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CONSTRAINED PARTITIONING PROBLEMS

Rainer E. BURKARD*
Institut für Mathematik, Technische Universität, Kopernikusgasse 24, A-8010 Graz, Austria

En-Yu YAO*
Mathematical Department, Zhejiang University, Hangzhou, People's Republic of China

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We consider partitioning problems subject to the constraint that the subsets in the partition are independent sets or bases of given matroids. We derive conditions for the functions F and f such that an optimal partition $(S_1^*, S_2^*, \dots, S_k^*)$ which minimizes $F(f(S_1), \dots, f(S_k))$ has certain order properties. These order properties allow to determine optimal partitions by Greedy-like algorithms. In particular balancing partitioning problems can be solved in this way.

Keywords. Partitioning problems, matroids, Greedy algorithm, balancing objective function, minimum variance.

1. Introduction

Partitioning of number sets have been studied by several authors, see e.g. Barnes and Hoffman [1], Chakravarty, Orlin and Rothblum [3, 4], Hwang [5], Hwang, Sun and Yao [6], and Tanaev [9]. This problem can be described as follows. Let $E = \{e_1, e_2, \dots, e_n\}$ be a finite set of numbers $e_1 \le e_2 \le \cdots \le e_n$ and let $P = (S_1, S_2, \dots, S_k)$ be a partition of E. That means the sets S_i, $i = 1, 2, \dots, k$ are nonempty, pairwise disjoint and their union equals E. Now, let f be a real-valued function defined on the subsets of E. We ask for a partition P^* which minimizes the objective function $F(P) := \sum_{i=1}^k f(S_i)$:

$$F(P^*) = \min\{F(P) \mid P \text{ partition of } E\}. \tag{1}$$

If no further condition is given, this problem is just called the *partitioning problem*. If the number k of sets in the partition is fixed, we call the problem the *k-partitioning problem*. Furthermore, not only k can be fixed, but also the sizes (shapes) $|S_1|, |S_2|, \dots, |S_k|$. In this case we call the problem the *k-shape partitioning problem*.

In this paper we study partitioning problems with additional constraints. We require that the sets $S_1, S_2, \dots, S_k$ of P are *independent* sets with respect to given matroids M_i, $i = 1, 2, \dots, k$. Moreover, we study a general form of the functions

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$f: 2^E \to \mathbb{R}$ and $F: \mathbb{R}^k \to \mathbb{R}$ which yields that an optimal solution of the partitioning problem exists showing special order properties. These order properties will then be used to solve such problems efficiently.

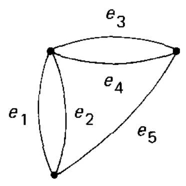
In Section 2 we introduce the notion of constrained partitioning problems and ordered solutions. In Section 3 we discuss a general form for the functions f and F in the case of constrained 2-partitioning problems. In particular balancing constrained partitioning problems and problems minimizing the variance will admit ordered solutions. Finally we extend these results in Section 4 to constrained k-partitioning problems, discuss the underlying combinatorial structure of these problems and derive a solution method. The solution method consists in an iterated application of a Greedy-like algorithm. If all matroids involved have the same structure, an optimal constrained k-partition can be found in $O(k^2)$ steps.

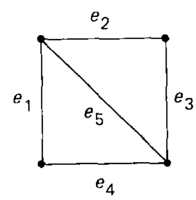
2. Constrained k-partitions

A 2-partitioning problem asks for a partition of the set $E=\{e_1, e_2, \ldots, e_n\}$ into two nonempty sets S_1 and S_2. In the following we will impose further constraints on the sets S_i, $i=1,2$ for example that they are independent sets with respect to given matroids $M_i=(E, \mathscr{F}_i)$, $i=1,2$, where $\mathscr{F}_i$ denotes the class of independent sets of matroid M_i. If we define $M_1=M_2$ by $\mathscr{F}:=2^E$ we get the classical partitioning of E into two sets. This leads to the following definition:

Definition 2.1 (*constrained k-partition*). Let k matroids $M_i(E, \mathscr{F}_i)$ be defined on E given by their classes $\mathscr{F}_i$ of independent sets. A partition $P=(S_1, S_2, \ldots, S_k)$ of E is called a *constrained k-partition*, if $S_i \in \mathscr{F}_i$ for $i=1,2,\ldots,k$.

Example 2.2. Let the following two graphic matroids M_1 and M_2 be given on the common ground set $E=\{e_1, e_2, e_3, e_4, e_5\}$.





A constrained 2-partition is e.g. $S_1=\{e_1, e_3\}$, $S_2=\{e_2, e_4, e_5\}$. On the other hand $\bar{S}_1=\{e_1, e_2\}$, $\bar{S}_2=\{e_3, e_4, e_5\}$ is not a constrained 2-partition, since $\bar{S}_1$ is not an independent set with respect to M_1.

Now we turn to shape partitions. Let $P=(S_1, S_2)$ be a 2-shape partition of E with $|S_1|=u$ and $|S_2|=v$, $u+v=|E|$. If we define matroids M_i, $i=1,2$ by their classes of bases as follows:

$$\mathscr{B}_1 = \{S \subseteq E\colon |S| = u\}, \qquad \mathscr{B}_2 = \{S \subseteq E\colon |S| = v\},$$

we can see that shape partitions are again very special constrained partitions of E. Moreover, the two matroids defined above are dual to each other. This motivates the following definition.

Definition 2.3 (*complementary class of matroids*). Let $M_i = (E, \mathscr{B}_i)$, $i = 1, 2, \ldots, k$ be matroids with ranks n_i on the set E, defined by their classes $\mathscr{B}_i$ of bases. These matroids are called a *complementary class on E* if and only if the following two properties hold:

(a) $$\sum_{i=1}^{k} n_i = |E|. \tag{2}$$

(b) If $S_j \in \mathscr{B}_j$, $j = 1, \ldots, n$; $j \neq i$ are pairwise disjoint sets, then

$$E \setminus \bigcup_{j \neq i} S_j \in \mathscr{B}_i. \tag{3}$$

Example 2.4. Let M_i be a matroid consisting of all $(m \times n)$ 0–1 matrices with at most n_i 1-entries in each column. If $\sum_{i=1}^{k} n_i = m$, these k matroids form a complementary class.

Now we define:

Definition 2.5 (*constrained k-shape partition*). Let $M_i = (E, \mathscr{B}_i)$, $i = 1, 2, \ldots, k$ be a complementary class of matroids defined on E. A partition $P = (S_1, S_2, \ldots, S_k)$ of E is called a *constrained k-shape partition*, if $S_i \in \mathscr{B}_i$ for $i = 1, 2, \ldots, k$.

Example 2.6. Shape partitioning problems play a role in matrix decomposition problems. Given a nonnegative integer matrix $T = (t_{ij})$, find 0–1 matrices S_i with at most one 1-entry (or in general n_i 1-entries) in each column, such that $T \leq \sum \lambda_i S_i$, with $\sum \lambda_i \to \min$. For example, let

$$T = \begin{pmatrix} 2 & 4 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$$

be given. Then

$$T \leq 2 \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and this is best possible, since the last column sum is 8. Obviously the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

represent a shape partition in the complementary class of matroids introduced in the previous example.

Now we endow the class of all subsets of $E=\{e_1,e_2,\dots,e_n\}$ with a total order. We say $S_1 \subseteq E$ is *lexicographically not greater* than $S_2 \subseteq E$, if either $S_1 \subseteq S_2$ or $\min\{k\colon e_k \in S_1 \setminus S_2\} < \min\{k\colon e_k \in S_2 \setminus S_1\}$.

Example 2.7. $\{e_1,e_2\} \leq \{e_1,e_2,e_3\} \leq \{e_1,e_4\}$.

This order enables us to introduce the notions of ordered sets and ordered partitions.

Definition 2.8 (*ordered pair of sets*). Let $M_1=(E,\mathscr{F}_1)$ and $M_2=(E,\mathscr{F}_2)$ be two matroids. Two sets $S_1 \in \mathscr{F}_1$ and $S_2 \in \mathscr{F}_2$ are called *not ordered*, if

$$\exists e_p \in S_1,\ e_r \in S_2 \text{ with } p<r,$$

and (4)

$$\exists e_s \in S_1,\ e_t \in S_2 \text{ with } s>t,$$

such that

$$S_1' := S_1 \setminus \{e_p\} \cup \{e_r\},\ S_1'' := S_1 \setminus \{e_s\} \cup \{e_t\} \in \mathscr{F}_1,$$

and

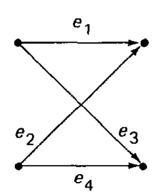
$$S_2' := S_2 \setminus \{e_r\} \cup \{e_p\},\ S_2'' := S_2 \setminus \{e_t\} \cup \{e_s\} \in \mathscr{F}_2.$$

Otherwise they are called *ordered.*

Definition 2.9 (*ordered partitions*). A constrained k-(shape) partition $P=(S_1,S_2,\dots,S_k)$ is called an *ordered partition* if and only if any two subsets S_i and S_j are ordered.

Example 2.10. Let $E=\{e_1,e_2,e_3,e_4\}$ be the ground set of a matroid M whose independent sets are arc sets in the graph G with the property that any two arcs have no starting point in common.

G



We define $M := M_1 := M_2$. Then $S_1 = \{e_1, e_4\}$, $S_2 = \{e_2, e_3\}$ is not an ordered shape partition, since we can exchange e_4 against e_2 and e_1 against e_3. By the first exchange we get $S_1' := \{e_1, e_2\}$, $S_2' = \{e_3, e_4\}$ and this partition is ordered.

Now let $f : 2^E \to \mathbb{R}$ be a real-valued function defined on the subsets of E and let $F : \mathbb{R}^k \to \mathbb{R}$ be a further real function. The *value of a partition* $P = (S_1, S_2, \ldots, S_k)$ is defined as

$$F(P) := F(f(S_1), f(S_2), \ldots, f(S_k)). \tag{5}$$

A partition P^* which is feasible (i.e., fulfills shape and matroid constraints) and minimizes $F(P)$ is called *optimal.*

Example 2.11. Let us consider the matrix decomposition example again. For any matrix S_i we define

$$f(S_i) := \text{maximum nonzero entry},$$

$$F(P) = \sum f(S_i).$$

Thus the matrix decomposition problem asks for a partition of the underlying set $\{(i,j)\colon 1 \le i \le m,\ 1 \le j \le n\}$ which minimizes the value $F(P)$. Problems of this kind play a role in the decomposition of traffic matrices arising in communication systems (see e.g. Burkard [2]).

Definition 2.12 (*OCP = OOCP*). If there exists an optimal constrained partition which is ordered, we say the problem has the property *OCP = OOCP.*

Originally, this notion was introduced in Hwang, Sun and Yao [6]. In many cases partitioning problems with this property can be well solved by efficient methods as will be shown in Section 4. In the next section we will discuss special problems and examples which yield property OCP = OOCP. Clearly, if OCP = OOCP holds for any shape of a constrained k-shape partitioning, then it holds also for the constrained k-partitioning, but not vice versa.

3. Constrained 2-partitioning problems

We are now interested in the question of which types of the functions f and F lead to partitioning problems with the property OOCP = OCP in the case of constrained 2-partitioning problems. Generalizing a result of Hwang, Sun and Yao [6] we get as an immediate consequence of Definition 2.5:

Proposition 3.1. *Let $P = (S_1, S_2)$ be a nonordered constrained partition. If for one of the two possible exchanges $S_1' = S_1 \setminus \{x\} \cup \{y\}$, $S_2' = S_2 \setminus \{y\} \cup \{x\}$ or $S_1'' = S_1 \setminus \{v\} \cup \{w\}$, $S_2'' = S_2 \setminus \{w\} \cup \{v\}$ the partition value does not increase*

$$F(f(S_1'), f(S_2')) \le F(f(S_1), f(S_2))$$

or (6)

$$F(f(S_1''), f(S_2'')) \leq F(f(S_1), f(S_2)),$$

then the partitioning problem has the property OCP = OOCP.

In the following we describe classes of functions f and F which fulfill the assumptions of Proposition 3.1. With any element $e \in E$ we associate a real number (cost) $c(e)$. Moreover, we index the elements of E such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_n)$. Now, let $g(x, y)$ be a symmetric real-valued function; g is called an *interval function*, if it is increasing with $\max(x, y)$ and decreasing with $\min(x, y)$. An interval function is called *super-additive*, if for x and y, both lying in the interval $[z, w]$ the inequality

$$g(z, w) + g(x, y) \geq g(z, x) + g(y, w) \tag{7}$$

holds.

Proposition 3.2. *Let g be a super-additive interval function. We assume that for any $S \subseteq E$ the value $u = u(S)$ is defined by*

$$\sum_{z \in S} g(c(z), u) = \min_{\bar{u}} \sum_{z \in S} g(c(z), \bar{u}). \tag{8}$$

Let $f(S) = \sum_{z \in S} g(c(z), u)$ and $F(X, Y) = X + Y$. Then the constrained 2-shape partitioning problem

$$F(P^*) = \min_{P} F(f(S_1), f(S_2))$$

has the property OCP = OOCP for any shape constraints on the sets S_1 and S_2.

Proof. Let $P = (S_1, S_2)$ be a nonordered constrained 2-shape partition and let $u = u(S_1)$, $\upsilon = \upsilon(S_2)$ be defined by

$$\sum_{z \in S_1} g(c(z), u) = \min_{\bar{u}} \sum_{z \in S_1} g(c(z), \bar{u}),$$

$$\sum_{z \in S_2} g(c(z), \upsilon) = \min_{\bar{\upsilon}} \sum_{z \in S_2} g(c(z), \bar{\upsilon}).$$

Let us assume $u \leq \upsilon$. Since S_1, S_2 are nonordered there exists an $x \in S_1$ and an element $w \in S_2$ with $c(x) \geq c(w)$ such that $S_1' := S_1 \setminus \{x\} \cup \{w\} \in \mathcal{F}_1$, $S_2' := S_2 \setminus \{w\} \cup \{x\} \in \mathcal{F}_2$. For the real numbers $c(w)$, $c(x)$, u, υ with $c(w) \leq c(x)$ and $u \leq \upsilon$ we have to consider six possible cases, namely

(I) $c(w) \leq c(x) \leq u < \upsilon$, $c(w) \leq u \leq c(x) \leq \upsilon$ and $u < \upsilon$,

$u < \upsilon \leq c(w) \leq c(x)$, $u \leq c(w) \leq \upsilon \leq c(x)$ and $u < \upsilon$,

(II) $c(w) \leq u < \upsilon \leq c(x)$, $u \leq c(w) \leq c(x) \leq \upsilon$ and $u < \upsilon$.

The superadditivity and monotonicity properties of the interval function g yield in any of these cases

$$g(c(x),u)+g(c(w),v)\geq g(c(w),u)+g(c(x),v). \tag{9}$$

Thus,

$$\begin{aligned}F(P)&=f(S_1)+f(S_2)\\&=\sum_{z\in S_1\setminus\{x\}} g(c(z),u)+g(c(x),u)+\sum_{z\in S_2\setminus\{w\}} g(c(z),v)+g(c(w),v)\\&\geq \sum_{z\in S_1\setminus\{x\}} g(c(z),u)+g(c(w),u)+\sum_{z\in S_2\setminus\{w\}} g(c(z),v)+g(c(x),v)\\&=\sum_{z\in S_1'} g(c(z),u)+\sum_{z\in S_2'} g(c(z),v)\\&\geq \sum_{z\in S_1'} g(c(z),u')+\sum_{z\in S_2'} g(c(z),v')\\&=f(S_1')+f(S_2')=F(P').\end{aligned} \tag{10}$$

Therefore the new solution is feasible and has no worse objective function value. We show that $u'\leq u$ and $v\leq v'$. This enables to continue this argument, until an optimal ordered solution is found. Suppose $u'>u$.

From the definition of u follows

$$f(S_1)=\sum_{z\in S_1\setminus\{x\}} g(c(z),u)+g(c(x),u)\leq \sum_{z\in S_1\setminus\{x\}} g(c(z),u')+g(c(x),u').$$

Similar for u':

$$f(S_1')=\sum_{z\in S_1'} g(c(z),u')<\sum_{z\in S_1'} g(c(z),u).$$

Merging these two inequalities yields

$$g(c(x),u)+g(c(w),u')<g(c(x),u')+g(c(w),u). \tag{11}$$

Again there are six possibilities for the real numbers $c(w)\leq c(x)$, $u<u'$. In case (I) the inequality (11) contradicts the superadditivity of g, in case (II) this inequality is a contradiction to the monotonicity properties of g. Therefore $u'\leq u$. A similar argument shows $v\leq v'$. This finishes the proof. □

Examples for functions $g(x,u)$ and u as specified in Proposition 3.2 are

$$g(x,y)=|x-y|, \qquad u=\frac{1}{|S|}\sum_{z\in S} c(z), \tag{12}$$

$$g(x,y)=(x-y)^{2k}, \quad k\in\mathbb{N}, \qquad u=\frac{1}{|S|}\sum_{z\in S} c(z), \tag{13}$$

$$g(x,y)=\frac{\max(x,y)}{\min(x,y)}, \qquad u=\left(\prod_{z\in S} c(z)\right)^{1/|S|}. \tag{14}$$

Proposition 3.2 yields in particular that the following two kinds of partitioning problems have the property OCP = OOCP.

An optimization problem is called *balancing*, if the difference between a maximum and a minimum coefficient in a solution is as small as possible (see Martello et al. [7]). This motivates the following definition.

Definition 3.3 (*balancing partitioning*). A constrained 2-(shape) partitioning problem is called *balancing*, if P is a constrained 2-(shape) partition with value

$$F(P)=\max\{f(S_1),f(S_2)\}$$

where

$$f(S_i)=\max_{z\in S_i}|c(z)-u|,\quad u=\frac{1}{|S_i|}\sum_{z\in S_i}c(z)\qquad \text{for } i=1,2.$$

Corollary 3.4. *Balancing constrained 2-(shape) partitioning problems have the property OCP = OOCP.*

Proof. Let $\bar{g}(x,y)=(x-y)^k$ for k sufficiently large. When

$$\bar{g}(c(x),u)+\bar{g}(c(w),v)\geq\bar{g}(c(w),u)+\bar{g}(c(x),v)$$

(cf. inequalities (9) and (10)), then also

$$|c(x)-u|^k+|c(w)-v|^k\geq|c(w)-u|^k+|c(x)-v|^k\quad \text{for all } k\geq k_0.$$

This implies for $k\to\infty$

$$\max\{|c(x)-u|,|c(w)-v|\}\geq\max\{|c(w)-u|,|c(x)-v|\}.$$

Therefore, relation (10) remains true, if the sum is replaced by maximum. But (10) is the essential argument in the proof of Proposition 3.2. Therefore the same kind of proof can be applied to balancing partitioning problems. □

The value $u=(1/|S|)\sum_{z\in S}c(z)$ can be interpreted as mean value of the cost elements of S. Then $(1/|S|)\sum_{z\in S}(c(z)-u)^2$ is the *variance* of the cost elements of S.

Corollary 3.5. *Variance minimizing constrained 2-shape partitioning problems with* $|S_1|=|S_2|$ *have the property OCP = OOCP.*

Proof. Let $g(x,y)=(x-y)^2$ and set $F(X,Y)=(1/\alpha)X+(1/\alpha)Y$ with $\alpha=|S_1|=|S_2|$. □

Before we discuss another class of problems with property OCP = OOCP, we state the following lemma (cf. Zimmermann [10]) which guarantees the existence of a "smallest" basis in a matroid. Let us recall that we ordered subsets $S\subseteq E=\{e_1,e_2,\dots,e_n\}$ lexicographically according to their indices.

Lemma 3.6 (Zimmermann). *Let* $M=(E,\mathcal{B})$ *be a matroid given by its class* $\mathcal{B}$ *of bases. Then there exists a basis* B^* *which is elementwise not greater than any other* *basis of M, i.e.,* $\exists B^* = (e_{i_1}, e_{i_2}, \dots, e_{i_k})$ *with* $i_1 < i_2 < \dots < i_k$ *such that for all* $B = (e_{j_1}, e_{j_2}, \dots, e_{j_k})$, $j_1 < j_2 < \dots < j_k$, $B \in \mathscr{B}$,

$$i_r \le j_r, \quad r = 1, 2, \dots, k \tag{15}$$

holds. B^* *can be found by applying the Greedy algorithm starting from* e_1 *and investigating the elements in increasing order.*

Proof. Let $B^* = \{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$ be the basis constructed in the above mentioned way by the Greedy algorithm. Obviously, B^* is lexicographically minimal. Let us now assume B^* is not the *smallest* basis, i.e., there exists a basis $\bar{B} = \{e_{j_1}, e_{j_2}, \dots, e_{j_k}\}$ with $i_r \le j_r$ for $1 \le r < k_0$ but $i_{k_0} > j_{k_0}$. Then $S^* := \{e_{i_1}, \dots, e_{i_{k_0-1}}\}$ and $\bar{S} := \{e_{j_1}, \dots, e_{j_{k_0}}\}$ are independent sets with $|S^*| < |\bar{S}|$. Therefore there exists an element $e_t \in \bar{S}$ such that $S^* \cup \{e_t\}$ is independent and lexicographically smaller than B^*. Since $S^* \cup \{e_t\}$ can be enlarged to a basis we get a contradiction that B^* is lexicographically minimal. □

The same argumentation yields:

Corollary 3.7. *In any matroid M there exists a largest basis, i.e., a basis which is elementwise not smaller than any other basis of M.*

As a consequence of Corollary 3.7 we get:

Corollary 3.8. *Let* $\bar{P} = (\bar{S}_1, \bar{S}_2)$ *be a constrained 2-shape partition of E. If* $\bar{S}_1$ *is the smallest basis of* $M_1 = (E, \mathscr{B}_1)$, *then* $\bar{S}_2$ *is the largest basis in the dual matroid* $M_2 = (E, \mathscr{B}_2)$.

Proof. For any basis S_1 of $M_1 = (E, \mathscr{B}_1)$, $S_2 := E \setminus S_1$ is a basis of M_2 and $P = (S_1, S_2)$ is a 2-shape partition. Let $\bar{S}_1$ be the smallest basis of M_1, i.e., for all $S_1 \in \mathscr{B}_1$: $\bar{S}_1 \le S_1$. This implies $E \setminus \bar{S}_1 = \bar{S}_2 \ge S_2$. □

Proposition 3.9. *If* $f(S)$ *has the property* $f(A) \ge f(B)$ *for* $|A| = |B|$, $A \ge B$, $A, B \subseteq E$ *and* $F: \mathbb{R}^2 \to \mathbb{R}$ *is increasing in the first and decreasing in the second argument, then the corresponding constrained 2-shape partitioning problem has the property* $OCP = OOCP$.

Proof. Under the conditions stated the solution $(\bar{S}_1, \bar{S}_2)$ is optimal where $\bar{S}_1$ is the smallest basis of M_1 and $\bar{S}_2$ is the largest basis of M_2 (cf. Corollary 3.8). This solution is an ordered optimal solution. □

Corollary 3.10. *The partitioning problem of Proposition* 3.9 *can be solved by applying the Greedy algorithm to matroid* M_1 *or* M_2.

Example 3.11. Given $E=\{c_i\colon i=1,2,\dots,n\}$, $c_i \in \mathbb{R}$, find that permutation φ which maximizes the function $\sum_{i=1}^{r} c_{\varphi(i)} - \sum_{i=r+1}^{n} c_{\varphi(i)}$. Obviously, this corresponds to the situation described by Proposition 3.9 and therefore we get an optimal solution by ordering the numbers c_i decreasingly. □

In the next section we generalize these results to constrained k-shape partitioning problems with $k>2$ and we derive an algorithm for problems with the property OCP = OOCP.

4. Constrained k-shape partitions

In this section we generalize the results of the previous section to constrained k-shape partitions with $k>2$. Moreover we analyze the corresponding combinatorial structure of constrained partitioning problems and we derive a Greedy-like algorithm for solving problems with the property OCP = OOCP.

At first we shall show that a k-shape partitioning problem has the property OCP = OOCP, if the problem restricted to any two matroids on a subset E' of the ground set E has this property. We deal with the following situation. Let $M_i := (E, \mathscr{B}_i)$, $i=1,2,\dots,k$ $(k\geq 2)$ be a complementary class of matroids. Let $f: 2^E \to \mathbb{R}$ and let $F: \mathbb{R}^k \to \mathbb{R}$ be a symmetric function with the following property. We denote the restriction of F to two variables by F^*. Then F must fulfill

$$F^*(S_i^*, S_j^*) \leq F^*(S_i, S_j)$$
$$\Rightarrow F(S_1,\dots,S_i^*,\dots,S_j^*,\dots,S_k) \leq F(S_1,\dots,S_k). \tag{16}$$

For any subset $E' \subseteq E$ and any pair (i,j) of different indices i and j, $1\leq i,j\leq k$ the problem $(\mathscr{P}_{i,j} \mid E')$

$$\min\{F^*(f(B_i), f(B_j))\colon B_i \in \mathscr{B}_i,\ B_j \in \mathscr{B}_j,\ B_i \cup B_j \subseteq E',\ B_i \cap B_j = \emptyset\}$$

is called *2-restriction of* $\mathscr{P}$.

If the set E' does not contain a pair of bases for the matroids M_i and M_j, we say $(\mathscr{P}_{i,j} \mid E')$ has no feasible solution and define the value of $(\mathscr{P}_{i,j} \mid E')$ to be $+\infty$. Notice that M_i restricted to any set E' still is a matroid. By Definition 2.8 we can define *ordered* solutions of $(\mathscr{P}_{i,j} \mid E')$. Now we prove:

Proposition 4.1. *Let* $\mathscr{P}$ *be a constrained k-shape partitioning problem with a symmetric value function F which fulfills* (16). *If any 2-restriction of* $\mathscr{P}$ *which has a feasible solution also has an ordered optimal solution, then* $\mathscr{P}$ *has the property OCP = OOCP.*

Proof. Let $(S_1,\dots,S_k)$ be an optimal constrained k-shape partition of $\mathscr{P}$. At first we consider the 2-restriction of $\mathscr{P}$ defined by matroids M_1 and M_2 and $E' := S_1 \cup S_2$. Since (S_1, S_2) is a feasible solution of this problem there exists according to our assumption an ordered solution $(S_1^{(1)}, S_2^{(1)})$ with $F^*(S_1^{(1)}, S_2^{(1)}) \le F^*(S_1, S_2)$ and $S_1^{(1)} < S_2^{(1)}$. Therefore (15) yields

$$F(S_1^{(1)}, S_2^{(1)}, S_3, \dots, S_k) \le F(S_1, S_2, \dots, S_k).$$

Next we consider the 2-restriction defined by M_1, M_3 and $E' := S_1^{(1)} \cup S_3$. The same argumentation as above yields an optimal solution $S_1^{(2)}$, $S_3^{(1)}$ with $F^*(S_1^{(2)}, S_3^{(1)}) \le F^*(S_1^{(1)}, S_3)$ and therefore $F(S_1^{(2)}, S_2^{(1)}, S_3^{(1)}, \dots, S_k) \le F(S_1, \dots, S_k)$. We apply this argument until we get $S_1^{(k-1)}, S_2^{(1)}, \dots, S_k^{(1)}$ with $S_1^{(k-1)} < S_j^{(1)}$ for all j and

$$F(S_1^{(k-1)}, S_2^{(1)}, \dots, S_k^{(1)}) \le F(S_1, S_2, \dots, S_k).$$

Next, this step is repeated with $S_2^{(1)}, \dots, S_k^{(1)}$. Finally, we arrive at an ordered solution $S_1^{(k-1)}, S_2^{(r)}, \dots, S_k^{(s)}$ of the k-shape partitioning problem with

$$F(S_1^{(k-1)}, S_2^{(r)}, \dots, S_k^{(s)}) \le F(S_1, \dots, S_k).$$

Since $(S_1, \dots, S_k)$ is optimal, also this ordered solution $(S_1^{(k-1)}, S_2^{(r)}, \dots, S_k^{(s)})$ is optimal. Thus OCP = OOCP. □

As a consequence of this theorem we get that the conditions stated in Proposition 3.2 and its Corollaries 3.4 and 3.5 also guarantee the property OCP = OOCP for constrained k-shape partitioning problems, since $F(X_1, \dots, X_k) = X_1 + \cdots + X_k$ as well as $F(X_1, \dots, X_k) = \max(X_1, \dots, X_k)$ are symmetric and fulfill (16). In particular balancing k-shape partitioning problems have this property. If the bases of all matroids M_i, $1 \le i \le k$, have the same cardinality, then also k-shape partitions which minimize the variance have the property OCP = OOCP.

Example 4.2. A soccer club has k goalkeepers, $4k$ defenders, $3k$ midfield players and $3k$ forwards. Thus the set of $11k$ members splits into 4 classes: E_1 goalkeepers, E_2 defenders, E_3 midfield players and E_4 forwards. Let the number $c(i)$ be a measure for the ability of player i. We want to organize teams T_r, $r = 1, 2, \dots, k$ such that the efficiency of the players in a team is about the same. This leads to the following variance minimization problem:

Let $M = (E, \mathscr{B})$ be defined as partition matroid with

$$\mathscr{B} = \{S \subseteq E \colon |S \cap E_j| = d_j \text{ for } j = 1, 2, 3, 4\},$$

$$d_1 = 1, \quad d_2 = 4, \quad d_3 = d_4 = 3.$$

Thus every basis corresponds to a soccer team. Further, let $M_i := M$ for $r = 1, 2, \dots, k$. We want to find teams $T_1, T_2, \dots, T_k$ such that $F(f(T_1), \dots, f(T_k))$ is minimum where

$$F = \sum_{r=1}^{k} f(T_r), \quad f(T_r) = \tfrac{1}{11} \sum_{i \in T_r} (c(i) - u_r)^2,$$

$$u_r := \tfrac{1}{11} \sum_{i \in T_r} c(i).$$

By Corollary 3.5 of Proposition 3.2 and Proposition 4.1 the optimal solution of partitioning the members into teams $T_1, T_2, \dots, T_k$ will be an ordered partitioning. We shall see later that such an ordered solution can be found by an algorithm with complexity $O(k^2)$.

Before we discuss an algorithm for solving problems with the property OCP = OOCP we prove two theorems which provide some insight in the structure of ordered partitions.

Proposition 4.3. *If $P=(S_1, S_2, \dots, S_k)$ is an ordered constrained k-shape partition and S_1 is that subset which contains the smallest element of E, then S_1 has the following properties*:

(i) *No element of S_1 can be exchanged against a smaller element in another set S_i, $i \geq 1$.*

(ii) *S_1 is elementwise smaller than any other basis in M_1, if we arrange the elements in any basis in increasing order.*

Proof. The subsets S_i, $i=1,2,\dots,k$ are lexicographically ordered if the elements in each S_i are arranged increasingly. Since $e_1 \in S_1$, we get $S_1 < S_i$ for all $i \geq 2$. Therefore e_1 can be exchanged with any element in every set S_i, $i \geq 2$. Since (S_1, S_i) is ordered, this implies that no element of S_1 can be exchanged against a smaller element of the set S_i. This proves (i).

Now let us assume there exists a basis B_1 of matroid M_1 such that $B_1 < S_1$. The components e_{i_r} of B_1 and e_{j_r} of S_1 fulfill

$$e_{i_r} = e_{j_r} \quad \text{for } r=1,2,\dots,p-1, \qquad e_{i_p} \leq e_{j_p}.$$

Since $e_1 \in S_1$, we get $p \geq 2$. Moreover, since S_1 is a basis and $e_{i_p} \notin S_1$, there is a unique circuit in $S_1 \cup \{e_{i_p}\}$. By (i) no element of S_1 can be exchanged against a smaller element. Therefore e_{i_p} is the largest element in this circuit. This implies $T := \{e_{j_1}, e_{j_2}, \dots, e_{j_{p-1}}, e_{i_p}\}$ is dependent in M_1. But $T \subset B_1$ which is a contradiction. Therefore S_1 is the smallest basis of matroid M_1 by Lemma 3.6. □

In a similar way we get by induction:

Proposition 4.4. *Let $P=(S_1, S_2, \dots, S_k)$ be an ordered constrained k-shape partition of E with $S_1 < S_2 < \cdots < S_k$. Then this partition has the following properties*:

(i) *Any S_i, $i=1,2,\dots,k$ contains the smallest element of $E' := E \setminus \bigcup_{j=1}^{i-1} S_j$.*

(ii) *Any element of S_i, $i=1,2,\dots,k-1$ cannot be exchanged against a smaller element in any S_j with $j>i$.*

(iii) *S_i is the lexicographically smallest basis of matroid M_i which does not intersect any set S_j with $j<i$.*

From the definition of ordered partitions and Propositions 4.3 and 4.4 we get immediately

Corollary 4.5. *A constrained k-shape partition* $P=(S_1, S_2, \ldots, S_k)$ *with* $S_1 \prec S_2 \prec \cdots \prec S_k$ *is ordered, if and only if*

(i) S_1 *is the smallest basis of matroid* M_1,
(ii) S_i, $2 \le i \le k$ *is the lexicographically smallest basis which contains only elements of* $E \setminus \bigcup_{j<i} S_j$.

Next we derive a solution method for k-shape partitioning problems with the property OCP = OOCP which is based on Propositions 4.3 and 4.4.

Let us fix the sequence of matroids $M_1, M_2, \ldots, M_k$. We can find an optimal ordered partition $(S_1, S_2, \ldots, S_k)$ subject to S_1 basis of M_1, S_2 basis of $M_2, \ldots, S_k$ basis of M_k by repeatedly applying the Greedy algorithm:

Step 1. Determine by the Greedy algorithm a smallest basis S_1 of M_1. Let $i=2$, $S := S \setminus S_1$.
Step 2. Let M be the matroid M_i restricted to S. Determine by the Greedy algorithm a smallest basis S_i of M.
Step 3. If $i<k$, let $S := S \setminus S_i$, $i := i+1$ and repeat Step 2.

Now let us analyze the complexity of this algorithm. We denote the rank of matroid M_i by r_i. For determining S_1 we need at most n calls of an independence oracle. Then S is reduced by r_1 elements. Thus we need not more than $n - r_1$ calls to determine S_2. Now S is reduced again. Therefore this algorithm yields after at most

$$(k-1)n - \sum_{i=1}^{k-2} (k-i-1)r_i \tag{17}$$

calls of the independence oracle an optimal solution. If all r_i are equal, say r, we need at most

$$K := (k-1)\left(n - r\frac{(k-2)}{2}\right) \tag{18}$$

steps to find an optimal solution. Since $n = r \cdot k$, this yields a complexity $O(k^2)$.

In general the order of matroids will not be fixed. Therefore we need to compare all $k!$ possible arrangements for finding a global optimum. But if all matroids M_i are equal, any of these arrangements yields the same (unique) solution. Therefore we can find the optimal ordered solution in this case in at most K steps. Moreover, any constrained 2-partitioning problem with OCP = OOCP can be solved by just investigating M_1 first and then M_2 and vice versa. Therefore it can be solved in at most $2n$ steps by applying the above algorithm just twice.

5. Conclusion

We consider in this paper partitioning problems subject to the constraint that the partition sets are bases of matroids. For several classes of functions it was possible to show that there exists an ordered optimal solution. Such an ordered optimal solution can be determined by repeatedly applying the Greedy algorithm. In particular, we get an algorithm of complexity $O(k^2)$ if all k given matroids are the same.

There is a number of questions for further research. One such question concerns the combinatorial structure of complementary classes of matroids and further examples for such classes. A second question concerns further classes of functions F and f which yield the property OCP = OOCP. A further question concerns the modifications necessary to find partitions, whose sets are in the intersection of two matroids. An important example for such a problem is the matrix decomposition problem (cf. Burkard [2]) which is, however, known to be NP-hard (Rendl [8]). Finally we can ask for efficient algorithms or good approximation algorithms, if the involved matroids are not all equal. In particular, if certain problems yield a special sequence for the matroids involved, they could be solved in polynomial time.

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