**, and the **yoga of commutators**, to stress the overwhelming feeling of technical strain and exertion. In the unitary case, due to the following circumstances,

- the presence of long and short roots,
- complicated elementary relations,
- non-commutativity,
- non-trivial involution,
- non-trivial form parameter,

these calculations tend to be especially lengthy, and highly involved.

A specific motivation for the present work was the desire to create tools to prove *relative* versions of structure results for unitary groups. One typical such result in which we were particularly interested, is description of subnormal subgroups, or, what is almost the same, description of subgroups of the unitary groups $\mathrm{GU}(2n, A, \Lambda)$, normalised by a relative elementary subgroup $\mathrm{EU}(2n, I, \Gamma)$, see [16,17,50–52].

Another one was generalisation of the mixed commutator formula

$$\big[E(n, R, I), \mathrm{GL}(n, R, J)\big] = \big[E(n, R, I), E(n, R, J)\big],$$

proved in the setting of general linear groups by Alexei Stepanov and the second author [48] where here R is a ring and I and J are two-sided ideals of R. This formula is a common generalisation of the standard commutator formulae. At the stable level, these formulae were first established in the work of Hyman Bass [11]. In another decade, Andrei Suslin, Leonid Vaserstein, Zenon Borewicz, and the second author [43–45,12,41] discovered that for commutative rings similar formulae hold for all $n \geqslant 3$. See also [8,9,14,15,21,22,25,41,47] for various proofs and non-commutative generalisations. However, for two relative subgroups such formulae were proven only at the stable level, by Alec Mason [30–33].

However, the proof in [48] relied on a strong form of decomposition of unipotents [41], and was not likely to directly generalise to other classical groups. The authors of [48] raised the problems of establishing this formula via localisation method, and to generalise it to the general setting of quadratic modules [48, Problems 1 and 2].

In the paper [24] the first and the third authors developed relative versions of conjugation calculus and commutator calculus in the general linear group $\mathrm{GL}(n, R)$, thus solving [48, Problem 1]. However, we believe that the importance and applicability of the method itself far surpass this immediate application.

In the present paper, which is a sequel of [24], we in a similar way evolve relative *unitary* conjugation calculus and commutator calculus, and, in particular, solve [48, Problem 2]. Actually, the present paper does not depend on the calculations from [19] and [20]. Instead, here we establish relative versions of these results from scratch, in a more general setting. The resulting versions of conjugation calculus and commutator calculus are both more general, and *substantially easier* than the ones available in the literature.

The overall scheme is always that devised by the first author in [19,20], which in turn follows Bak's localisation–completion method [3], whose distinguishing feature is that principal t-localisations are injective on small t-adic neighbourhoods. However, we propose several important technical innovations, and simplifications. Some such simplifications are similar to those proposed by the first and the second authors in [22]. Most importantly, following [24] we introduce certain subgroups of relative elementary quadratic groups, and prove all results not at the absolute, but at the relative level. Another important improvement is that we notice that the case analysis in the proof of Lemmas 8 and 12 which provide the base of induction, can be cut in half.

As an immediate application of our methods we prove the following mixed commutator formula.

Theorem 1. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I, Γ) and (J, Δ) be two form ideals of a form ring (A, Λ). Then*

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] = \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big]. \tag{1}$$

This theorem is a very broad generalisation of many preceding results, including the following ones—which, in turn, generalise a lot previous of results!

- Absolute standard unitary commutator formulae, Bak and Vavilov [9, Theorem 1.1], and Vaserstein and Hong You [46].
- Relative unitary commutator formula at the stable level, under some additional stability assumptions, Habdank [16,17].
- Relative commutator formula for the general linear group $\mathrm{GL}(n, R)$, Stepanov and Vavilov [48,49], and Hazrat and Zhang [24]. This case is obtained, by setting in our theorem, $A = R \oplus R^0$.

Observe, that in the above generality (relative, without stability conditions) our results are new already for the following familiar cases.

- The case of symplectic groups $\mathrm{Sp}(2l, R)$, when the involution is trivial, and $\Lambda = R$.
- The case of split orthogonal groups $\mathrm{SO}(2l, R)$, when the involution is trivial and $\Lambda = 0$.
- The case of classical unitary groups $\mathrm{SU}(2l, R)$, when $\Lambda = \Lambda_{\max}$.

See [18, §5.2B] for further discussion on the generalised unitary groups.

Actually, in Sections 8, 9 we give *another* proof of Theorem 1, imitating that of [49]. Namely, we show, that Theorem 1 can be deduced from the *absolute* standard commutator formula by careful calculation of levels of the above commutator groups, and some group-theoretic arguments.

Nevertheless, we believe that our localisation proof, based on the relative conjugation calculus and commutator calculus, which we develop in Sections 5, 6 of the present paper, and especially the calculations themselves, are of independent value, and will be used in many further applications.

The paper is organised as follows. In Sections 2–4 we recall basic notation, and some background facts, used in the sequel. The next two sections constitute the technical core of the paper. Namely, in Section 5, and in Section 6 we develop relative unitary conjugation calculus, and relative unitary commutator calculus, respectively. After that we are in a position to give a localisation proof of Theorem 1 in Section 7. In Section 8 we calculate the levels of the mixed commutator subgroups. Using these calculations in Section 9 we give another proof of Theorem 1, deducing it from the *absolute* standard commutator formula. There we also obtain slightly more precise results in some special situations, for instance, when A itself is commutative or when I and J are comaximal, $I + J = A$. Finally, in Section 10 we state and briefly discuss some further related problems.

2. Form rings and form ideal

The notion of Λ-quadratic forms, quadratic modules and generalised unitary groups over a form ring (A, Λ) were introduced by Anthony Bak in his thesis, see [1,2]. In this section, and the next one, we *very briefly* review the most fundamental notation and results that will be constantly used in the present paper. We refer to [1,2,18,9,13,7,19,20,5,34,23] for details, proofs, and further references.

2.1. Let R be a commutative ring with 1, and A be a (not necessarily commutative) R-algebra. An involution, denoted by $\bar{\ }$, is an anti-morphism of A of order 2. Namely, for $\alpha, \beta \in A$, one has $\overline{\alpha+\beta} = \bar{\alpha} + \bar{\beta}$, $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ and $\bar{\bar{\alpha}} = \alpha$. Fix an element $\lambda \in \mathrm{Cent}(A)$ such that $\lambda\bar{\lambda} = 1$. One may define two additive subgroups of A as follows:

$$\Lambda_{\min} = \{\alpha - \lambda\alpha \mid a \in A\}, \qquad \Lambda_{\max} = \{\alpha \in A \mid \alpha = -\lambda\bar{\alpha}\}.$$

A *form parameter* Λ is an additive subgroup of A such that

(1) $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$,
(2) $\alpha\Lambda\bar{\alpha} \subseteq \Lambda$ for all $\alpha \in A$.

The pair (A, Λ) is called a *form ring*.

2.2. Let $I \trianglelefteq A$ be a two-sided ideal of A. We assume I to be involution invariant, i.e. such that $\bar{I} = I$. Set

$$\Gamma_{\max}(I) = I \cap \Lambda, \qquad \Gamma_{\min}(I) = \{\xi - \lambda\bar{\xi} \mid \xi \in I\} + \langle \xi\alpha\bar{\xi} \mid \xi \in I,\ \alpha \in \Lambda\rangle.$$

A *relative form parameter* Γ in (A, Λ) of level I is an additive group of I such that

(1) $\Gamma_{\min}(I) \subseteq \Gamma \subseteq \Gamma_{\max}(I)$,
(2) $\alpha\Gamma\bar{\alpha} \subseteq \Gamma$ for all $\alpha \in A$.

The pair (I, Γ) is called a *form ideal*.

In the level calculations we will use sums and products of form ideals. Let (I, Γ) and (J, Δ) be two form ideals. Their sum is artlessly defined as $(I + J, \Gamma + \Delta)$, it is immediate to verify that this is indeed a form ideal.

Guided by analogy, one is tempted to set $(I, \Gamma)(J, \Delta) = (IJ, \Gamma\Delta)$. However, it is considerably harder to correctly define the product of two relative form parameters. The papers [16,17,19,20] introduce the following definition

$$\Gamma\Delta = \Gamma_{\min}(IJ) + {}^J\Gamma + {}^I\Delta,$$

where

$${}^J\Gamma = \langle \xi\Gamma\bar{\xi} \mid \xi \in J\rangle, \qquad {}^I\Delta = \langle \xi\Delta\bar{\xi} \mid \xi \in I\rangle.$$

One can verify that this is indeed a relative form parameter of level IJ if $IJ = JI$. Otherwise one needs to consider the symmetrised product

$$(I, \Gamma)(J, \Delta) + (J, \Delta)(I, \Gamma) = \big(IJ + JI, \Gamma_{\min}(IJ + JI) + {}^J\Gamma + {}^I\Delta\big).$$

2.3. A *form algebra over a commutative ring* R is a form ring (A, Λ), where A is an R-algebra and the involution leaves R invariant, i.e., $\overline{R} = R$.

- A form algebra (A, Λ) is called *module finite*, if A is finitely generated as an R-module.
- A form algebra (A, Λ) is called *quasi-finite*, if there is a direct system of module finite R-subalgebras A_i of A such that $\varinjlim A_i = A$.

However, in general Λ is not an R-module. This forces us to replace R by its subring R_0, generated by all $\alpha\overline{\alpha}$ with $\alpha \in R$. Clearly, all elements in R_0 are invariant with respect to the involution, i.e. $\bar{r} = r$, for $r \in R_0$.

It is immediate, that any form parameter Λ is an R_0-module. This simple fact will be used throughout. This is precisely why we have to localise in multiplicative subsets of R_0, rather than in those of R itself.

2.4. Let (A, Λ) be a form algebra over a commutative ring R with 1, and let S be a multiplicative subset of R_0 (see Section 2.3). For any R_0-module M one can consider its localisation $S^{-1}M$ and the corresponding localisation homomorphims $F_S : M \to S^{-1}M$. By definition of the ring R_0 both A and Λ are R_0-modules, and thus can be localised in S.

In the present paper, we mostly use localisation with respect to the following two types of multiplication systems of R_0.

- *Principal localisation*: for any $s \in R_0$ with $\bar{s} = s$, the multiplicative system generated by s is defined as $\langle s\rangle = \{1, s, s^2, \ldots\}$. The localisation of the form algebra (A, Λ) with respect to multiplicative system $\langle s\rangle$ is usually denoted by (A_s, Λ_s), where as usual $A_s = \langle s\rangle^{-1}A$ and $\Lambda_s = \langle s\rangle^{-1}\Lambda$ are the usual principal localisations of the ring A and the form parameter Λ. Notice that, for each $\alpha \in A_s$, there exists an integer n and an element $a \in A$ such that $\alpha = \frac{a}{s^n}$, and for each $\xi \in \Lambda_s$, there exists an integer m and an element $\zeta \in \Lambda$ such that $\xi = \frac{\zeta}{s^m}$.
- *Maximal localisation*: consider a maximal ideal $\mathfrak{m} \in \operatorname{Max}(R_0)$ of R_0 and the multiplicative closed set $S_\mathfrak{m} = R_0 \backslash \mathfrak{m}$. We denote the localisation of the form algebra (A, Λ) with respect to $S_\mathfrak{m}$ by $(A_\mathfrak{m}, \Lambda_\mathfrak{m})$, where $A_\mathfrak{m} = S_\mathfrak{m}^{-1}A$ and $\Lambda_\mathfrak{m} = S_\mathfrak{m}^{-1}\Lambda$ are the usual maximal localisations of the ring A and the form parameter, respectively.

In these cases the corresponding localisation homomorphisms will be denoted by F_s and by $F_\mathfrak{m}$, respectively.

The following fact is verified by a straightforward computation.

Lemma 1. *For any $s \in R_0$ and for any $\mathfrak{m} \in \operatorname{Max}(R_0)$ the pairs (A_s, Λ_s) and $(A_\mathfrak{m}, \Lambda_\mathfrak{m})$ are form rings.*

3. Unitary groups

In the present section we recall basic notation and facts related to Bak's generalised unitary groups and their elementary subgroups.

3.1. Let, as above, A be an associative ring with 1. For natural m, n we denote by $M(m,n,A)$ the additive group of $m \times n$ matrices with entries in A. In particular $M(m, A) = M(m,m, A)$ is the ring of matrices of degree n over A. For a matrix $x \in M(m,n, A)$ we denote by x_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, its entry in the position (i, j). Let e be the identity matrix and e_{ij}, $1 \leqslant i, j \leqslant n$, be a standard matrix unit, i.e. the matrix which has 1 in the position (i, j) and zeros elsewhere.

As usual, $\mathrm{GL}(m, A) = M(m, A)^*$ denotes the general linear group of degree m over A. The group $\mathrm{GL}(m, A)$ acts on the free right A-module $V \cong A^m$ of rank m. Fix a base $e_1, \ldots, e_m$ of the module V. We may think of elements $v \in V$ as columns with components in A. In particular, e_i is the column whose i-th coordinate is 1, while all other coordinates are zeros.

Actually, in the present paper we are only interested in the case, when $m = 2n$ is even. We usually number the base as follows: $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$. All other occurring geometric objects will be numbered accordingly. Thus, we write $v = (v_1, \dots, v_n, v_{-n}, \dots, v_{-1})^t$, where $v_i \in A$, for vectors in $V \cong A^{2n}$.

The set of indices will be always ordered accordingly, $\Omega = \{1, \dots, n, -n, \dots, -1\}$. Clearly, $\Omega = \Omega^+ \sqcup \Omega^-$, where $\Omega^+ = \{1, \dots, n\}$ and $\Omega^- = \{-n, \dots, -1\}$. For an element $i \in \Omega$ we denote by $\varepsilon(i)$ the sign of Ω, i.e. $\varepsilon(i) = +1$ if $i \in \Omega^+$, and $\varepsilon(i) = -1$ if $i \in \Omega^-$.

3.2. For a form ring (A, Λ), one considers the *hyperbolic unitary group* $\mathrm{GU}(2n, A, \Lambda)$, see [9, §2]. This group is defined as follows:

One fixes a symmetry $\lambda \in \mathrm{Cent}(A)$, $\lambda\bar{\lambda} = 1$ and supplies the module $V = A^{2n}$ with the following λ-hermitian form $h : V \times V \to A$,

$$h(u, v) = \bar{u}_1 v_{-1} + \dots + \bar{u}_n v_{-n} + \lambda \bar{u}_{-n} v_n + \dots + \lambda \bar{u}_{-1} v_1,$$

and the following Λ-quadratic form $q : V \to A/\Lambda$,

$$q(u) = \bar{u}_1 u_{-1} + \dots + \bar{u}_n u_{-n} \mod \Lambda.$$

In fact, both forms are engendered by a sesquilinear form f,

$$f(u, v) = \bar{u}_1 v_{-1} + \dots + \bar{u}_n v_{-n}.$$

Now, $h = f + \lambda \bar{f}$, where $\bar{f}(u, v) = \overline{f(v, u)}$, and $q(v) = f(u, u) \bmod \Lambda$.

By definition, the hyperbolic unitary group $\mathrm{GU}(2n, A, \Lambda)$ consists of all elements from $\mathrm{GL}(V) \cong \mathrm{GL}(2n, A)$ preserving the λ-hermitian form h and the Λ-quadratic form q. In other words, $g \in \mathrm{GL}(2n, A)$ belongs to $\mathrm{GU}(2n, A, \Lambda)$ if and only if

$$h(gu, gv) = h(u, v) \quad \text{and} \quad q(gu) = q(u), \quad \text{for all } u, v \in V.$$

When the form parameter is not maximal or minimal, these groups are not algebraic. However, their internal structure is very similar to that of the usual classical groups. They are also oftentimes called general quadratic groups, or classical-like groups.

3.3. *Elementary unitary transvections* $T_{ij}(\xi)$ correspond to the pairs $i, j \in \Omega$ such that $i \neq j$. They come in two stocks. Namely, if, moreover, $i \neq -j$, then for any $\xi \in A$ we set

$$T_{ij}(\xi) = e + \xi e_{ij} - \lambda^{(\varepsilon(j)-\varepsilon(i))/2} \bar{\xi} e_{-j,-i}.$$

These elements are also often called *elementary short root unipotents*. On the other side for $j = -i$ and $\alpha \in \lambda^{-(\varepsilon(i)+1)/2} \Lambda$ we set

$$T_{i,-i}(\alpha) = e + \alpha e_{i,-i}.$$

These elements are also often called *elementary long root elements*.

Note that $\overline{\Lambda} = \bar{\lambda}\Lambda$. In fact, for any element $\alpha \in \Lambda$ one has $\bar{\alpha} = -\bar{\lambda}\alpha$ and thus $\overline{\Lambda}$ coincides with the set of products $\bar{\lambda}\alpha$, $\alpha \in \Lambda$. This means that in the above definition $\alpha \in \overline{\Lambda}$ when $i \in \Omega^+$ and $\alpha \in \Lambda$ when $i \in \Omega^-$.

Subgroups $X_{ij} = \{T_{ij}(\xi) \mid \xi \in A\}$, where $i \neq \pm j$, are called *short root subgroups*. Clearly, $X_{ij} = X_{-j,-i}$. Similarly, subgroups $X_{i,-i} = \{T_{ij}(\alpha) \mid \alpha \in \lambda^{-(\varepsilon(i)+1)/2} \Lambda\}$ are called *long root subgroups*.

The *elementary unitary group* $\mathrm{EU}(2n, A, \Lambda)$ is generated by elementary unitary transvections $T_{ij}(\xi)$, $i \neq \pm j$, $\xi \in A$, and $T_{i,-i}(\alpha)$, $\alpha \in \Lambda$, see [9, §3].

3.4. Elementary unitary transvections $T_{ij}(\xi)$ satisfy the following *elementary relations*, also known as *Steinberg relations*. These relations will be used throughout this paper.

(R1) $T_{ij}(\xi) = T_{-j,-i}(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\xi})$,
(R2) $T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi+\zeta)$,
(R3) $[T_{ij}(\xi), T_{hk}(\zeta)] = 1$, where $h \neq j, -i$ and $k \neq i, -j$,
(R4) $[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta)$, where $i, h \neq \pm j$ and $i \neq \pm h$,
(R5) $[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(\xi\zeta - \lambda^{-\varepsilon(i)}\overline{\zeta}\overline{\xi})$, where $i \neq \pm j$,
(R6) $[T_{i,-i}(\xi), T_{-i,j}(\zeta)] = T_{ij}(\xi\zeta)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\zeta}\xi\zeta)$, where $i \neq \pm j$.

Relation (R1) coordinates two natural parametrisations of the same short root subgroup $X_{ij} = X_{-j,-i}$. Relation (R2) expresses additivity of the natural parametrisations. All other relations are various instances of the Chevalley commutator formula. Namely, (R3) corresponds to the case, where the sum of two roots is not a root, whereas (R4), and (R5) correspond to the case of two short roots, whose sum is a short root, and a long root, respectively. Finally, (R6) is the Chevalley commutator formula for the case of a long root and a short root, whose sum is a root. Observe that any two long roots are either opposite, or orthogonal, so that their sum is never a root.

3.5. Let G be a group. For any $x, y \in G$, ${}^{x}y = xyx^{-1}$ and $y^{x} = x^{-1}yx$ denote the left conjugate and the right conjugate of y by x, respectively. As usual, $[x, y] = xyx^{-1}y^{-1}$ denotes the left-normed commutator of x and y. Throughout the present paper we repeatedly use the following commutator identities:

(C1) $[x, yz] = [x, y] \cdot {}^{y}[x, z]$,
(C2) $[xy, z] = {}^{x}[y, z] \cdot [x, z]$,
(C3) ${}^{x}[[y, x^{-1}]^{-1}, z] = {}^{y}[x, [y^{-1}, z]] \cdot {}^{z}[y, [z^{-1}, x]]$,
(C4) $[x, {}^{y}z] = {}^{y}[{}^{y^{-1}}x, z]$,
(C5) $[{}^{y}x, z] = {}^{y}[x, {}^{y^{-1}}z]$.

Especially important is (C3), the celebrated *Hall–Witt identity*. Sometimes it is used in the following form, known as the *three subgroup lemma*.

Lemma 2. *Let $F, H, L \trianglelefteq G$ be three normal subgroups of G. Then*

$$\big[[F, H], L\big] \leqslant \big[[F, L], H\big] \cdot \big[F, [H, L]\big].$$

4. Relative subgroups

In this section we recall definitions and basic facts concerning relative subgroups.

4.1. One associates with a form ideal (I, Γ) the following four relative subgroups.

- The subgroup $\mathrm{FU}(2n, I, \Gamma)$ generated by elementary unitary transvections of level (I, Γ),

$$\mathrm{FU}(2n, I, \Gamma) = \big\langle T_{ij}(\xi) \bigm| \xi \in I \text{ if } i \neq \pm j \text{ and } \xi \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma \text{ if } i = -j \big\rangle.$$

- The *relative elementary subgroup* $\mathrm{EU}(2n, I, \Gamma)$ of level (I, Γ), defined as the normal closure of $\mathrm{FU}(2n, I, \Gamma)$ in $\mathrm{EU}(2n, A, \Lambda)$,

$$\mathrm{EU}(2n, I, \Gamma) = \mathrm{FU}(2n, I, \Gamma)^{\mathrm{EU}(2n, A, \Lambda)}.$$

- The *principal congruence subgroup* $\mathrm{GU}(2n, I, \Gamma)$ of level (I, Γ) in $\mathrm{GU}(2n, A, \Lambda)$ consists of those $g \in \mathrm{GU}(2n, A, \Lambda)$, which are congruent to e modulo I and preserve $f(u, u)$ modulo Γ,

$$f(gu, gu) \in f(u, u) + \Gamma, \quad u \in V.$$

- The full congruence subgroup $\mathrm{CU}(2n, I, \Gamma)$ of level (I, Γ), defined as

$$\mathrm{CU}(2n, I, \Gamma) = \{g \in \mathrm{GU}(2n, A, \Lambda) \mid [g, \mathrm{GU}(2n, A, \Lambda)] \subseteq \mathrm{GU}(2n, I, \Gamma)\}.$$

In some books, including [18], the group $\mathrm{CU}(2n, I, \Gamma)$ is defined differently. However, in many important situations these definitions yield the same group. Starting from Lemma 6, this is certainly the case for rings considered in the present paper.

4.2. Let us collect several basic facts, concerning relative groups, which will be used in the sequel. The first one of them asserts that the relative elementary groups are $\mathrm{EU}(2n, A, \Lambda)$-perfect.

Lemma 3. *Suppose either $n \geqslant 3$ or $n = 2$ and $I = \Lambda I + I\Lambda$. Then*

$$\mathrm{EU}(2n, I, \Gamma) = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, A, \Lambda)].$$

The next lemma gives generators of the relative elementary subgroup $\mathrm{EU}(2n, I, \Gamma)$ as a subgroup. With this end, consider matrices

$$Z_{ij}(\xi, \zeta) = {}^{T_{ji}(\zeta)}T_{ij}(\xi) = T_{ji}(\zeta)T_{ij}(\xi)T_{ji}(-\zeta),$$

where $\xi \in I$, $\zeta \in A$, if $i \neq \pm j$, and $\xi \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma$, $\zeta \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$, if $i = -j$. The following result is [9, Proposition 5.1].

Lemma 4. *Suppose $n \geqslant 3$. Then*

$$\mathrm{EU}(2n, I, \Gamma) = \langle Z_{ij}(\xi, \zeta) \mid \xi \in I,\ \zeta \in \Lambda \text{ if } i \neq \pm j \text{ and } \xi \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma,\ \zeta \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda, \text{ if } i = -j\rangle.$$

The following lemma was first established in [1], but remained unpublished. See [18] and [9, Lemma 4.4], for published proofs.

Lemma 5. *The groups $\mathrm{GU}(2n, I, \Gamma)$ and $\mathrm{CU}(2n, I, \Gamma)$ are normal in $\mathrm{GU}(2n, A, \Lambda)$.*

The following lemma is the main result of [8,9]. It is usually referred as the *absolute standard commutator formula*. Its role in the present paper is two-fold. On the one hand, here we develop a new and more powerful relative version of the conjugation calculus and the commutator calculus, which allow, among other things, to give a new proof of this result. In other words, the localisation proof of Theorem 1 proceeds directly in the relative case, and does not depend on the absolute case. On the other hand, in Sections 8, 9 we show that using level calculations one can deduce Theorem 1 directly from the absolute case.

Lemma 6. *Let (A, Λ) be a quasi-finite form ring and $n \geqslant 3$. Then for any form ideal (I, Γ) the corresponding elementary subgroup $\mathrm{EU}(2n, I, \Gamma)$ is normal in the hyperbolic unitary group $\mathrm{GU}(2n, A, \Lambda)$, in other words,*

$$\mathrm{EU}(2n, I, \Gamma) = [\mathrm{GU}(2n, A, \Lambda), \mathrm{EU}(2n, I, \Gamma)].$$

Moreover,

$$\mathrm{EU}(2n, I, \Gamma) = \bigl[\mathrm{EU}(2n, A, \Lambda), \mathrm{CU}(2n, I, \Gamma)\bigr].$$

4.3. The proofs in the present paper critically depend on the fact that the functors GU_{2n} and EU_{2n} commute with direct limits. This idea is used twice.

- Analysis of the quasi-finite case can be reduced to the case, where A is module finite over R_0, whereas R_0 itself is Noetherian. Indeed, if (A, Λ) is quasi-finite (see Section 2.3), it is a direct limit $\varinjlim ((A_j)_{R_j}, \Lambda_j)$ of an inductive system of form sub-algebras $((A_j)_{R_j}, \Lambda_j) \subseteq (A_R, \Lambda)$ such that each A_j is module finite over R_j, $R_0 \subseteq R_j$ and R_j is finitely generated as an R_0-module. It follows that A_j is finitely generated as an R_0-module, see [19, Cor. 3.8]. This reduction to module finite algebras will be used in Lemma 17 and Theorem 1.
- Analysis of any localisation can be reduced to the case of principal localisations. Indeed, let S be a multiplicative system in a commutative ring R. Then R_s, $s \in S$, is an inductive system with respect to the localisation maps $F_t : R_s \to R_{st}$. Thus, for any functor $\mathcal{F}$ commuting with direct limits one has $\mathcal{F}(S^{-1}R) = \varinjlim \mathcal{F}(R_s)$.

The following crucial lemma relies on both of these reductions. In fact, starting from the next section, we will be mostly working in the principal localisation A_t. However, eventually we shall have to return to the algebra A itself. In general, localisation homomorphism F_S is not injective, so we cannot pull elements of $\mathrm{GU}(2n, S^{-1}A, S^{-1}\Lambda)$ back to $\mathrm{GU}(2n, A, \Lambda)$. However, over a *Noetherian* ring, *principal* localisation homomorphims F_t are indeed injective on small t-adic neighbourhoods of identity!

Lemma 7. *Let R be a commutative Noetherian ring and let A be a module finite R-algebra. Then for any $t \in R$ there exists a positive integer l such that restriction*

$$F_t : \mathrm{GU}\bigl(2n, t^l A, t^l \Lambda\bigr) \to \mathrm{GU}(2n, A_t, \Lambda_t),$$

of the localisation map to the principal congruence subgroup of level $(t^l A, t^l \Lambda)$ is injective.

Proof. Follows from the injectivity of the localisation map $F_t : t^l A \to A_t$, see [3, Lemma 4.10] or [22, Lemma 5.1]. $\square$

5. Conjugation calculus

In the present section we develop a relative version of unitary conjugation calculus. Throughout this section, we assume that $n \geqslant 3$, that (A, Λ) is a form ring over a commutative ring R with involution, that R_0 is the subring of R, generated by $a\bar{a}$, where $a \in R$, as in Section 2.3, and, finally, that (I, Γ) and (J, Δ) are two form ideals of (A, Λ).

Clearly, for any $t \neq 0 \in R_0$ and any given positive integer l, the set $t^l A$ is in fact an ideal of the algebra A. Similarly, it is straightforward to verify that $t^l \Lambda = \{t^l \alpha \mid \alpha \in \Lambda\}$ is in fact relative form parameter for $t^l A$, and, thus, $(t^l A, t^l \Lambda)$ is a form ideal.

By the same token, any form ideal (I, Γ) gives rise to the form ideal $(t^l I, t^l \Gamma)$. In particular, we have the corresponding groups $\mathrm{FU}(2n, t^l A, t^l \Lambda)$ and $\mathrm{FU}(2n, I, \Gamma)$.

Starting from Lemma 8 up to Lemma 17, all calculations actually take place inside the elementary group $\mathrm{EU}(2n, A_t, \Lambda_t)$, for some $t \in R_0$. Thus, when we write something like $\mathrm{FU}^1(2n, t^l I, t^l \Gamma)$ or $T_{ij}(t^l \alpha)$ what we *really* mean is $F_t(\mathrm{FU}^1(2n, t^l I, t^l \Gamma))$ or $T_{ij}(F_t(t^l \alpha))$, respectively.

The overall intention of what we are doing in this section, and the next one, is to perfect the art of getting rid of denominators. We consider conjugates ${}^x y$ or commutators $[x, y]$, where x may be fractional in t, whereas y is at our disposal. We wish to show that for a given x and any y from a *very* small t-adic neighbourhood of 1 the elements ${}^{x}y$ and $[x, y]$ still fall in a reasonably small t-adic neighbourhood of 1. Actually, we aim at such neighbourhood, where F_t is injective, as in Lemma 7.

For the group $\mathrm{EU}(2n, A_t, \Lambda_t)$ itself, such calculations have been performed before in the Doktorarbeit of the first author [19,20], and have been later used by ourselves, Anthony Bak, Victor Petrov, and others [4,34–36,50].

What we want to do now, is to develop similar techniques inside the relative group $\mathrm{EU}(2t, I_t, \Gamma_t)$, where (I, Γ) is a form ideal of the form algebra (A, Λ). However, a direct imitation of the existing proof leads to awkward and unwieldy calculations.

Before, one always carried such calculations in the familiar neighbourhoods of 1, namely in $\mathrm{FU}(2n, t^l I, t^l \Gamma)$ or in $\mathrm{EU}(2n, t^l I, t^l \Gamma)$. However, as it turns out, the first one of them is a bit too small, whereas the second one is a bit too large. A major new technical point of the present paper, suggested by the method of our paper [24], is that calculations become much less cumbersome if one works inside the subgroup

$$\mathrm{FU}\big(2n, t^l I, t^l \Gamma\big) \leqslant \mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big) \leqslant \mathrm{EU}\big(2n, t^l I, t^l \Gamma\big),$$

instead.

By definition, it is the normal closure of $\mathrm{FU}(2n, t^l I, t^l \Gamma)$ in $\mathrm{FU}(2n, t^l A, t^l \Lambda)$,

$$\mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big) = {}^{\mathrm{FU}(2n, t^l A, t^l \Lambda)}\mathrm{FU}\big(2n, t^l I, t^l \Gamma\big) \trianglelefteqslant \mathrm{FU}\big(2n, t^l A, t^l \Lambda\big).$$

Normality of $\mathrm{FU}(2n, t^l A, t^l I, t^l \Gamma)$ in $\mathrm{FU}(2n, t^l A, t^l \Lambda)$ will be repeatedly used in the sequel. Notice, that $\mathrm{FU}(2n, t^l A, t^l A, t^l \Lambda) = \mathrm{FU}(2n, t^l A, t^l \Lambda)$.

Let us introduce a further piece of notation. For a form ideal (I, Γ) and an element $t \in R_0$, the set $\mathrm{FU}^1(2n, \frac{I}{t^m}, \frac{\Gamma}{t^m})$ consists of elementary unitary transvections $T_{ij}(a)$, such that $a \in \frac{I}{t^m}$ if $i \neq \pm j$ and $a \in \lambda^{(\varepsilon(i)+1)/2} \frac{\Gamma}{t^m}$ if $i = -j$. The set $\mathrm{FU}^1(2n, t^m I, t^m \Gamma)$ is defined similarly. By $\mathrm{FU}^K(2n, t^m I, t^m \Gamma)$, we mean a product of K (or fewer) elements of $\mathrm{FU}^1(2n, t^m I, t^m \Gamma)$.

The following result is based on an induction. As everyone knows, a journey of a thousand miles starts with the first step, which is usually also the hardest one. In this case it certainly is.

Lemma 8. *For any given l, m there exists a sufficiently large integer p such that*

$${}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m})}\,\mathrm{FU}^1\big(2n, t^{4p} I, t^{4p} \Gamma\big) \subseteq \mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big).$$

Proof. Suppose that

$$g = {}^{T_{ij}(a/t^m)}T_{hk}\big(t^{4p}\alpha\big) \in {}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m})}\,\mathrm{FU}^1\big(2n, t^{4p} I, t^{4p} \Gamma\big).$$

The proof is divided into four cases depending on whether the root elements $T_{ij}(a/t^m)$ and $T_{hk}(t^{4p}\alpha)$ are short or long.

Case I: Both $T_{hk}(t^{4p}\alpha)$ and $T_{ij}(a/t^m)$ are short root elements, in other words $h \neq \pm k$, $i \neq \pm j$, and, as above, $\alpha \in I$ and $a \in A$.

The proof breaks into four subcases:

(1) $i \neq k$ and $j \neq h$;
(2) $i = k$ and $j \neq h$;
(3) $i \neq k$ and $j = h$;
(4) $i = k$ and $j = h$.

We shall prove subcases (1) and (2) and leave it to the reader to reduce subcases (3)–(4) to subcase (1). In subcase (1), we have further four subcases.

(i) $i \neq -h$ and $j \neq -k$. Then $T_{hk}(t^{4p}\alpha)$ commutes with $T_{ij}(a/t^m)$ by identity (R3). Therefore, $\rho = T_{hk}(t^{4p}\alpha)$ and we are done.

(ii) $i = -h$ and $j \neq -k$. In this subcase, $g = {}^{T_{ij}(a/t^m)}T_{-ik}(t^{4p}\alpha)$.
If $j = k$, then using (R5) we get

$$\begin{aligned} g &= {}^{T_{ij}(a/t^m)}T_{-i,j}\big(t^{4p}\alpha\big) = T_{-i,j}\big(t^{4p}\alpha\big)\big[T_{-i,j}\big(-t^{4p}\alpha\big), T_{i,j}\big(a/t^m\big)\big] \\ &= T_{-i,j}\big(t^{4p}\alpha\big)T_{-j,j}\big(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{\alpha}at^{4p-m} + \lambda^{(\varepsilon(j)-\varepsilon(i))/2}\bar{a}\alpha t^{4p-m}\big) \\ &\in \mathrm{FU}\big(2n, t^{4p-m}A, t^{4p-m}I, t^{4p-m}\Gamma\big). \end{aligned}$$

If $j \neq k$, then using (R4) we get

$$\begin{aligned} g &= {}^{T_{ij}(a/t^m)}T_{-i,k}\big(t^{4p}\alpha\big) = T_{-i,k}\big(t^{4p}\alpha\big)\big[T_{-i,k}\big(-t^{4p}\alpha\big), T_{i,j}\big(a/t^m\big)\big] \\ &= T_{-i,j}\big(t^{4p}\alpha\big)T_{-k,j}\big(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{\alpha}at^{4p-m}\big) \in \mathrm{FU}\big(2n, t^{4p-m}A, t^{4p-m}I, t^{4p-m}\Gamma\big). \end{aligned}$$

(iii) $i \neq -h$ and $j = -k$. In this subcase,

$$g = {}^{T_{ij}(a/t^m)}T_{h,-j}\big(t^{4p}\alpha\big).$$

If $i = h$ then using (R5) we get

$$\begin{aligned} g &= {}^{T_{ij}(a/t^m)}T_{i,-j}\big(t^{4p}\alpha\big) = T_{i,-j}\big(t^{4p}\alpha\big)\big[T_{i,-j}\big(-t^{4p}\alpha\big), T_{i,j}\big(a/t^m\big)\big] \\ &= T_{i,-j}\big(t^{4p}\alpha\big)T_{i,-i}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\alpha}at^{4p-m} + \lambda^{(\varepsilon(-j)-\varepsilon(i))/2}\bar{a}\alpha t^{4p-m}\big) \\ &\in \mathrm{FU}\big(2n, t^{4p-m}A, t^{4p-m}I, t^{4p-m}\Gamma\big). \end{aligned}$$

If $i \neq h$ then using (R4) we get

$$\begin{aligned} g &= {}^{T_{ij}(a/t^m)}T_{h,-j}\big(t^{4p}\alpha\big) = T_{h,-j}\big(t^{4p}\alpha\big)\big[T_{h,-j}\big(-t^{4p}\alpha\big), T_{ij}\big(a/t^m\big)\big] \\ &= T_{h,-j}\big(t^{4p}\alpha\big)T_{h,-i}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\alpha}at^{4p-m}\big) \in \mathrm{FU}\big(2n, t^{4p-m}A, t^{4p-m}I, t^{4p-m}\Gamma\big). \end{aligned}$$

(iv) $i = -h$ and $j = -k$. In this subcase, $g = {}^{T_{ij}(a/t^m)}T_{-i,-j}(t^{4p}\alpha)$. By (R1),

$$g = {}^{T_{ij}(a/t^m)}T_{ji}\big(\lambda^{(\varepsilon(-i)-\varepsilon(-j))/2}t^{4p}\alpha\big).$$

To simplify notation, we denote $\lambda^{(\varepsilon(-i)-\varepsilon(-j))/2}\alpha$ by α.
Take an index $q \neq \pm i, \pm j$. Then,

$$\begin{aligned} g &= {}^{T_{ij}(a/t^m)}T_{ji}\big(t^{4p}\alpha\big) = {}^{T_{ij}(a/t^m)}\big[T_{jq}\big(t^{2p}\big), T_{qi}\big(t^{2p}\alpha\big)\big] \\ &= \big[{}^{T_{ij}(a/t^m)}T_{jq}\big(t^{2p}\big), {}^{T_{i,j}(a/t^m)}T_{qi}\big(t^{2p}\alpha\big)\big] \\ &= \big[T_{iq}\big(t^{2p-m}a\big)T_{jq}\big(t^{2p}\big), T_{qi}\big(t^{2p}\alpha\big)T_{qj}\big(-t^{2p-m}\alpha a\big)\big]. \end{aligned}$$

Denote the first and the second factors on the right-hand side by x and y respectively. Clearly,

$$y \in \mathrm{FU}\big(2n, t^{2p-m}I, t^{2p-m}\Gamma\big) \quad \text{and} \quad x \in \mathrm{FU}\big(2n, t^{2p-m}A, t^{2p-m}\Lambda\big),$$

and thus

$$[x, y] \in \mathrm{FU}\big(2n, t^{2p-m}A, t^{2p-m}I, t^{2p-m}\Gamma\big).$$

Now, taking any $p \geqslant (l+m)/2$ we see that $g \in \mathrm{FU}(2n, t^l A, t^l I, t^l \Gamma)$. This finishes the proof of subcase.

In subcase (2), we have $g = {}^{T_{ij}(a/t^m)}T_{hi}(t^{4p}\alpha) = {}^{T_{ij}(a/t^m)}T_{-i,-h}(\lambda^{(\varepsilon(i)-\varepsilon(h))/2}t^{4p}\alpha)$. It follows by subcase (1)(ii) that $g \in \mathrm{FU}(2n, t^l A, t^l I, t^l \Gamma)$ for some suitable p.

Subcases (3) and (4) can be reduced to subcase (1) in a similar fashion.

Case II: $T_{hk}(t^{4p}\alpha)$ is a short root element and $T_{ij}(a/t^m)$ is a long root element, i.e., $i = -j$, $h \neq \pm k$, $\alpha \in I$ and $a/t^m \in \frac{\Lambda}{t^m}$. This case is handled by dividing into three subcases:

(1) $h \neq -i$ and $k \neq i$. By (R3), $T_{hk}(t^{4p}\alpha)$ commutes with $T_{i,-i}(a/t^m)$. Therefore, $g = T_{hk}(t^{4p}\alpha)$ and we are done.
(2) $h = -i$ and $k \neq i$. By (R6) we have

$$\begin{aligned} g &= {}^{T_{i,-i}(a/t^m)}T_{-i,k}\big(t^{4p}\alpha\big) = T_{-i,k}\big(t^{4p}\alpha\big)\big[T_{-i,k}\big(-t^{4p}\alpha\big), T_{i,-i}\big(a/t^m\big)\big] \\ &= T_{-i,k}\big(t^{4p}\alpha\big)T_{-k,k}\big(\lambda^{(\varepsilon(k)-\varepsilon(-i))/2}t^{8p-m}\bar{\alpha}a\alpha\big)T_{i,k}\big(t^{4p-m}a\alpha\big) \\ &\in \mathrm{FU}\big(2n, t^{4p-m}A, t^{4p-m}I, t^{4p-m}\Gamma\big). \end{aligned}$$

(3) $h \neq -i$ and $k = i$. Our claim follows from an argument similar to that used in subcase (2).

Case III: $T_{hk}(t^{4p}\alpha)$ is a long root element and $T_{ij}(a/t^m)$ is a short root element. Namely, $i \neq \pm j$, $h = -k$, $\alpha \in \Gamma$ and $a \in A$. This case is treated by dividing into three subcases:

(1) $i \neq -h$ and $j \neq h$. By (R3), $T_{h,-h}(t^{4p}\alpha)$ commutes with $T_{ij}(a/t^m)$. Therefore, $g = T_{h,-h}(t^{4p}\alpha)$ and we are done.
(2) $i = -h$ and $j \neq h$. By (R6) we have

$$\begin{aligned} g &= {}^{T_{i,j}(a/t^m)}T_{-i,i}\big(t^{4p}\alpha\big) = T_{-i,i}\big(t^{4p}\alpha\big)\big[T_{-i,i}\big(-t^{4p}\alpha\big), T_{i,j}\big(a/t^m\big)\big] \\ &= T_{-i,i}\big(t^{4p}\alpha\big)T_{-i,j}\big(-t^{4p-m}\alpha a\big)T_{-j,j}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{4p-2m}\bar{a}\alpha a\big) \\ &\in \mathrm{FU}\big(2n, t^{4p-2m}A, t^{4p-2m}I, t^{4p-2m}\Gamma\big). \end{aligned}$$

(3) $i \neq -h$ and $j = h$. It follows from an argument similar to that used in subcase (2).

Case IV: Both $T_{hk}(t^{4p}\alpha)$ and $T_{ij}(a/t^m)$ are long root elements. Namely, $i = -j$, $h = -k$, $\alpha \in \Gamma$ and $a/t^m \in \frac{\Lambda}{t^m}$. This case is handled by dividing into further two subcases:

(1) $i \neq -h$. By (R3), $T_{h,-h}(t^{4p}\alpha)$ commutes with $T_{i,-i}(a/t^m)$. Therefore, $g = T_{h,-h}(t^{4p}\alpha)$ and we are done.

(2) $i=-h$. Pick a $q\neq\pm i$. Without loss of generality, we may assume that $\varepsilon(q)=\varepsilon(-i)$. Then by (R6) we have

$$\begin{aligned} g &= {}^{T_{i,-i}(a/t^m)}T_{-i,i}\big(t^{4p}\alpha\big) = {}^{T_{i,-i}(a/t^m)}\big(T_{q,i}\big(t^{3p-m}\alpha\big)\big[T_{-q,i}\big(t^p\big),T_{q,-q}\big(t^{2p}\alpha\big)\big]\big) \\ &= \big({}^{T_{i,-i}(a/t^m)}T_{q,i}\big(t^{3p-m}\alpha\big)\big)\big[{}^{T_{i,-i}(a/t^m)}T_{-q,i}\big(t^p\big),{}^{T_{i,-i}(a/t^m)}T_{q,-q}\big(t^{2p}\alpha\big)\big]. \end{aligned}$$

Now, ${}^{T_{i,-i}(a/t^m)}T_{q,-q}(t^{2p}\alpha)$ is trivial by (R3). By Case II, there is a sufficiently large p such that

$${}^{T_{i,-i}(a/t^m)}T_{q,i}\big(t^{3p-m}\alpha\big)\in \mathrm{FU}\big(2n,t^lA,t^lI,t^l\Gamma\big),$$

and

$${}^{T_{i,-i}(a/t^m)}T_{-q,i}\big(t^p\big)\in \mathrm{FU}\big(2n,t^lA,t^l\Lambda\big).$$

By definition, $\mathrm{FU}(2n,t^lA,t^lI,t^l\Gamma)$ is normalised by $\mathrm{FU}(2n,t^lA,t^l\Lambda)$. Hence, there is a sufficiently large p such that $g\in\mathrm{FU}(2n,t^lA,t^lI,t^l\Gamma)$. This finishes the proof of Case IV, hence the whole proof. $\square$

The next lemma immediately follows from Lemma 8 by induction.

Lemma 9. *For any given m, l there exists a sufficiently large p such that*

$${}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\mathrm{FU}\big(2n,t^pI,t^p\Gamma\big)\leqslant \mathrm{FU}\big(2n,t^lA,t^lI,t^l\Gamma\big).$$

For further applications we need a stronger fact with $\mathrm{FU}(2n,t^pI,t^p\Gamma)$ on the left-hand side replaced by its normal closure $\mathrm{FU}(2n,t^pA,t^pI,t^p\Gamma)$ in $\mathrm{FU}(2n,t^pA,t^p\Lambda)$.

Lemma 10. *For any given m, l there exists a sufficiently large p such that*

$${}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\mathrm{FU}\big(2n,t^pA,t^pI,t^p\Gamma\big)\leqslant \mathrm{FU}\big(2n,t^lA,t^lI,t^l\Gamma\big).$$

Proof. We have

$$\begin{aligned} {}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\mathrm{FU}\big(2n,t^pA,t^pI,t^p\Gamma\big) &= {}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\big({}^{\mathrm{FU}(2n,t^pA,t^p\Lambda)}\mathrm{FU}\big(2n,t^pI,t^p\Gamma\big)\big) \\ &\subseteq {}^{{}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\mathrm{FU}(2n,t^pA,t^p\Lambda)}\big({}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\mathrm{FU}\big(2n,t^pI,t^p\Gamma\big)\big). \end{aligned}$$

By Lemma 9, there exists a sufficiently large p such that the conjugate in the exponent is contained in $\mathrm{FU}(2n,t^lA,t^lA,t^l\Lambda)=\mathrm{FU}(2n,t^lA,t^l\Lambda)$, whereas the conjugate in the base is contained in $\mathrm{FU}(2n,t^lA,t^lI,t^l\Gamma)$. Since the group $\mathrm{FU}(2n,t^lA,t^lI,t^l\Gamma)$ is normalised by $\mathrm{FU}(2n,t^lA,t^l\Lambda)$, our claim follows. $\square$

The next lemma is a direct consequence of Lemma 10. Observe, that here we start working with *two* form ideals (I,Γ) and (J,Δ).

Lemma 11. *For any give m, l there exists a sufficiently large p such that*

$$\begin{aligned} &{}^{\mathrm{FU}^1(2n,\frac{A}{t^m},\frac{\Lambda}{t^m})}\big[\mathrm{FU}\big(2n,t^pA,t^pI,t^p\Gamma\big),\mathrm{FU}\big(2n,t^pA,t^pJ,t^p\Delta\big)\big] \\ &\quad\subseteq \big[\mathrm{FU}\big(2n,t^lA,t^lI,t^l\Gamma\big),\mathrm{FU}\big(2n,t^lA,t^lJ,t^l\Delta\big)\big]. \end{aligned}$$

However, in this lemma, denominators occur in the conjugating elements, not inside the commutators. To prove our main results, we will have to face denominators *inside* the commutator. This is done in the next section.

6. Commutator calculus

In the present section we develop a relative version of unitary commutator calculus. As above, we always assume that $n \geqslant 3$, that (A, Λ) is a form ring over a commutative ring R with involution, that R_0 is the subring of R, generated by $a\bar{a}$, where $a \in R$, and, finally, that (I, Γ) and (J, Δ) are two form ideals of (A, Λ). As before, all calculations take place inside the group $\mathrm{EU}(2n, A_t, \Lambda_t)$.

Lemma 12. *Suppose m, l, K are given. For any $t \in R$ there is an integer p, independent of K, such that*

$$\left[\mathrm{FU}^K\big(2n, t^{4p}I, t^{4p}\Gamma\big), \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right] \subseteq \big[\mathrm{FU}\big(2n, t^lA, t^lI, t^l\Gamma\big), \mathrm{FU}\big(2n, t^lA, t^lJ, t^l\Delta\big)\big].$$

Proof. An easy induction, using identity (C2), shows that

$$\left[\prod_{i=1}^{K} u_i, x\right] = \prod_{i=1}^{K} {}^{\prod_{j=1}^{K-i} u_j}[u_{K-i+1}, x],$$

where by convention $\prod_{j=1}^{0} u_j = 1$. This, with the fact that $\mathrm{FU}(2n, t^lA, t^lI, t^l\Gamma)$ and $\mathrm{FU}(2n, t^lA, t^lJ, t^l\Delta)$ are normalised by $\mathrm{FU}(2n, t^pA, t^p\Lambda)$, where $p \geqslant l$, show that it is enough to establish the lemma for $K = 1$, namely,

$$\left[\mathrm{FU}^1\big(2n, t^{4p}I, t^{4p}\Gamma\big), \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right] \subseteq \big[\mathrm{FU}\big(2n, t^lA, t^lI, t^l\Gamma\big), \mathrm{FU}\big(2n, t^lA, t^lJ, t^l\Delta\big)\big].$$

Let

$$T_{ij}\big(t^{4p}\alpha\big) \in \mathrm{FU}^1\big(2n, t^{4p}I, t^{4p}\Gamma\big), \qquad T_{hk}\left(\frac{\beta}{t^m}\right) \in \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right),$$

and set

$$g = \left[T_{ij}\big(t^{4p}\alpha\big), T_{hk}\left(\frac{\beta}{t^m}\right)\right].$$

As in Lemma 8, we divide the proof into four cases according to whether root elements $T_{ij}(t^{4p}\alpha)$ and $T_{hk}(\frac{\beta}{t^m})$ are long or short.

Case I: Both $T_{ij}(t^{4p}\alpha)$ and $T_{hk}(\frac{\beta}{t^m})$ are short root elements, i.e., $i \neq \pm j$, $h \neq \pm k$, $\alpha \in I$ and $\beta \in J$. The proof breaks further into following four subcases:

(1) $i \neq k$ and $j \neq h$;
(2) $i = k$ and $j \neq h$;
(3) $i \neq k$ and $j = h$;
(4) $i = k$ and $j = h$.

We shall prove subcases (1) and (2) and leave it to the reader to reduce subcases (3) and (4) to subcase (1). In subcase (1), we have further four subcases:

(i) $i \neq -h$ and $j \neq -k$. By identity (R3), $T_{ij}(t^{4p}\alpha)$ commutes with $T_{hk}(\frac{\beta}{t^m})$. Therefore, $g = 1$ and we are done.

(ii) $i = -h$ and $j \neq -k$. In this subcase,

$$g = \left[T_{ij}\big(t^{4p}\alpha\big), T_{-i,k}\left(\frac{\beta}{t^m}\right)\right].$$

If $j = k$, then by (R5) one has

$$\begin{aligned} g &= \left[T_{ij}\big(t^{4p}\alpha\big), T_{-i,j}\left(\frac{\beta}{t^m}\right)\right] \\ &= T_{-j,j}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\alpha}\beta t^{4p-m} + \lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{\beta}\alpha t^{4p-m}\big) \\ &= \big[T_{ij}\big(t^{2p}\alpha\big), T_{-i,j}\big(t^{2p-m}\beta\big)\big] \\ &\in \big[\mathrm{FU}\big(2n, t^{2p}A, t^{2p}I, t^{2p}\varGamma\big), \mathrm{FU}\big(2n, t^{2p-m}A, t^{2p-m}J, t^{2p-m}\Delta\big)\big]. \end{aligned}$$

If $j \neq k$, then by (R4) one has

$$\begin{aligned} g &= \left[T_{ij}\big(t^{4p}\alpha\big), T_{-i,k}\left(\frac{\beta}{t^m}\right)\right] = T_{-j,k}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\alpha}\beta t^{4p-m}\big) \\ &= \big[T_{ij}\big(t^{2p}\alpha\big), T_{-i,k}\big(t^{2p-m}\beta\big)\big] \\ &\in \big[\mathrm{FU}\big(2n, t^{2p}A, t^{2p}I, t^{2p}\varGamma\big), \mathrm{FU}\big(2n, t^{2p-m}A, t^{2p-m}J, t^{2p-m}\Delta\big)\big]. \end{aligned}$$

(iii) $i \neq -h$ and $j = -k$.
It follows from an argument similar to that used in subcase (ii).

(iv) $i = -h$ and $j = -k$.
In this subcase, $g = [T_{ij}(t^{4p}\alpha), T_{-i,-j}(\frac{\beta}{t^m})]$. By (R1) one has

$$g = \left[T_{ij}\big(t^{4p}\alpha\big), T_{j,i}\left(\lambda^{(\varepsilon(-i)-\varepsilon(-j))/2}\frac{\beta}{t^m}\right)\right].$$

To simplify notation, we denote $\lambda^{(\varepsilon(-i)-\varepsilon(-j))/2}\beta$ by β. Let $q \neq \pm i, \pm j$. Then by (C3) we have

$$\begin{aligned} g &= \left[T_{ij}\big(t^{4p}\alpha\big), T_{ji}\left(\frac{\beta}{t^m}\right)\right] = \left[\big[T_{i,q}\big(t^{2p}\alpha\big), T_{q,j}\big(t^{2p}\big)\big], T_{ji}\left(\frac{\beta}{t^m}\right)\right] \\ &= {}^{T_{i,q}(t^{2p}\alpha)T_{i,q}(-t^{2p}\alpha)}\left[\big[T_{i,q}\big(t^{2p}\alpha\big), T_{q,j}\big(t^{2p}\big)\big], T_{ji}\left(\frac{\beta}{t^m}\right)\right]. \end{aligned}$$

Applying Hall–Witt identity, we get

$$\begin{aligned} g = {}^{T_{iq}(t^{2p}\alpha)}\Bigg(& {}^{T_{qj}(t^{2p})}\left[T_{iq}\big(-t^{2p}\alpha\big), \left[T_{qj}\big(-t^{2p}\big), T_{ji}\left(\frac{\beta}{t^m}\right)\right]\right] \\ & \times {}^{T_{ji}(\frac{\beta}{t^m})}\left[T_{qj}\big(t^{2p}\big), \left[T_{ji}\left(-\frac{\beta}{t^m}\right), T_{iq}\big(-t^{2p}\alpha\big)\right]\right]\Bigg). \end{aligned}$$

By (R4) this expression can be further rewritten as

$$g = {}^{T_{iq}(t^{2p}\alpha)}\left({}^{T_{qj}(t^{2p})}\left[T_{iq}\left(-t^{2p}\alpha\right), T_{qi}\left(-t^{2p-m}\beta\right)\right] \cdot {}^{T_{ji}(\frac{\beta}{t^m})}\left[T_{qj}\left(t^{2p}\right), T_{jq}\left(t^{2p-m}\alpha\beta\right)\right]\right).$$

Clearly, for all p such that $2p - m > l$ the first factor in the base belongs to

$$\left[\mathrm{FU}\left(2n, t^l A, t^l I, t^l \Gamma\right), \mathrm{FU}\left(2n, t^l A, t^l J, t^l \Delta\right)\right].$$

On the other hand, the second factor equals

$$\begin{aligned} y &= {}^{T_{ji}(\frac{\beta}{t^m})}\left[T_{qj}\left(t^{2p}\right), T_{jq}\left(t^{2p-m}\alpha\beta\right)\right] \\ &= {}^{T_{ji}(\frac{\beta}{t^m})}\left[T_{qj}\left(t^{2p}\right), \left[T_{ji}\left(t^{\lfloor\frac{2p-m}{2}\rfloor}\beta\right), T_{iq}\left(t^{2p-m-\lfloor\frac{2p-m}{2}\rfloor}\alpha\right)\right]\right]. \end{aligned}$$

Set

$$p' = \max\left(\left\lfloor \frac{2p-m}{2} \right\rfloor, 2p - m - \left\lfloor \frac{2p-m}{2} \right\rfloor\right).$$

Normality of $\mathrm{FU}(2n, t^l A, t^l I, t^l \Gamma)$ implies that

$$\begin{aligned} &\left[T_{qj}\left(t^{2p}\right), \left[T_{ji}\left(t^{\lfloor\frac{2p-m}{2}\rfloor}\beta\right), T_{iq}\left(t^{2p-m-\lfloor\frac{2p-m}{2}\rfloor}\alpha\right)\right]\right] \\ &\quad \in \left[\mathrm{FU}\left(2n, t^{p'} A, t^{p'} I, t^{p'} \Gamma\right), \mathrm{FU}\left(2n, t^{p'} A, t^{p'} J, t^{p'} \Delta\right)\right]. \end{aligned} \tag{2}$$

Hence

$$y \in {}^{T_{ji}(\frac{\beta}{t^m})}\left[\mathrm{FU}\left(2n, t^{p'} A, t^{p'} I, t^{p'} \Gamma\right), \mathrm{FU}\left(2n, t^{p'} A, t^{p'} J, t^{p'} \Delta\right)\right].$$

Therefore, by Lemma 11, for any given l, there is a sufficiently large p' such that,

$$y \in \left[\mathrm{FU}\left(2n, t^l A, t^l I, t^l \Gamma\right), \mathrm{FU}\left(2n, t^l A, t^l J, t^l \Delta\right)\right].$$

Summarising the above inclusions for the first and the second factors, we see that for a sufficiently large p, one has

$$g \in \left[\mathrm{FU}\left(2n, t^l A, t^l I, t^l \Gamma\right), \mathrm{FU}\left(2n, t^l A, t^l J, t^l \Delta\right)\right].$$

This finishes the proof of subcase (1).

In subcase (2), we have

$$g = \left[T_{ij}\left(t^{4p}\alpha\right), T_{hi}\left(\frac{\beta}{t^m}\right)\right] = \left[T_{ij}\left(t^{4p}\alpha\right), T_{-i,-h}\left(\lambda^{(\varepsilon(i)-\varepsilon(h))/2}\frac{\beta}{t^m}\right)\right].$$

By subcase (1)(ii) it follows that

$$g \in \left[\mathrm{FU}\left(2n, t^l A, t^l I, t^l \Gamma\right), \mathrm{FU}\left(2n, t^l A, t^l J, t^l \Delta\right)\right]$$

for a suitable p.

Case II: $T_{ij}(t^{4p}\alpha)$ is a short root element and $T_{hk}(\frac{\beta}{t^m})$ is a long root element, i.e., $i \neq \pm j$, $h = -k$, $\alpha \in I$ and $\beta \in \lambda^{-(\varepsilon(h)+1)/2}\Delta$. This case is handled by dividing into three subcases:

(1) $i \neq -h$ and $j \neq h$. By (R3), T_{ij} commutes with T_{hk}. Therefore, $g = 1$ and we are done.
(2) $i = -h$ and $j \neq h$. By (R6) we have

$$g = \left[T_{ij}\big(t^{4p}\alpha\big), T_{-i,i}\left(\frac{\beta}{t^m}\right)\right] = \big(T_{-i,j}\big(\beta\alpha t^{4p-m}\big)T_{-j,j}\big(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{\alpha}\beta\alpha t^{8p-m}\big)\big)^{-1}.$$

Further, set

$$M = \left\lfloor \frac{8p-m}{3} \right\rfloor, \qquad M' = \left(8p - m - 2\left\lfloor \frac{8p-m}{3} \right\rfloor\right).$$

Then by (R6) one has

$$\begin{aligned} g^{-1} &= T_{-j,i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{4p-m}\alpha\beta\big)T_{-j,j}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{8p-m}\alpha\beta\overline{\alpha}\big) \\ &= T_{-j,i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{4p-m}\alpha\beta\big)T_{-j,j}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{M}\alpha t^{M'}\beta\overline{t^{M}\alpha}\big) \\ &= T_{-j,i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{4p-m}\alpha\beta\big)T_{-j,i}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{M'+M}\alpha\beta\big) \\ &\quad \times \big[T_{-j,-i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{M}\alpha\big), T_{-i,i}\big(t^{M'}\beta\big)\big]. \end{aligned}$$

Picking a $q \neq \pm i, \pm j$, we see that the first factor of the above expression equals

$$\begin{aligned} &T_{-j,i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{4p-m}\alpha\beta\big)T_{-j,i}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{M'+M}\alpha\beta\big) \\ &\quad = \big[T_{-j,q}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{\lfloor 4p-m\rfloor/2}\alpha\big), T_{q,i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{4p-m-\lfloor 4p-m\rfloor/2}\beta\big)\big] \\ &\qquad \times \big[T_{-j,q}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{M'}\alpha\big), T_{q,i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}t^{M}\beta\big)\big]. \end{aligned}$$

Therefore, for any

$$p \geqslant \max\left(\frac{m+l}{4} + 1, \frac{3l+m}{8} + 1\right),$$

both factors of g^{-1}, and thus also g^{-1} and g themselves, belong to

$$\big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

(3) $i \neq -h$ and $i = k$. It follows from an argument similar to that used in subcase (2).

Case III: $T_{ij}(t^{4p}\alpha)$ is a long root element and $T_{hk}(\frac{\beta}{t^m})$ is a short root element. Namely, $i = -j$, $h \neq \pm k$, $\alpha \in \Gamma$ and $a \in A$. This case is treated by dividing into three subcases:

(1) $i \neq -h$ and $i \neq k$. By (R3), $T_{i,-i}$ commutes with T_{hk}. Therefore, $g = 1$ and we are done.
(2) $i = -h$ and $i \neq k$. By (R6) we have

$$g = \left[T_{i,-i}\big(t^{4p}\alpha\big), T_{-i,k}\left(\frac{\beta}{t^m}\right)\right] = T_{i,k}\big(\alpha\beta t^{4p-m}\big)T_{-k,j}\big(-\lambda^{(\varepsilon(k)-\varepsilon(-i))/2}\overline{\beta}\alpha\beta t^{4p-2m}\big)$$
$$= T_{i,k}\big(\alpha\beta t^{4p-m}\big)T_{ik}\big(-t^{3p-m}\alpha\beta\big)\cdot\big[T_{i,-i}\big(t^{2p}\alpha\big), T_{-i,k}\big(t^{p-m}\beta\big)\big].$$

When $p \geqslant m + l$, both factors of the above expression belong to

$$\big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

(3) $i \neq -h$ and $j = h$. It follows from an argument similar to that used in subcase (2).

Case IV: Both $T_{ij}(t^{4p}\alpha)$ and $T_{hk}(\frac{\beta}{t^m})$ are long root elements. Namely, $i = -j$, $h = -k$, $\alpha \in \Gamma$ and $\beta \in \Delta$. This case is handled by further subdividing it into two subcases.

(1) $i \neq -h$. By (R3), two non-opposite long root elements commute, and thus $g = 1$.
(2) $i = -h$. Pick a $q \neq \pm i$. Without loss of generality, we may assume that $\varepsilon(q) = \varepsilon(-i)$. Then by (R6) we have

$$g = \left[T_{i,-i}\big(t^{4p}\alpha\big), T_{-i,i}\left(\frac{\beta}{t^m}\right)\right] = \left[T_{i,-i}\big(t^p t^{2p}\alpha\overline{t^p}\big), T_{-i,i}\left(\frac{\beta}{t^m}\right)\right]$$
$$= \left[T_{i,-q}\big(-t^{3p}\alpha\big)\big[T_{iq}\big(t^p\big), T_{q,-q}\big(t^{2p}\alpha\big)\big], T_{-i,i}\left(\frac{\beta}{t^m}\right)\right].$$

By (C2) one has

$$g = {}^{T_{i,-q}(-t^{3p}\alpha)}\left[\big[T_{iq}\big(t^p\big), T_{q,-q}\big(t^{2p}\alpha\big)\big], T_{-i,i}\left(\frac{\beta}{t^m}\right)\right]\cdot\left[T_{i,-q}\big(-t^{3p}\alpha\big), T_{-i,i}\left(\frac{\beta}{t^m}\right)\right].$$

We claim that for a sufficiently large p both factors on the right-hand side belong to

$$\big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

For the second factor this follows from Case II. Thus, it remains to show that

$$\left[\big[T_{i,q}\big(t^p\big), T_{q,-q}\big(t^{2p}\alpha\big)\big], T_{-i,i}\left(\frac{\beta}{t^m}\right)\right] \in \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

But

$$\left[\big[T_{i,q}\big(t^p\big), T_{q,-q}\big(t^{2p}\alpha\big)\big], T_{-i,i}\left(\frac{\beta}{t^m}\right)\right]$$
$$= {}^{T_{q,-q}(-t^{2p}\alpha)T_{q,-q}(t^{2p}\alpha)}\left[\big[T_{i,q}\big(t^p\big), T_{q,-q}\big(t^{2p}\alpha\big)\big], T_{-i,i}\left(\frac{\beta}{t^m}\right)\right].$$

By the Hall–Witt identity one has

$$ {}^{T_{q,-q}(-t^{2p}\alpha)}\left({}^{T_{i,q}(-t^p)}\left[T_{q,-q}\left(-t^{2p}\alpha\right),\left[T_{-i,i}\left(\frac{\beta}{t^m}\right),T_{iq}\left(-t^p\right)\right]\right]\right. $$
$$ \left.\times\,{}^{T_{-i,i}(-\frac{\beta}{t^m})}\left[T_{i,q}\left(t^p\right),\left[T_{q,-q}\left(-t^{2p}\alpha\right),T_{-i,i}\left(-\frac{\beta}{t^m}\right)\right]\right]\right). $$

By (R3) this can be further rewritten as

$$ {}^{T_{q,-q}(-t^{2p}\alpha)}\left({}^{T_{i,q}(t^p)}\left[T_{q,-q}\left(-t^{2p}\alpha\right),\left[T_{-i,i}\left(\frac{\beta}{t^m}\right),T_{i,q}\left(-t^p\right)\right]\right]\right). $$

In turn, by (R6) this is equal to

$$ {}^{T_{q,-q}(-t^{2p}\alpha)}\left({}^{T_{i,q}(t^p)}\left[T_{q,-q}\left(-t^{2p}\alpha\right),T_{-i,j}\left(-t^{p-m}\beta\right)T_{-q,q}\left(-\lambda t^{2p-m}\beta\right)\right]\right). $$

When $p > l+m$, the commutator in the base belongs to

$$ \left[\mathrm{FU}\left(2n,t^lA,t^lI,t^l\Gamma\right),\mathrm{FU}\left(2n,t^lA,t^lJ,t^l\Delta\right)\right]. $$

Both $\mathrm{FU}(2n,t^lA,t^lI,t^l\Gamma)$ and $\mathrm{FU}(2n,t^lA,t^lJ,t^l\Delta)$ are normalised by $\mathrm{FU}(2n,t^lA,t^l\Lambda)$. As $T_{q,-q}(-t^{2p}\alpha)$ and $T_{i,q}(t^p)$ belong to $\mathrm{FU}(2n,t^{l'}A,t^{l'}\Lambda)$, it follows that the first factor also belongs to

$$ \left[\mathrm{FU}\left(2n,t^lA,t^lI,t^l\Gamma\right),\mathrm{FU}\left(2n,t^lA,t^lJ,t^l\Delta\right)\right], $$

as claimed.

This finishes the proof of Case IV, and thus the whole proof. $\square$

Lemma 13. *Suppose m, l, K are given. For any $t \in R$ there is an integer p, independent of K, such that*

$$ \left[{}^{\mathrm{FU}^K(2n,t^pA,t^p\Lambda)}\mathrm{FU}^1\left(2n,t^pI,t^p\Gamma\right),\mathrm{FU}^1\left(2n,\frac{J}{t^m},\frac{\Delta}{t^m}\right)\right] $$
$$ \subseteq\left[\mathrm{FU}\left(2n,t^lA,t^lI,t^l\Gamma\right),\mathrm{FU}\left(2n,t^lA,t^lJ,t^l\Delta\right)\right]. $$

Proof. Let a, b and c be arbitrary elements in $\mathrm{FU}^K(2n,t^pA,t^p\Lambda)$, $\mathrm{FU}^1(2n,t^pI,t^p\Gamma)$ and $\mathrm{FU}^1(2n,\frac{J}{t^m},\frac{\Delta}{t^m})$, respectively. Then by (C2) one has

$$ \left[{}^ab,c\right]=\left[b\left[b^{-1},a\right],c\right]=\left({}^b\left[\left[b^{-1},a\right],c\right]\right)[b,c]. \tag{3} $$

By Lemma 12, we may find a sufficiently large p, such that for the second factor of Eq. (3),

$$ [b,c]\in\left[\mathrm{FU}\left(2n,t^lA,t^lI,t^l\Gamma\right),\mathrm{FU}\left(2n,t^lA,t^lJ,t^l\Delta\right)\right]. $$

Applying Hall–Witt identity to the first of the above factors, we get

$$ {}^b\left[\left[b^{-1},a\right],c\right]={}^{ba^{-1}}\left({}^a\left[\left[b^{-1},a\right],c\right]\right)={}^{ba^{-1}}\left({}^b\left[a^{-1},[c,b]\right]\times{}^{c^{-1}}\left[b^{-1},\left[a^{-1},c^{-1}\right]\right]\right). $$

By Lemma 12, there is a sufficiently large p, such that

$$[c,b] \in \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \varGamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

Furthermore, $a \in \mathrm{FU}^K(2n, t^p A, t^p \varLambda)$ implies that

$${}^b\big[a^{-1}, [c,b]\big] \in \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \varGamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

Again, Lemma 12 implies that for any given l', there is a sufficiently large p such that

$$\big[a^{-1}, c^{-1}\big] \in \big[\mathrm{FU}\big(2n, t^{l'} A, t^{l'} \varLambda\big), \mathrm{FU}\big(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta\big)\big] \subseteq \mathrm{FU}\big(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta\big).$$

It follows immediately that

$$\big[b^{-1}, \big[a^{-1}, c^{-1}\big]\big] \in \big[\mathrm{FU}\big(2n, t^{l'} A, t^{l'} I, t^{l'} \varGamma\big), \mathrm{FU}\big(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta\big)\big].$$

Therefore,

$${}^{c^{-1}}\big[b^{-1}, \big[a^{-1}, c^{-1}\big]\big] \subseteq {}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\varLambda}{t^m})}\big[\mathrm{FU}\big(2n, t^{l'} A, t^{l'} I, t^{l'} \varGamma\big), \mathrm{FU}\big(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta\big)\big].$$

Then by Lemma 11, we may find a sufficiently large l', such that

$$\begin{aligned} &{}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\varLambda}{t^m})}\big[\mathrm{FU}\big(2n, t^{l'} A, t^{l'} I, t^{l'} \varGamma\big), \mathrm{FU}\big(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta\big)\big] \\ &\quad \subseteq \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \varGamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big]. \end{aligned}$$

Hence we may find a sufficiently large p, such that

$${}^{c^{-1}}\big[b^{-1}, \big[a^{-1}, c^{-1}\big]\big] \in \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \varGamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

This finishes the proof. □

Lemma 14. *Suppose that m, l are given. For any $t \in R$ there is an integer p such that*

$$\left[\mathrm{FU}\big(2n, t^p A, t^p I, t^p \varGamma\big), \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right] \subseteq \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \varGamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

Proof. Since $\mathrm{FU}(2n, t^p A, t^p I, t^p \varGamma)$ is a group generated by elements of the form

$${}^{\mathrm{FU}^K(2n, t^p A, t^p \varLambda)}\,\mathrm{FU}^1\big(2n, t^p I, t^p \varGamma\big)$$

for all natural numbers K and since in Lemma 13, p is independent of K, the lemma follows from Lemma 13 and identity (C2) by induction. □

Lemma 15. *Suppose m, l are given. For any $t \in R$ there is an integer p such that*

$$\left[\mathrm{FU}\big(2n, t^p I, t^p \varGamma\big), {}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\varLambda}{t^m})}\,\mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right] \subseteq \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \varGamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big].$$

Proof. Let

$$a \in \mathrm{FU}\big(2n, t^p I, t^p \Gamma\big), \qquad b \in \mathrm{FU}^1\left(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m}\right), \qquad c \in \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right).$$

We consider the commutator $[a, {}^b c] = {}^b[{}^{b^{-1}}a, c]$. Lemma 9 implies that for any p' there is a sufficiently large p such that

$${}^{b^{-1}}a \in \mathrm{FU}\big(2n, t^{p'}A, t^{p'}I, t^{p'}\Gamma\big).$$

Therefore,

$$\big[{}^{b^{-1}}a, c\big] \in \left[\mathrm{FU}\big(2n, t^{p'}A, t^{p'}I, t^{p'}\Gamma\big), \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right].$$

By Lemma 14, for any p'' there is a sufficiently large p' such that

$$\begin{aligned}\big[{}^{b^{-1}}a, c\big] &\in \left[\mathrm{FU}\big(2n, t^{p'}A, t^{p'}I, t^{p'}\Gamma\big), \mathrm{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right] \\ &\subseteq \big[\mathrm{FU}\big(2n, t^{p''}A, t^{p''}I, t^{p''}\Gamma\big), \mathrm{FU}\big(2n, t^{p''}A, t^{p''}J, t^{p''}\Delta\big)\big].\end{aligned}$$

Hence ${}^b[{}^{b^{-1}}a, c]$ belongs to

$${}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m})}\big[\mathrm{FU}\big(2n, t^{p''}A, t^{p''}I, t^{p''}\Gamma\big), \mathrm{FU}\big(2n, t^{p''}A, t^{p''}J, t^{p''}\Delta\big)\big].$$

Finally, Lemma 11 implies that there is a sufficient large p'' such that

$$\begin{aligned}&{}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m})}\big[\mathrm{FU}\big(2n, t^{p''}A, t^{p''}I, t^{p''}\Gamma\big), \mathrm{FU}\big(2n, t^{p''}A, t^{p''}J, t^{p''}\Delta\big)\big] \\ &\quad\subseteq \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l\Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l\Delta\big)\big].\end{aligned}$$

This finishes the proof. □

In the following lemma we use the set $\mathrm{EU}^K(2n, \frac{J}{t^m}, \frac{\Delta}{t^m})$ defined as the set of products of K or fewer elements of ${}^{\mathrm{FU}^1(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m})}\,\mathrm{FU}^1(2n, \frac{J}{t^m}, \frac{\Delta}{t^m})$.

Lemma 16. *Suppose m, l, K are given. For any $t \in R$ there is an integer p such that*

$$\left[\mathrm{FU}^1\big(2n, t^p I, t^p\Gamma\big), \mathrm{EU}^K\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)\right] \subseteq \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l\Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l\Delta\big)\big].$$

Proof. The lemma follows from Lemmas 15 and 11 and identity formulae (C1), (C2) by an easy induction. □

7. Mixed commutator formula: localisation proof

In this section we continue to assume that $n \geqslant 3$, R is a commutative ring, (A, Λ) is a form ring such that A is a module-finite R-algebra, and, finally, (I, Γ) and (J, Δ) are two form ideals of (A, Λ).

So far all calculations were taking place in the elementary group $\mathrm{EU}(2n, A_t, \Lambda_t)$. Now we start to pull them back to the group $\mathrm{GU}(2n, A, \Lambda)$. The key ingredient is Lemma 7, which asserts that for a suitable positive integer l, the restriction

$$F_t : \mathrm{GU}\big(2n, t^l A, t^l \Lambda\big) \to \mathrm{GU}(2n, A_t, \Lambda_t),$$

of the localisation homomorphism F_t to the congruence subgroup $\mathrm{GU}(2n, t^l A, t^l \Lambda)$ is injective.

Recall, that the functors EU_{2n} and GU_{2n} commute with direct limits. By Section 4.3, proofs of the following results are reduced to the case, where A is finite over R_0 and R_0 itself is Noetherian.

Lemma 17. *Let $\mathfrak{m} \in \mathrm{Max}(R_0)$ be a maximal ideal of R_0. For any $g \in \mathrm{GU}(2n, J, \Delta)$, there exists a $t \in R_0 \backslash \mathfrak{m}$ and an integer p, such that*

$$[e, g] \in \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big],$$

where $e \in \mathrm{FU}^1(2n, t^p I, t^p \Gamma)$. (Here p depends on the choice of e.)

Proof. For any maximal ideal $\mathfrak{m} \in \mathrm{Max}(R_0)$, the form ring $(A_\mathfrak{m}, \Lambda_\mathfrak{m})$ contains $(J_\mathfrak{m}, \Delta_\mathfrak{m})$ as a form ideal. Consider the localisation homomorphism $F_\mathfrak{m} : A \to A_\mathfrak{m}$ which induces homomorphisms on the level of unitary groups,

$$F_\mathfrak{m} : \mathrm{GU}(2n, A, \Lambda) \to \mathrm{GU}(2n, A_\mathfrak{m}, \Lambda_\mathfrak{m}),$$

and

$$F_\mathfrak{m} : \mathrm{GU}(2n, J, \Delta) \to \mathrm{GU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m}).$$

Therefore, for $g \in \mathrm{GU}(2n, J, \Delta)$, $F_\mathfrak{m}(g) \in \mathrm{GU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m})$. Since $A_\mathfrak{m}$ is module finite over the local ring $R_\mathfrak{m}$, $A_\mathfrak{m}$ is semi-local [10, III(2.5), (2.11)], therefore its stable rank is 1. It follows by (see [18, 9.1.4]) that

$$\mathrm{GU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m}) = \mathrm{EU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m})\,\mathrm{GU}(2, J_\mathfrak{m}, \Delta_\mathfrak{m}).$$

Thus, $F_\mathfrak{m}(g)$ can be decomposed as $F_\mathfrak{m}(g) = \varepsilon h$, where $\varepsilon \in \mathrm{EU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m})$ and $h \in \mathrm{GU}(2, J_\mathfrak{m}, \Delta_\mathfrak{m})$ is a 2×2 matrix embedded in $\mathrm{GU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m})$ and this embedding can be arranged modulo $\mathrm{EU}(2n, J_\mathfrak{m}, \Delta_\mathfrak{m})$.

Now, by (4.3), we may reduce the problem to the case A_t with $t \in R_0 \backslash \mathfrak{m}$. Namely, $F_t(g)$ is a product of ε and h, where $\varepsilon \in \mathrm{EU}(2n, J_t, \Delta_t)$, and $h \in \mathrm{GL}(2, J_t, \Delta_t)$.

Therefore ε is a product of the elementary matrices, thus one has (see [9, Prop. 5.1])

$$\varepsilon \in \mathrm{EU}^K\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right).$$

Let $e \in \mathrm{FU}^1(2n, t^p I, t^p \Gamma)$. We choose h such that it commutes with $F_t(e)$. By Lemma 16, for any given l, there is a sufficiently large p such that

$$\big[F_t(e), F_t(g)\big] = \big[F_t(e), \varepsilon\big] \in \big[\mathrm{FU}\big(2n, t^l A, t^l I, t^l \Gamma\big), \mathrm{FU}\big(2n, t^l A, t^l J, t^l \Delta\big)\big]. \tag{4}$$

Since $e \in \mathrm{EU}(2n, t^p I, t^p \Gamma) \leqslant \mathrm{GU}(2n, t^l A, t^l \Lambda)$ and $\mathrm{GU}(2n, t^l A, t^l \Lambda)$ is normal in $\mathrm{GU}(2n, A, \Lambda)$, it follows $[e, g] \in \mathrm{GU}(2n, t^l A, t^l \Lambda)$. On the other hand, using (4), one can find

$$x \in \bigl[\mathrm{FU}\bigl(2n, t^l A, t^l I, t^l \Gamma\bigr), \mathrm{FU}\bigl(2n, t^l A, t^l J, t^l \Delta\bigr)\bigr]$$

in $\mathrm{EU}(2n, A, \Lambda)$ such that $F_t(x) = [F_t(e), F_t(g)]$. Since for suitable l, the restriction of F_t to $\mathrm{GL}_n(t^l A, t^l \Lambda)$ is injective by Lemma 7, it follows $[e, g] = x$ and thus

$$[e, g] \in \bigl[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\bigr]. \qquad \square$$

Now, we are prepared to patch the local data. The following lemma is a key step in the proof of Theorem 1, after that the proof is finished by an easy induction.

Lemma 18. *One has*

$$\bigl[\mathrm{FU}^1(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\bigr] \subseteq \bigl[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\bigr]. \tag{5}$$

Proof. Let $T_{hk}(\alpha) \in \mathrm{FU}^1(2n, I, \Gamma)$, and $g \in \mathrm{GU}(2n, J, \Gamma)$. For any maximal ideal $\mathfrak{m}_i \lhd R_0$, choose a $t_i \in R_0 \backslash \mathfrak{m}_i$ and a positive integer p_i according to Lemma 17. Since the collection of all $t_i^{p_i}$ is not contained in any maximal ideal, we may find a finite number of t_i and $x_i \in R_0$ such that

$$\sum_i t_i^{p_i} x_i = 1.$$

We have,

$$T_{hk}(\alpha) = T_{hk}\biggl(\sum_i t_i^{p_i} x_i \cdot \alpha\biggr) = \prod_i T_{hk}\bigl(t_i^{p_i} x_i \alpha\bigr).$$

By Lemma 17, it follows immediately that for each i,

$$\begin{aligned}\bigl[T_{hk}\bigl(t_i^{p_i} x_i \alpha\bigr), g\bigr] &\in \bigl[\mathrm{EU}\bigl(2n, t_i^l I, t_i^l \Gamma\bigr), \mathrm{EU}\bigl(2n, t_i^l J, t_i^l \Delta\bigr)\bigr] \\ &\leqslant \bigl[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\bigr].\end{aligned} \tag{6}$$

A direct computation using (6) and formula (C2) and the fact that $\mathrm{EU}(2n, I, \Gamma)$ and $\mathrm{EU}(2n, J, \Delta)$ are normal in $\mathrm{EU}(2n, A, \Lambda)$, shows that

$$\bigl[T_{hk}(\alpha), g\bigr] = \biggl[\prod_i T_{hk}\bigl(t_i^{p_i} x_i \alpha\bigr), g\biggr] \in \bigl[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\bigr],$$

as claimed. $\square$

Now we are in a position to finish the proof of Theorem 1.

First proof of Theorem 1. Since $\mathrm{FU}(n, I, \Gamma)$ is generated by $\mathrm{FU}^1(2n, I, \Gamma)$ whereas $\mathrm{EU}(2n, I, \Gamma)$ and $\mathrm{EU}(2n, J, \Delta)$ are normalised by $\mathrm{EU}(2n, A, \Lambda)$, repeated use of (5) along with formula (C2), gives the inclusion

$$\bigl[\mathrm{FU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\bigr] \leqslant \bigl[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\bigr].$$

Since $\mathrm{EU}(2n, I, \Gamma)$ is the normal closure of $\mathrm{FU}(2n, I, \Gamma)$ in $\mathrm{EU}(2n, A, \Lambda)$, while both $\mathrm{GU}(2n, J, \Delta)$ and the right-hand side of the above formula are normalised by $\mathrm{EU}(2n, A, \Lambda)$, we get the inclusion

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big].$$

The opposite inclusion is obvious. □

8. Level of the mixed commutators

In this section we calculate lower and upper levels of mixed commutators

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big].$$

Lemma 19. *Let $n \geqslant 2$. Then for any two form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ) one has*

$$\mathrm{EU}(2n, I, \Gamma)\,\mathrm{EU}(2n, J, \Delta) = \mathrm{EU}(2n, I + J, \Gamma + \Delta).$$

Proof. Additivity of the elementary unitary transvections $T_{ij}(\alpha + \beta) = T_{ij}(\alpha)T_{ij}(\beta)$, where $i, j \in \Omega$ and $i \neq j$, while $\alpha \in I$, $\beta \in J$ for $i \neq -j$ and $\alpha \in \Gamma$, $\beta \in \Delta$ for $i = -j$, implies that the left-hand side contains *generators* of the right-hand side. The product of two normal subgroups is normal in $\mathrm{EU}(2n, A, \Lambda)$. □

As a preparation to the calculation of lower level, let us observe that together with [19, Theorem 2.3] this lemma implies the following corollary. Observe that in its turn the proof of [19, Theorem 2.3] heavily depends on Lemma 4.

Lemma 20. *Let $n \geqslant 3$ and further let (I, Γ) and (J, Δ) be two form ideals of (A, Λ). Then*

$$\mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big) \leqslant \mathrm{FU}(2n, I + J, \Gamma + \Delta).$$

Proof. In [19, Theorem 2.3] this lemma is proved for the case that $IJ = JI$. The similar proof shows that elements of the form

$$\mathrm{EU}\big(2n, IJ, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ)\big), \quad \mathrm{EU}\big(2n, JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(JI)\big),$$

are contained in $\mathrm{FU}(2n, I + J, \Gamma + \Delta)$. By the previous lemma, the group on the left-hand side is their product. □

In the next lemma we calculate the *lower* level of the mixed commutator subgroup.

Lemma 21. *Let $n \geqslant 3$. Then for any two form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ) one has the following inclusions*

$$\mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big) \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big].$$

Proof. Let $i \neq j$. Take an arbitrary index $h \neq \pm i, \pm j$. Then the right-hand side contains all elementary transvections of the form

$$T_{ij}(\alpha\beta) = \big[T_{ih}(\alpha), T_{hj}(\beta)\big] \quad \text{and} \quad T_{ij}(\beta\alpha) = \big[T_{ih}(\beta), T_{hj}(\alpha)\big],$$

for all $\alpha \in I$, $\beta \in J$.

Moreover, being the mutual commutator of two normal subgroups it is normal in the absolute elementary group $\mathrm{EU}(n, A, \Lambda)$. Thus, a similar argument as in Lemma 20 shows that

$$\mathrm{EU}\big(n, IJ+JI, \Gamma_{\min}(IJ+JI)\big) \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big].$$

Furthermore, the right-hand side contains

$$T_{i,-i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\beta\alpha\beta\big) = T_{i,-j}(-\beta\alpha)\big[T_{i,j}(\beta), T_{j,-j}(\alpha)\big]$$

and

$$T_{i,-i}\big(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\alpha\beta\alpha\big) = T_{i,-j}(-\alpha\beta)\big[T_{i,j}(\alpha), T_{j,-j}(\beta)\big].$$

It immediately follows that

$$\mathrm{EU}\big(2n, IJ+JI, {}^{J}\Gamma + {}^{I}\Delta + \Gamma_{\min}(IJ+JI)\big) \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big]. \qquad \square$$

Observe, that the above calculation crucially depended on the fact that $n \geqslant 3$ and we do not know how to estimate the lower level for $n = 2$ without some strong additional assumptions on the ring A. In the following lemmas we estimate the *upper* level.

Lemma 22. *Let $n \geqslant 2$. Then for any two form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ) one has the following inclusion*

$$\big[\mathrm{GU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] \leqslant \mathrm{GU}\big(2n, IJ+JI, \Gamma_{\max}(IJ+JI)\big).$$

Proof. Take arbitrary $x \in \mathrm{GU}(2n, I, \Gamma)$ and $y \in \mathrm{GU}(2n, J, \Delta)$. Then $x = e + x_1$, $x^{-1} = e + x_2$ for some $x_1, x_2 \in M(2n, I)$ such that $x_1 + x_2 + x_1x_2 = 0$ and $y = e + y_1$, $y^{-1} = e + y_2$ for some $y_1, y_2 \in M(2n, J)$ such that $y_1 + y_2 + y_1y_2 = 0$. Modulo $IJ + JI$ one has

$$[x, y] = (e + x_1)(e + y_1)(e + x_2)(e + y_2) \equiv e + x_1 + x_2 + x_1x_2 + y_1 + y_2 + y_1y_2 = e.$$

This shows that $[x, y] \in \mathrm{GL}(2n, A, IJ + JI)$. Clearly, $x \in \mathrm{GU}(2n, I, \Gamma)$ and $y \in \mathrm{GU}(2n, J, \Delta)$ preserve the sesquilinear form f modulo Γ and Δ, respectively, see Section 4.1. Now, an easy calculation shows that $[x, y]$ preserves f modulo $\Gamma + \Delta$. On the other hand, since $x \in \mathrm{GL}(2n, A, I)$ it follows that $f(xu, xu) - f(u, u) \in I$. Putting these observations together, we see that $[x, y]$ preserves f modulo

$$(IJ + JI) \cap (\Gamma + \Delta) \subseteq (IJ + JI) \cap \Lambda = \Gamma_{\max}(IJ + JI).$$

This finishes the proof. $\square$

Lemma 23. *Let $n \geqslant 3$. Then for any two form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ) one has the following inclusion*

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] \leqslant \mathrm{GU}\big(2n, IJ+JI, {}^{J}\Gamma + {}^{I}\Delta + \Gamma_{\min}(IJ+JI)\big).$$

Proof. By the commutator identities (C1) and (C2) and Lemma 5, it suffices to verify that

$$g = \big[T_{lk}(\alpha), h\big] \in \mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big),$$

where $h = (h_{i,j}) \in \mathrm{GU}(2n, J, \Delta)$ and $\alpha \in I$ for $l \neq -k$, and $\alpha \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma$ for $l = -k$.

By the previous lemma, we already have a similar inclusion with the maximal value of relative form parameter. Thus, it only remains to verify that

$$\sum_{1\leqslant i\leqslant n} \bar{g}_{ij} g_{-i,j} \in {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI).$$

The proof is divided into two cases depending on whether the root element $T_{lk}(\alpha)$ is of long or of short type, respectively. We attach a detailed calculation for the case of a long root type element. The case of a short root type element is settled by a similar calculation which will be omitted.

Let $T_{l,-l}(\alpha)$ be a long root element, where $\alpha \in \lambda^{-(\varepsilon(l)+1)/2}\Gamma$. In this case

$$g = \big[T_{l,-l}(\alpha), h\big] = T_{l,-l}(\alpha)\left(e - \sum_{i,j} h_{i,l}\alpha\bar{h}_{-j,l}\right).$$

Let us have a closer look at the sum $\sum_{1\leqslant i\leqslant n} \bar{g}_{ij} g_{-i,j}$. When $j \neq -l$, we may, without loss of generality, assume that $l \geqslant 0$ and $j \geqslant 0$, and thus this sum can be rewritten in the form

$$\begin{aligned}
&\sum_{1\leqslant i\leqslant n} \overline{h_{i,l}\alpha\bar{h}_{-j,l}} h_{-i,l}\alpha\bar{h}_{-j,l} - \lambda^{(\varepsilon(j)-\varepsilon(-l))/2} h_{-j,l}\alpha\bar{h}_{-j,l} + \overline{\alpha h_{-l,l}\alpha\bar{h}_{-j,l}} h_{-l,l}\alpha\bar{h}_{-j,l} \\
&\quad = \sum_{1\leqslant i\leqslant n} h_{-j,l}\lambda\bar{\alpha}\bar{h}_{i,l}h_{-i,l}\alpha\bar{h}_{-j,l} - h_{-j,l}\bar{\lambda}\alpha\bar{h}_{-j,l} + h_{-j,l}\bar{\alpha}\bar{h}_{-l,l}\bar{\alpha}h_{-l,l}\alpha\bar{h}_{-j,l},
\end{aligned}$$

where the first summand belongs to ${}^I\Delta$, whereas the second and the third ones belong to ${}^J\Gamma$, as claimed.

On the other hand, when $j = -l$, this sum equals

$$\sum_{1\leqslant i\leqslant n} \overline{h_{il}\alpha\bar{h}_{ll}} h_{-i,l}\alpha\bar{h}_{ll} - h_{ll}\bar{\alpha}\bar{h}_{ll} + (\bar{\alpha} - \overline{\alpha h_{-l,l}\alpha\bar{h}_{ll}})(1 - h_{-l,l}\alpha\bar{h}_{ll}),$$

where the first sum belongs to ${}^I\Delta$, while the rest equals

$$\begin{aligned}
x &= -h_{ll}\bar{\alpha}\bar{h}_{ll} + (\bar{\alpha} - h_{ll}\bar{\alpha}\bar{h}_{-l,l}\bar{\alpha})(1 - h_{-l,l}\alpha\bar{h}_{ll}) \\
&= -h_{ll}\bar{\alpha}\bar{h}_{ll} + \bar{\alpha} - \bar{\alpha}h_{-l,l}\alpha\bar{h}_{ll} - h_{l,l}\bar{\alpha}\bar{h}_{-l,l}\bar{\alpha} + h_{l,l}\bar{\alpha}\bar{h}_{-l,l}\bar{\alpha}h_{-l,l}\alpha\bar{h}_{ll} \\
&= -(1 + h_{ll} - 1)\bar{\alpha}(1 + \overline{h_{ll} - 1}) + \bar{\alpha} + (\lambda\bar{\alpha}h_{-l,l}\bar{\alpha}\bar{h}_{ll} - h_{ll}\bar{\alpha}\bar{h}_{-l,l}\bar{\alpha}) + h_{ll}\bar{\alpha}\bar{h}_{-l,l}\bar{\alpha}h_{-l,l}\alpha\bar{h}_{ll}
\end{aligned}$$

where the two last summands belong to $\Gamma_{\min}(IJ + JI)$ and to ${}^J\Gamma$, respectively. Thus, modulo ${}^J\Gamma + \Gamma_{\min}(IJ + JI)$ one has

$$x = -(h_{ll} - 1)\bar{\alpha} + \lambda\alpha\overline{(h_{ll} - 1)} - (h_{l,l} - 1)\alpha\overline{(h_{l,l} - 1)},$$

where the first summand also belongs to $\Gamma_{\min}(IJ + JI)$, whereas the second one belongs to ${}^J\Gamma$, respectively.

Thus, in both cases the desired sum belongs to ${}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)$, as claimed. $\square$

9. Relative versus absolute, and variations

Now we are in a position to give another proof of Theorem 1.

Second proof of Theorem 1. By Lemma 3 one has

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] = \big[\big[\mathrm{EU}(2n, A, \Lambda), \mathrm{EU}(2n, I, \Gamma)\big], \mathrm{GU}(2n, J, \Delta)\big].$$

Since (A, Λ) is a quasi-finite form ring and $n \geqslant 3$, by Lemma 6, all the subgroups above are normal in $\mathrm{GU}(2n, A, \Lambda)$. Now Lemma 2 implies that

$$\begin{aligned}
&\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big] \\
&\quad \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \big[\mathrm{EU}(2n, A, \Lambda), \mathrm{GU}(2n, J, \Delta)\big]\big] \\
&\qquad \cdot \big[\mathrm{EU}(2n, A, \Lambda), \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)\big]\big].
\end{aligned}$$

Applying to the first factor on the right-hand side the *absolute* standard commutator formula we immediately see that it *coincides* with $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$.

On the other hand, applying Lemma 23 followed by Lemma 21 to the second factor on the right-hand side, we can conclude that it is *contained* in

$$\begin{aligned}
&\big[\mathrm{EU}(2n, A, \Lambda), \mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big)\big] \\
&\quad = \mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big) \\
&\quad \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big].
\end{aligned}$$

Thus, the left-hand side is contained in the right-hand side, the inverse inclusion is obvious. □

It turns out, that for commutative form rings one can prove a slightly stronger result.

Theorem 2. *Let $n \geqslant 3$, and (R, Λ) be a commutative form ring. Then for any two form ideals (I, Γ) and (J, Δ) of the form ring (R, Λ) one has*

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{CU}(2n, J, \Delta)\big] = \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big].$$

The proof of Theorem 2 repeats this proof word for word, but the reference to Lemma 23 should be replaced by the reference to the following slightly stronger lemma.

Lemma 24. *Let $n \geqslant 3$ and (R, Λ) be a commutative form ring. Then for any two form ideals (I, Γ) and (J, Δ) of the form ring (R, Λ) one has the following inclusion*

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{CU}(2n, J, \Delta)\big] \leqslant \mathrm{GU}\big(2n, IJ, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ)\big).$$

This lemma is verified by calculations closely imitating those used to establish Lemmas 22 and 23. However, the difference is that now the element y figuring in the proof of Lemma 22 is congruent modulo J not to e itself, but to some βe, where β is a unit of the ring R/J. It remains to observe that when β is *central* in R, the argument goes through without any changes.

One can show by examples that Lemma 24 definitely fails for non-commutative rings. The reason is as follows. By the very definition of $\mathrm{CU}(2n, J, \Delta)$, the above element β is central modulo J. However, it does not have to be central in the ring R itself, and the summands in the proof of Lemma 22 do not cancel. As a result, the level may be much higher than expected.

Lemma 3 asserts that the commutator of two elementary subgroups, one of which is absolute, is itself an elementary subgroup. One can ask, whether one always has

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big] = \mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big).$$

Easy examples show that in general this equality may fail quite spectacularly. In fact, when $I = J$, one can only conclude that

$$\mathrm{EU}\big(2n, I^2, \Gamma^2\big) \leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, I, \Gamma)\big] \leqslant \mathrm{EU}(2n, I, \Gamma),$$

with right bound attained for some *proper* ideals, such as an ideal A generated by a central idempotent.

Nevertheless, the true reason, why the equality in Lemma 3 holds, is not the fact that one of the ideals I or J coincides with A, but only the fact that I and J are comaximal.

Theorem 3. *Let $n \geqslant 3$, and (A, Λ) be an arbitrary form ring for which absolute standard commutator formulae are satisfied. Then for any two comaximal form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ), $I + J = A$, one has the following equality*

$$\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big] = \mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big).$$

Proof. First of all, observe that by Lemmas 3 and 20 one has

$$\begin{aligned}\mathrm{EU}(2n, I, \Gamma) &= \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(n, A, \Lambda)\big]\\ &= \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, I, \Gamma) \cdot \mathrm{EU}(2n, J, \Delta)\big].\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{EU}(2n, I, \Gamma) &\leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, I, \Gamma)\big] \cdot \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big]\\ &\leqslant \big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, I, \Gamma)\big] \cdot \mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big).\end{aligned}$$

Commuting this inclusion with $\mathrm{EU}(2n, J, \Delta)$, we see that

$$\begin{aligned}\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big] &\leqslant \big[\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, I, \Gamma)\big], \mathrm{EU}(2n, J, \Delta)\big]\\ &\quad\cdot \big[\mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big), \mathrm{EU}(2n, J, \Delta)\big].\end{aligned}$$

The absolute standard commutator formula, applied to the second factor, shows that its is contained in

$$\begin{aligned}&\big[\mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big), \mathrm{EU}(2n, J, \Delta)\big]\\ &\quad\leqslant \big[\mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big), \mathrm{EU}(n, A, \Lambda)\big]\\ &\quad= \mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big).\end{aligned}$$

On the other hand, applying to the first factor Lemma 23, and then again the absolute standard commutator formula, we see that it is contained in

$$\begin{aligned}
&\big[\big[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)\big], \mathrm{EU}(2n, I, \Gamma)\big] \\
&\quad\leqslant \big[\mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big), \mathrm{EU}(2n, I, \Gamma)\big] \\
&\quad\leqslant \big[\mathrm{GU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big), \mathrm{EU}(2n, A, \Lambda)\big] \\
&\quad= \mathrm{EU}\big(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)\big).
\end{aligned}$$

Together with Lemma 20 this finishes the proof. □

10. Where next?

In this section we state and very briefly discuss some further relativisation problems, related to the results of the present paper. We are convinced that these problems can be successfully addressed with our methods.

In the following problems we propose to generalise results by Sivatski and Stepanov [39], and Stepanov and Vavilov [42] to Bak's unitary groups.

Problem 1. Obtain explicit length estimates in the relative conjugation calculus and commutator calculus.

Problem 2. Let $\text{j-dim}(R) < \infty$. Prove that the width of commutators in elementary generators is bounded, and estimate this width.

Alexei Stepanov (unpublished) established that the above width is bounded, without actually producing any specific bound. We believe that the methods of the present paper allow to give an *exponential* bound, similar to the one obtained for Chevalley groups over commutative rings [42], by developing a constructive version of the localisation method from Hazrat and Vavilov [22]. We believe that obtaining a similar constructive version of the results of the present paper would be simply a matter of patience. On the other hand, to obtain a *polynomial* bound, similar to that obtained for $\mathrm{GL}(n, A)$ in [39], one would need to combine our methods with a *full-scale* generalisation of decomposition of unipotents [41], including the explicit polynomial formulae for the conjugates of root unipotents. This seems to be somewhat remote.

In the main results of the present paper we always assume that $n \geqslant 3$. Obviously, due to the exceptional behaviour of the orthogonal group $\mathrm{SO}(4, A)$, these results do not fully generalise to the case, where $n = 2$. However, we believe they do generalise under appropriate additional assumptions on the form ring, such as $\Lambda A + A\Lambda = A$. Known results, including the work by Vyacheslav Kopeiko [26] and the work by Bak and Vavilov [8] clearly indicate both that this should be possible, and that the analysis of the case $n = 2$ be considerably harder from a technical viewpoint, than that of the case $n \geqslant 3$.

Problem 3. Develop conjugation calculus and commutator calculus for the group $\mathrm{GU}(4, A, \Lambda)$, provided $\Lambda A + A\Lambda = A$.

Problem 4. Prove relative standard commutator formula for the group $\mathrm{GU}(4, A, \Lambda)$, provided $\Lambda A + A\Lambda = A$.

Solution of the following problem would be a broad generalisation of Bak [3], Hazrat [19,20], and Bak, Hazrat and Vavilov [4]. Clearly, it will require the full force of localisation–completion.

Problem 5. Let R be a ring of finite Bass–Serre dimension $\delta(R) = d < \infty$, and let (I_i, Γ_i), $1 \leqslant i \leqslant m$, be form ideals of (A, Λ). Prove that for any $m > d$ one has

$$\begin{aligned}&\big[\big[\ldots\big[\mathrm{GU}(2n, I_1, \Gamma_1), \mathrm{GU}(2n, I_2, \Gamma_2)\big], \ldots\big], \mathrm{GU}(2n, I_m, \Gamma_m)\big]\\&\quad = \big[\big[\ldots\big[\mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2)\big], \ldots\big], \mathrm{EU}(2n, I_m, \Gamma_m)\big].\end{aligned}$$

Let us mention also generalisation of the results of the present paper to other types of groups. In view of [22,4,21,42] the first of the problems below seems almost immediate, and it is our intention to address it in a subsequent paper.

Problem 6. Obtain results similar to those of the present paper for Chevalley groups.

The other two problems, especially the last one, seem to be *much* more challenging, from a technical viewpoint. In both cases root subgroups are not abelian, and the analogues of the Chevalley commutator formula are much fancier, than in the familiar cases of Chevalley groups, or Bak's unitary groups. As a matter of fact, the required version of localisation has not been developed in either of these contexts, even at the absolute level.

The following problem refers to the context of odd unitary groups, as created by Victor Petrov [34–36].

Problem 7. Generalise results of the present paper to odd unitary groups.

The last problem refers to the recent context of isotropic reductive groups. Of course, it only makes sense over commutative rings, but on the other hand, a lot of new complications occur, due to the fact that relative roots do not form a root system, and the interrelations of the elementary subgroup with the group itself are abstruse even over fields (the Kneser–Tits problem). Still, we are convinced that most of necessary tools are already there, in the remarkable recent papers by Victor Petrov, Anastasia Stavrova and Alexander Luzgarev [37,40,29]. Of course, one will have to develop the whole conjugation and commutator calculus almost from scratch.

Problem 8. Obtain results similar to those of the present paper for [groups of points of] isotropic reductive groups.

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