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Reorientations of covering graphs*

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Abstract

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The aim of this paper is to construct a graph G on n vertices which is a connected covering graph but has $2^{o(n)}$ diagram orientations. This provides a negative answer to a question of I. Rival.

Given a poset $P = (X, <)$ we denote by $G(P)$ the *covering graph* of P, i.e. the undirected graph (X, E) where $\{x, y\} \in E$ iff $x < y$ and $x < z < y$ for no z. The *diagram* $D(P)$ of P is the orientation of $G(P)$ given by the ordering of P.

One can characterize covering graphs as those graphs G which possess an acyclic orientation which does not contain a *quasicycle* (or a *bypass*), i.e. a set of edges of the form $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_1, x_n)$ for some $n > 2$.

Given a graph G denote by $o(G)$ the number of diagram orientations of G. It is a nontrivial question whether $o(G) > 0$, see e.g. [1, 4, 6]. However it seems that if $o(G) > 0$ then $o(G)$ is actually rather large.

During the recent Oberwolfach meeting on Partially Ordered Sets (organized by M. Aigner and R. Wille), I. Rival asked whether every covering graph has at least $2^{n/3}$ orientations. In this paper we construct diagrams of size n which have $2^{o(n)}$ orientations thus answering Rival's question in the negative. More exactly we prove the following theorem.

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Theorem. *For every sufficiently large n there exists a covering graph G of size n with at most* $2^{14N\cdot\log\log\log N/\log N}$ *orientations.*

Our approach is based on the lexicographic construction, combined with a lemma which is perhaps of an independent interest. It yields the existence of a covering graph with the maximal size of an independent set $o(n)$. This implies the existence of highly chromatic covering graphs (see [1–2, 5]) and provides an analogue with [3]:

Lemma *For n sufficiently large, there is a covering graph on n vertices with independence number at most* $6n\log\log\log n/\log n$.

Proof. Let $r = 2n\log\log\log n/\log n$. We shall assume for convenience that r divides n (clearly this does not materially affect the proof).

Consider the following probability space of random graphs on a totally ordered set $(V, <)$ with n vertices. We partition V into sets $V_1, \ldots, V_{n/r}$, each of r vertices, and join each pair of vertices by an edge with probability $p = \log n\log\log n/n\log\log\log n$, each pair considered independently. Then we remove all edges joining pairs of vertices in the same V_i.

Firstly, we prove that almost every such random graph G has independence number at most $2r$. Indeed, the probability that there is an independent set of size $2r$ in G is at most

$$\binom{n}{2r}(1-p)^{r^2},$$

since for every $2r$-subset of V there are at least r^2 potential edges, each in G with probability p. This quantity is at most

$$\left(\left(\frac{ne}{2r}\right)^2 e^{-pr}\right)^r = \left(\frac{e}{4\log\log\log n}\right)^{2r},$$

which is certainly less than $\frac{1}{4}$ for sufficiently large n.

Next we claim that the expected number of *bypasses* in $(G, <)$ is at most $r/2$. Clearly the only possible bypasses have all vertices in distinct V_i, so the expected number of bypasses is

$$\sum_{g=3}^{n/r}\binom{n/r}{g}r^g p^g \leq (1+rp)^{n/r} \leq (4\log\log n)^{\log n/(2\log\log\log n)}$$
$$\leq n^{1/2+1/\log\log\log n} \leq r/2 \quad \text{for sufficiently large } n.$$

So with probability at least $\frac{1}{2}$, a randomly chosen such G will have no more than r bypasses. We now take a G with independence number at most $2r$ and at most r

bypasses, and delete an edge of each bypass. The resulting graph is a covering graph with independence number at most $3r$, as desired. □

Proof of Theorem. It seems likely that almost every graph in the probability space of the above lemma has the properties we require: i.e., it has at most $2^{o(n)}$ diagram orientations. However, we choose to alter our graph by taking multiple copies of each vertex. It will be readily seen that the graph we construct in this manner does have the desired properties.

Let G be a covering graph of size n with independence number at most $6n \log\log\log n/\log n$, the existence of which is guaranteed by the lemma. The graph produced in the manner of the lemma is in fact almost surely connected, but in any case we can make it connected by adding single edges joining components, not increasing the independence number and keeping our graph a covering graph. Thus without loss of generality G is connected.

We now form a new graph G' by taking n copies $u_1, u_2, \ldots, u_n$ of each vertex u of G, and joining u_i and v_j by an edge iff u and v are adjacent in G. We claim that G' has the properties we seek. Certainly G' is a covering graph, with an orientation induced from G, and G' is connected. Thus it remains to prove that G' has few orientations as a diagram.

Given a diagram orientation of G', we call a vertex u of G *split* if there exists a vertex v of G and indices i, j, k such that (u_i, v_k) and (v_k, u_j) are edges of the orientation of G', Particularly $u_i < u_j$ in the poset generated by this orientation and we can see easily that the set $\{1, \ldots, n\}$ is partitioned into two sets L and V such that $u_i < u_j$ whenever $i \in L$, $j \in U$.

It is not difficult to see that u cannot split three ways, i.e. there cannot be i, j, k, such that $u_i < u_k < u_1$, but we shall prove a stronger statement in the following claim.

Claim. *Let G' be as above, with a given diagram orientation. Then if u is split, each u_i is either maximal or minimal in the poset generated by the orientation.*

Proof. Suppose not, and let u be a split vertex with u_i neither maximal nor minimal. Without loss of generality there is a j with $u_j > u_i$. But u_i has a lower cover v_k say. The edge $u_j v_k$ is also an edge of G', but it is not a covering relation, contradicting the assertion that we have a diagram orientation. □

Clearly if each u_i is maximal (minimal) then u is not split, so for each split u there is a u_i minimal in the orientation. Given a diagram orientation of G' we can obtain another one by *pushing down* all the maximal u_i for each split u, i.e., by reorienting all edges incident with the u_i. The reorientation has no split vertices, and so induces a diagram orientation of G, since the orientation of each edge $u_i v_j$ is independent of i and j.

Now we can count the reorientations of G'. Crudely, G has at most $n! \leqslant 2^{2n \log n}$ diagram orientations. In each such orientation, the set of minimals is an independent set, so has at most $6n \log\log\log n/\log n$ elements. Each orientation of G' inducing our chosen orientation of G is specified by a subset of the $n \times 6n \log\log\log n/\log n$ minimal elements of G' which have been pushed down. Thus there are at most $2^{6n^2 \log\log\log n/\log n}$ orientations of G' giving rise to each orientation of G. Setting $N = n^2$, the total number of vertices of G', we thus see that the total number of diagram orientations of G' is at most $2^{14N \log\log\log N/\log N}$. □ (Theorem)

We do not know what the correct asymptotic expression is for the minimum number of orientations of a covering graph. It could conceivably be as small as $2^{c\sqrt{n}}$, but the most plausible value is $2^{cn/\log n}$.

References

[1] B. Bollobás, Covering lattices. Alg. Universalis 7 (1977) 313–319.
[2] B. Bollobás, G. Brightwell and J. Nešetřil, Random graphs and covering graphs of posets, Order 3 (1985) 245–255.
[3] P. Erdős, Graph theory and probability. Canad. J. Math. 2 (1959) 34–38.
[4] J. Nešetřil, V. Rödl, On a probabilistic graph theoretical method, Proc. Amer. Math. Soc. 72 (1978) 417–421.
[5] J. Nešetřil, V. Rödl, Combinatorial partitions of finite lattices and posets Ramsey lattices, Algebra Univ. 19 (1984) 106–119.
[6] J. Nešetřil, V. Rödl, Complexity of diagrams, Order 3 (1987) 321–330.