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Trivial lagrangians in field theory*

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Abstract: The paper presents a complete description of trivial lagrangians in field theory. It is shown that any higher order trivial lagrangian can be expressed as the horizontal component of the exterior derivative of a projectable form.

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1. Introduction

A lagrangian of order r is called *trivial*, or *null*, if its Euler–Lagrange form vanishes identically. The problem of finding all trivial lagrangians belongs to the most difficult problems of the geometrical variational theory. It can be easily seen that the well-known classical result stating that each trivial lagrangian is of the *divergence type* should be reformulated more precisely because the problem is connected, via the Stokes integral theorem, rather with the exterior derivative than the divergence operator.

Partial results, concerning the *first order* trivial lagrangians, have been obtained by several authors (see, e.g., Hojman [7], Krupka [8, 12], Rund [19], and the references in Olver [18]). A complete characterization of trivial lagrangians of the 1st order has been given, within the geometric variational theory on fibered manifolds, by Krupka [11] (see also [9]). According to this theory, a lagrangian λ defined on the first jet prolongation J^1Y of a fibered manifold Y over an n-dimensional base X is trivial if and only if it has the form of the horizontal component of a closed n-form defined on Y, i.e., $\lambda = h\eta$, $d\eta = 0$. This result shows, in particular, that there are much more first order trivial lagrangians than the divergencies.

Several partial results for higher order lagrangians have been obtained by Aldersley [1], Ball, Currie, and Olver [4], Olver [18] (polynomial lagrangians), Krupka [11] (relations with Lepage forms). Implicit characterization of trivial lagrangians has been provided by the variational bicomplex theory (Anderson [2], Anderson and Duchamp [3], Chrastina [5], Dedecker and

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Tulczyjew [6], Saunders [20], Tulczyjew [23], Vinogradov [24], and others), and by the theory of finite order variational sequences (Krupka [15, 16]).

The aim of this paper is to prove a complete analog of the above mentioned result [11] for lagrangians of all orders. This improves a description of trivial lagrangians as given by Anderson and Duchamp [3], and corrects the proof of the same result given by Krupka in [15].

2. Decompositions of forms

In this section we summarize some results on the decomposition of forms on jet spaces into their contact components. For more details we refer the reader to [17].

As the underlying space we use an $(n+m)$-dimensional fibered manifold Y over an n-dimensional base X, with projection $\pi : Y \to X$. The r-jet prolongation of Y is denoted by J^rY, $\pi^r : J^rY \to X$ and $\pi^{r,s} : J^rY \to J^sY$, $r \geqslant s \geqslant 0$, being the corresponding canonical projections. A fibered chart on Y, the associated chart on the base and the associated fibered chart on J^rY are denoted by (V, ψ), $\psi = (x^i, y^\sigma)$, (U, φ), $\varphi = (x^i)$, and (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_r})$, respectively. If $W \subset Y$ is an open set, we denote $\Omega^r_0 W$ the ring of smooth functions on $W^r = (\pi^{r,0})^{-1}(W)$, and $\Omega^r_q W$ the $\Omega^r_0 W$-module of smooth q-forms on W^r. The forms $(dx^i, \omega^\sigma_{j_1}, \ldots, \omega^\sigma_{j_1 \ldots j_{r-1}}, dy^\sigma_{j_1 \ldots j_r})$, where

$$\omega^\sigma_{j_1 \ldots j_k} = dy^\sigma_{j_1 \ldots j_k} - y^\sigma_{j_1 \ldots j_k i} dx^i, \tag{1}$$

define a basis of 1-forms on V^r.

Let $\varrho \in \Omega^r_q V$ be a form. There is a unique decomposition

$$(\pi^{r+1,r})^* \varrho = h\varrho + p\varrho = h\varrho + p_1\varrho + \cdots + p_q\varrho \tag{2}$$

of the form ϱ into its *horizontal*, or 0-*contact*, *component* $h\varrho = p_0\varrho$ and the k-*contact components* $p_k\varrho$, $1 \leqslant k \leqslant q$. Denote by $\binom{\sigma}{I}$ the multiindices $\binom{\sigma}{j_1 \ldots j_s}$ for $0 \leqslant s \leqslant r$, $s = |I|$ being the *length* of the multiindex I. We also use the following notations

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{|J|<r} \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Ji} + \frac{\partial f}{\partial y^\sigma_I} y^\sigma_{Ii} = d'_i f + \frac{\partial f}{\partial y^\sigma_I} y^\sigma_{Ii},$$

$$pdf = p'df + \frac{\partial f}{\partial y^\sigma_I} \omega^\sigma_I, \quad |I| = r,$$

where $f \in \Omega^r_0 V$; d_i is the *total derivative* with respect to the variable x^i.

Lemma 1. *Let $W \subset Y$ be an open set, $q \geqslant 1$ an integer, and $\varrho \in \Omega^r_q W$ a q-form. Let (V, ψ) be a fibered chart on Y for which $V \subset W$. Suppose that the chart expression of ϱ is*

$$\varrho = \sum_{s=0}^{q} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_s}_{\sigma_s\, i_{s+1} i_{s+2} \ldots i_q} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_s}_{J_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \ldots \wedge dx^{i_q} \tag{3}$$

with coefficients antisymmetric in all multiindices $(\binom{I_1}{\sigma_1}, \ldots, \binom{I_s}{\sigma_s})$, $0 \leqslant |I_p| \leqslant r$, $1 \leqslant p \leqslant s$, antisymmetric in all indices $(i_{s+1}, \ldots, i_q)$ and symmetric in all indices within each multiindex I_p.

Then the k-contact component of ϱ has the chart expression

$$p_k\varrho = B^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_k}_{\sigma_k\, i_{k+1}i_{k+2}\dots i_q}\,\omega^{\sigma_1}_{I_1}\wedge\omega^{\sigma_2}_{I_2}\wedge\dots\wedge\omega^{\sigma_k}_{I_k}\wedge dx^{i_{k+1}}\wedge dx^{i_{k+2}}\wedge\dots\wedge dx^{i_q}, \tag{4}$$

$$B^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_k}_{\sigma_k\, i_{k+1}i_{k+2}\dots i_q} = \sum_{s=k}^{q}\binom{q-k}{q-s}A^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_k}_{\sigma_k}\cdots{}^{I_s}_{\sigma_s\, i_{s+1}i_{s+2}\dots i_q}\,y^{\sigma_{k+1}}_{I_{k+1}i_{k+1}}\cdots y^{\sigma_s}_{I_s i_s}, \tag{5}$$
$$\mathrm{alt}(i_{k+1}i_{k+2}\dots i_q),\quad 0\leqslant k\leqslant q.$$

A proof of Lemma 1 is given in [17].

A form $\varrho\in\Omega^r_q V$ is called π^r*-horizontal* if $(\pi^{r+1,r})^*\varrho = h\varrho$. It is called *contact* if $h\varrho = 0$. Every q-form with $q > n$ is contact. A q-form ϱ, $n < q \leqslant N$, is called *strongly contact* if $p_{q-n}\varrho = 0$. A q-form is called *decomposable* if it is the sum of a horizontal form and a contact form (for $1\leqslant q\leqslant n$), or the sum of a $(q-n)$-contact and a strongly contact form (for $n < q\leqslant N$).

Lemma 2. *Let $W\subset Y$ be an open set, q an integer, $1\leqslant q\leqslant n$, and $\varrho\in\Omega^r_q W$ a form. Let (V,ψ) be any fibered chart on Y for which $V\subset W$. Then the form ϱ is contact if and only if it can be expressed as*

$$\begin{aligned}\varrho &= \Phi^J_\sigma\,\omega^\sigma_J, & q&=1, & 0&\leqslant|J|\leqslant r-1,\\ \varrho &= \omega^\sigma_J\wedge\Psi^J_\sigma + d\Psi, & 2&\leqslant q\leqslant n, & 0&\leqslant|J|\leqslant r-1,\end{aligned} \tag{6}$$

$\Phi^J_\sigma\in\Omega^r_0 V$ being some functions, $\Psi^J_\sigma\in\Omega^r_{q-1}V$ some $(q-1)$-forms, and $\Psi\in\Omega^r_{q-1}V$ is a contact $(q-1)$-form which can be expressed as $\omega^\sigma_I\wedge\chi^I_\sigma$ for some $(q-2)$-forms $\chi^I_\sigma\in\Omega^r_{q-2}V$, $|I| = r-1$.

We say that the forms $\varrho_0 = \omega^\sigma_J\wedge\Psi^J_\sigma$, $\Psi = \omega^\sigma_I\wedge\chi^I_\sigma$ in (6) are *generated* by the forms ω^σ_J, $0\leqslant|J|\leqslant r-1$.

3. Projectability of forms

In this section we show that a q-form η defined on J^rY, $1\leqslant q\leqslant n-1$, whose exterior derivative is decomposable, can be locally expressed as the sum of a closed form, a contact form and a $\pi^{r,r-1}$-projectable form.

In what follows, alt (resp. sym) denotes antisymmetrization (resp. symmetrization).

Lemma 3. *Let $W\subset Y$ be an open set, q an integer, $1\leqslant q\leqslant n$. Let (V,ψ) be any fibered chart on Y for which $V\subset W$. A π^r-horizontal form $\varrho\in\Omega^r_q W$ with the chart expression*

$$\varrho = A_{i_1 i_2\dots i_q}\,dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_q}$$

is $\pi^{r,r-1}$-projectable if and only if

$$\frac{\partial A_{i_1 i_2\dots i_q}}{\partial y^\sigma_I}\,\delta^j_{i_0} = 0,\quad \mathrm{alt}(i_0 i_1 i_2\dots i_q),\quad \mathrm{sym}(Ij),\quad |I| = r. \tag{7}$$

Proof. If a π^r-horizontal form ϱ is $\pi^{r,r-1}$-projectable, then its coefficients $A_{i_1 i_2 \dots i_q}$ are defined on V^{r-1}. Thus, conditions (7) are satisfied trivially. Consequently, only the converse needs proof. Since ϱ is π^r-horizontal we can write

$$(\pi^{r+1,r})^* d\varrho = hd\varrho + p_1 d\varrho,$$

$$hd\varrho = hdA_{i_1 i_2 \dots i_q} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q}$$

$$= \left(d'_{i_0} A_{i_1 i_2 \dots i_q} + \frac{\partial A_{i_1 i_2 \dots i_q}}{\partial y^\sigma_{j_1 j_2 \dots j_r}} y^\sigma_{j_1 j_2 \dots j_r i_0} \right) dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q},$$

$$p_1 d\varrho = \sum_{k=0}^{r} \frac{\partial A_{i_1 i_2 \dots i_q}}{\partial y^\nu_{j_1 j_2 \dots j_k}} \omega^\nu_{j_1 j_2 \dots j_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q}.$$

Suppose that conditions (7) are satisfied. This immediately implies that

$$\frac{\partial A_{i_1 i_2 \dots i_q}}{\partial y^\sigma_{j_1 j_2 \dots j_r}} y^\sigma_{j_1 j_2 \dots j_r i_0} dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} = 0$$

and thus $hd\varrho$ is $\pi^{r+1,r}$-projectable. This leads to the $\pi^{r+1,r}$-projectability of $p_1 d\varrho$ as well. Then

$$\frac{\partial A_{i_1 i_2 \dots i_q}}{\partial y^\nu_{j_1 j_2 \dots j_r}} = 0,$$

proving $\pi^{r,r-1}$-projectability.

Theorem 1. *Let $W \subset Y$ be an open set, q an integer, $1 \leqslant q \leqslant n-1$, and $\eta \in \Omega^r_q V$ a form. The following two conditions are equivalent*:

(i) *$hd\eta$ is $\pi^{r,r-1}$-projectable.*

(ii) *For every fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y such that $V \subset W$ there exist a form $\chi \in \Omega^{r-1}_q V$, a contact form $\nu \in \Omega^r_q V$ and a form $\tau \in \Omega^r_{q-1} V$ such that*

$$\eta = (\pi^{r,r-1})^* \chi + \nu + d\tau. \tag{8}$$

Proof. Since the condition (ii) obviously implies (i), only the converse needs proof. We proceed in several steps.

1. Let $\eta \in \Omega^r_q V$, and let $hd\eta$ be $\pi^{r+1,r}$-projectable. Then the form $pd\eta = p_1 d\eta + \dots + p_q d\eta$ is $\pi^{r+1,r}$-projectable and contact, and thus, by Lemma 2, it is of the form $pd\eta = \varrho_0 + d\nu_0$ where both ϱ_0 and ν_0 are contact and generated by the 1-forms ω^σ_J with $0 \leqslant |J| \leqslant r-1$. Consequently, $d\eta = hd\eta + \varrho_0 + d\nu_0$, so that $hd\eta + \varrho_0 = d\eta_0$ for some form $\eta_0 \in \Omega^r_{q-1}$. Integrating we obtain $\eta = \eta_0 + \nu_0 + d\tau_0$ where τ_0 is a $(q-1)$-form. The form η_0 has the following basic properties:

(a) $hd\eta_0 = hd\eta$, i.e., $hd\eta_0$ is $\pi^{r+1,r}$-projectable,

(b) $p_s d\eta_0 = p_s \varrho_0$ for $1 \leqslant s \leqslant q+1$, which implies that the forms $p_1 d\eta_0, \dots, p_{q+1} d\eta_0$ are generated by the forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$.

In what follows, we construct a suitable sequence of triples (η_k, ν_k, τ_k), $1 \leqslant k \leqslant q$, such that ν_k and $p_1 d\eta_k, \dots, p_{q-k+1} d\eta_k$ are generated by the forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$, and $p_{q-k+1}\eta_k = \dots = p_q \eta_k = 0$, and, moreover, $\eta_{k-1} = \eta_k + \nu_k + d\tau_k$ (indeed, $p_{q-k+2} d\eta_k = \dots = p_{q+1} d\eta_k = 0$). By construction, we get for $k = q$ a π^r-horizontal form η_q which satisfies the relation $hd\eta_q = hd\eta_{q-1} = \cdots = hd\eta_1 = hd\eta_0 = hd\eta$. Finally, we use the $\pi^{r+1,r}$-projectability of $hd\eta_q$, and we prove that the form η_q can be expressed as $(\pi^{r,r-1})^*\chi$.

2. Taking into account that $(\pi^{r+1,r})^* d\eta_0 = d(\pi^{r+1,r})^*\eta_0$, we can easily obtain the identities

$$\begin{aligned}
&(\pi^{r+2,r+1})^* hd\eta_0 = hdh\eta_0,\\
&(\pi^{r+2,r+1})^* p_k d\eta_0 = p_k dp_{k-1}\eta_0 + p_k dp_k\eta_0, \quad 1 \leqslant k \leqslant q,\\
&(\pi^{r+2,r+1})^* p_{q+1} d\eta_0 = p_{q+1} dp_q \eta_0.
\end{aligned} \tag{9}$$

Suppose the form η_0 to be expressed as in (3) and decomposed into its contact components by Lemma 1. Then by the first part of this proof, and by the last of equations (9), the form

$$\begin{aligned}
p_{q+1} d\eta_0 &= p d B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_q}_{I_q}\\
&= p' d B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_q}_{I_q}\\
&\quad + \frac{\partial B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q}}{\partial y^{\sigma_0}_{I_0}} \omega^{\sigma_0}_{I_0} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_q}_{I_q}
\end{aligned} \tag{10}$$

where $|I_0| = r$, should be generated by the forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$. Taking the terms with $|I_0|, |I_1|, \ldots, |I_q| = r$, we obtain

$$\frac{\partial B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q}}{\partial y^{\sigma_0}_{I_0}} \omega^{\sigma_0}_{I_0} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_q}_{I_q} = 0 \tag{11}$$

which implies

$$\frac{\partial B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q}}{\partial y^{\sigma_0}_{I_0}} = 0, \qquad \operatorname{alt}\left(\binom{\sigma_0}{I_0}\binom{\sigma_1}{I_1}\cdots\binom{\sigma_q}{I_q}\right). \tag{12}$$

But by (5), $B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q} = A^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q}$ and thus

$$\frac{\partial A^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_q}_{\sigma_q}}{\partial y^{\sigma_0}_{I_0}} = 0, \qquad \operatorname{alt}\left(\binom{\sigma_0}{I_0}\binom{\sigma_1}{I_1}\cdots\binom{\sigma_q}{I_q}\right), \qquad |I_0|, |I_1|, \ldots, |I_q| = r. \tag{13}$$

Define a mapping $\chi : \mathbb{R} \times V^r \to V^r$ by

$$\chi(s, (x^i, y^\sigma, y^{\sigma_1}_{j_1}, \ldots, y^\sigma_{j_1\ldots j_{r-1}}, y^\sigma_{j_1\ldots j_r})) = (x^i, y^\sigma, y^{\sigma_1}_{j_1}, \ldots, y^\sigma_{j_1\ldots j_{r-1}}, s y^\sigma_{j_1\ldots j_r}).$$

and consider the $(q-1)$-form

$$\tau_1 = C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_q}_{\sigma_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}, \tag{14}$$

where

$$C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_q}_{\sigma_q} = q y^\nu_J \int_0^1 (A^{J I_2 I_3}_{\nu\sigma_2\sigma_3} \cdots {}^{I_q}_{\sigma_q} \circ \chi) \cdot s^{q-1}.$$

Then

$$d\tau_1 = d'C^{I_2 I_3}_{\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}$$
$$+ \frac{\partial C^{I_2 I_3}_{\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^{\sigma_1}_{I_1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}, \qquad |I_1| = r.$$

The second term in $d\tau_1$ can be expressed as

$$\frac{\partial C^{I_2 I_3}_{\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^{\sigma_1}_{I_1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}$$
$$= q\int_0^1 \left(A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q} + \frac{\partial A^{J I_2 I_3}_{\nu\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^{\sigma_1}_{I_1}} y^\nu_J \right) \circ \chi \cdot s^{q-1}\, ds \tag{15}$$
$$\times\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}.$$

The expression in parentheses can be replaced by the antisymmetrized one which has the form

$$A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q} + \frac{1}{q}\left(\frac{\partial A^{J I_2 I_3}_{\nu\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^{\sigma_1}_{I_1}} - \frac{\partial A^{J I_1 I_3}_{\nu\sigma_1\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^{\sigma_2}_{I_2}} - \ldots - \frac{\partial A^{J I_2 I_3}_{\nu\sigma_2\sigma_3}\cdots{}^{I_{q-1} I_1}_{\sigma_{q-1}\sigma_1}}{\partial y^{\sigma_q}_{I_q}} \right) y^\nu_J.$$

Taking into account (13) and interchanging the multiindices $\binom{I_1}{\sigma_1}$, $\binom{J}{\nu}$, we finally obtain

$$\frac{\partial C^{I_2 I_3}_{\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^{\sigma_1}_{I_1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}$$
$$= q\int_0^1 \left(A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q} + \frac{1}{q}\frac{\partial A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}}{\partial y^\nu_J} y^\nu_J \right) \circ \chi \cdot s^{q-1}\, ds \cdot dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}$$
$$= q\int_0^1 \frac{d}{ds}\left(A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q} \circ \chi \cdot \frac{1}{q} s^q \right) ds \cdot dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}$$
$$= A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q}\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q}.$$

Consequently,

$$\eta_0 = A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q}$$
$$+ \sum_{k=1}^{q-1} A^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2}\ldots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_q}$$
$$+ \sum A^{J_1 J_2}_{\sigma_1\sigma_2}\cdots{}^{J_q}_{\sigma_q} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_q}_{J_q} \tag{16}$$
$$+ d\tau_1 - d'C^{I_2 I_3}_{\sigma_2\sigma_3}\cdots{}^{I_q}_{\sigma_q} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_q}_{I_q},$$

where at least one of the summation multiindices in the sum $\sum A^{J_1 J_2}_{\sigma_1\sigma_2}\cdots{}^{J_q}_{\sigma_q} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_q}_{J_q}$ is of the length lower than r. Now we write (16) in terms of the basis $(dx^i, \omega^\sigma_J, dy^\sigma_I)$, where $0 \leqslant |J| \leqslant r-1$, $|I| = r$. Using the decompositions

$$dy^\sigma_J = \omega^\sigma_J + y^\sigma_{Jj}dx^j, \quad d'C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_q}_{\sigma_q} = hd'C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_q}_{\sigma_q} + pd'C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_q}_{\sigma_q}$$

we get for η_0 the expression $\eta_0 = \eta_1 + \nu_1 + d\tau_1$, where

$$\eta_1 = \tilde{A}_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q} + \sum_{k=1}^{q-1} \tilde{A}^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+1} \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_q} \tag{17}$$

with proper coefficients, and ν_1 is a contact form generated by forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$. Clearly $hd\eta_0 = hd\eta_1$, which implies that the form $hd\eta_1$ is $\pi^{r+1,r}$-projectable.

3. Consider the form η_1 in (17) written, for simplicity, with the coefficients denoted again as $A^{I_1 I_2}_{\sigma_2\sigma_3} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \ldots i_q}$ instead of $\tilde{A}^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \ldots i_q}$. Recall that $hd\eta_1$ is $\pi^{r+1,r}$-projectable. It can be easily seen, from (17), that $p_q\eta_1 = 0$, and thus

$$(\pi^{r+1,r})^*\eta_1 = h\eta_1 + \sum_{k=1}^{q-1} p_k\eta_1 .$$

The k-contact components of the form η_1, with $0 \leqslant k \leqslant q-1$, have again the form (4) with coefficients $B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \ldots i_q}$ related to $A^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_s}_{\sigma_s\, i_{s+1} i_{s+2} \ldots i_q}$, $k \leqslant s \leqslant q-1$, by the expressions (5). Especially, for $k = q-1$ we have

$$p_{q-1}\eta_1 = B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}\, \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q}.$$

Using the relation $(\pi^{r+2,r+1})^* p_q d\eta_1 = p_q dp_{q-1}\eta_1$, resulting from the decompositions (9) written for η_1 and the fact that $p_q\eta_1 = 0$, we can write

$$\begin{aligned} p_q d\eta_1 &= pdB^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ &= p'dB^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ &\quad + \frac{\partial B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}}}{\partial y^{\sigma_0}_{I_0}} \omega^{\sigma_0}_{I_0} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \ldots \wedge \omega^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q}, \qquad |I_0| = r. \end{aligned}$$

But, by $\pi^{r+1,r}$-projectability, $p_q d\eta_1$ should be generated by the forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$. Taking the terms labelled by multiindices such that $|I_0|, |I_1|, |I_2| \ldots, |I_{q-1}| = r$ we obtain for the coefficients $B^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ and $A^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ the relations completely analogous to (11–13). The same procedure as in the part 2 of this proof leads to the following decomposition of η_1

$$\eta_1 = \eta_2 + \nu_2 + d\tau_2 \tag{18}$$

in which ν_2 is contact and generated by forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$, the form τ_2 is given by

$$\tau_2 = C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q}$$

$$C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q} = (q-1) y^\nu_J \int_0^1 A^{J I_2 I_3}_{\nu\sigma_2\sigma_3} \cdots {}^{I_{q-1}}_{\sigma_{q-1},\, i_q} \circ \chi \cdot s^{q-2}\, ds$$

and η_2 is of the form

$$\eta_2 = \tilde{A}_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} + \sum_{k=1}^{q-2} \tilde{A}^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_q}. \tag{19}$$

It holds $hd\eta_1 = hd\eta_2$ and thus $hd\eta_2$ is $\pi^{r+1,r}$-projectable.

4. Now, we proceed by induction. As the induction hypothesis, we suppose that

$$\eta_{q-p} = A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} + \sum_{k=1}^{p} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_q}, \tag{20}$$

where $0 \leqslant |I_1|, |I_2|, \dots, |I_p| \leqslant r$, is a form for which $hd\eta_{q-p}$ is $\pi^{r+1,r}$-projectable. We wish to show that η_{q-p} can be written as

$$\eta_{q-p} = \eta_{q-p+1} + \nu_{q-p+1} + d\tau_{q-p+1}, \tag{21}$$

where τ_{q-p+1} is a q-form, ν_{q-p+1} is a contact form generated by the forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$, and η_{q-p+1} is given by

$$\eta_{q-p+1} = \tilde{A}_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} + \sum_{k=1}^{p-1} \tilde{A}^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_q}. \tag{22}$$

For $p = q$ and $p = q-1$ this hypothesis is satisfied (see parts 2 and 3 of this proof). For the form η_{q-p} given by (22) it holds $p_{p+1}\eta_{q-p} = 0$, thus

$$(\pi^{r+1,r})^* \eta_{q-p} = h\eta_{q-p} + \sum_{k=1}^{p} p_k \eta_{q-p}$$

with components $h\eta_{q-p}, p_1\eta_{q-p}, \dots, p_p\eta_{q-p}$ given by (4), the coefficients being expressed by (5). Then, by analogy with (9), we have $(\pi^{r+2,r+1})^* p_{p+1} d\eta_{q-p} = p_{p+1} dp_p \eta_{q-p}$. Thus, the form

$$\begin{aligned} p_{p+1} d\eta_{q-p} &= p' dB^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \dots \wedge dx^{i_q} \\ &+ \frac{\partial B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q}}{\partial y^{\sigma_0}_{I_0}} \omega^{\sigma_0}_{I_0} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \dots \wedge dx^{i_q}, \end{aligned}$$

where $|I_0| = r$, should be generated by the forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$. Taking the terms with $|I_0|, |I_1|, \dots, |I_q| = r$, we obtain

$$\frac{\partial B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q}}{\partial y^{\sigma_0}_{I_0}} \omega^{\sigma_0}_{I_0} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \dots \wedge dx^{i_q} = 0.$$

This implies, together with (5),

$$B^{I_1 I_2 \cdots I_p}_{\sigma_1\sigma_2 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q} = A^{I_1 I_2 \cdots I_p}_{\sigma_1\sigma_2 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q},$$

$$\frac{\partial A^{I_1 I_2 \cdots I_p}_{\sigma_1\sigma_2 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}}{\partial y^{\sigma_0}_{I_0}} = 0, \quad \mathrm{alt}(\tbinom{\sigma_0}{I_0}\tbinom{\sigma_1}{I_1}\ldots\tbinom{\sigma_p}{I_p}), \quad |I_0|, |I_1|, \ldots, |I_p| = r. \tag{23}$$

Consider the $(q-1)$-form

$$\tau_{q-p+1} = C^{I_2 I_3 \cdots I_p}_{\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}\, dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q},$$

where

$$C^{I_2 I_3 \cdots I_p}_{\sigma_2\sigma_3 \cdots \sigma_p} = p y^{\nu}_J \int_0^1 (A^{J I_2 I_3 \cdots I_p}_{\nu\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q} \circ \chi) \cdot s^{p-1}\, ds.$$

Then

$$d\tau_{q-p+1} = d' C^{I_2 I_3 \cdots I_p}_{\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}$$
$$+ \frac{\partial C^{I_2 I_3 \cdots I_p}_{\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}}{\partial y^{\sigma_1}_{I_1}}\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q},$$

where $|I_1| = r$. The second term in $d\tau_{q-p+1}$ can be expressed as

$$\frac{\partial C^{I_2 I_3 \cdots I_p}_{\sigma_2\sigma_3 \cdots \sigma_p\, i+p+1 i_{p+2}\ldots i_q}}{\partial y^{\sigma_1}_{I_1}}\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}$$
$$= p \int_0^1 \left(A^{I_1 I_2 I_3 \cdots I_p}_{\sigma_1\sigma_2\sigma_3 \cdots \sigma_p} + \frac{\partial A^{J I_2 I_3 \cdots I_p}_{\nu\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}}{\partial y^{\sigma_1}_{I_1}}\, y^{\nu}_J \right) \circ \chi \cdot s^{p-1} ds$$
$$\times dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p-1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}.$$

The expression in parenthesis can be replaced by the antisymmetrized one which has the form

$$A^{I_1 I_2 I_3 \cdots I_p}_{\sigma_1\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}$$
$$+ \frac{1}{p}\left(\frac{\partial A^{J I_2 I_3 \cdots I_p}_{\nu\sigma_2\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}}{\partial y^{\sigma_1}_{I_1}} - \frac{\partial A^{J I_1 I_3 \cdots I_p}_{\nu\sigma_1\sigma_3 \cdots \sigma_p\, i_{p+1} i_{p+2}\ldots i_q}}{\partial y^{\sigma_2}_{I_2}} - \ldots \right.$$
$$\left. - \frac{\partial A^{J I_2 I_3 \cdots I_{p-1} I_1}_{\nu\sigma_2\sigma_3 \cdots \sigma_{p-1}\sigma_1\, i_{p+1} i_{p+2}\ldots i_q}}{\partial y^{\sigma_p}_{I_p}} \right) y^{\nu}_J.$$

Taking into account (23) and interchanging the multiindices $\binom{I_1}{\sigma_1}$, $\binom{J}{\nu}$, we obtain (see the same procedure in the part 2 of this proof)

$$\frac{\partial C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q}}{\partial y^{\sigma_1}_{I_1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}$$

$$= p \int_0^1 \frac{d}{ds} \left(A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_Q} \circ \chi \cdot \frac{1}{p} s^p \right) ds$$

$$\times\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}$$

$$= A^{I_1 I_2 I_3}_{\sigma_1\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}.$$

Finally

$$\begin{aligned} \eta_{q-p} &= A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q} \\ &+ \sum_{k=1}^{p-1} A^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k\, i_{k+1} i_{k+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_q} \\ &+ \sum A^{J_1 J_2}_{\sigma_1\sigma_2} \cdots {}^{J_q}_{\sigma_q} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_p}_{J_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q} + d\tau_{q-p+1} \\ &- d' C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q} \wedge dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q} \end{aligned} \tag{24}$$

where at least one of the multiindices in the sum $\sum A^{J_1 J_2}_{\sigma_1\sigma_2} \cdots {}^{J_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \ldots \wedge dy^{\sigma_p}_{J_p} \wedge dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \ldots \wedge dx^{i_q}$ is of length lower than r. Now we write (24) in terms of the basis $(dx^i, \omega^\sigma_J, dy^\sigma_I)$, where $0 \leqslant |J| \leqslant r-1$, $|I| = r$. Using the decompositions

$$dy^\sigma_J = \omega^\sigma_J + y^\sigma_{Jj}\, dx^j$$

and

$$d' C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q} = hd' C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q} + pd' C^{I_2 I_3}_{\sigma_2\sigma_3} \cdots {}^{I_p}_{\sigma_p\, i_{p+1} i_{p+2} \dots i_q}$$

we get for η_{q-p} the expression (21) in which η_{q-p+1} has the form (22) and ν_{q-p+1} is the contact form generated by forms ω^σ_J, $0 \leqslant |J| \leqslant r-1$. Clearly $hd\eta_{q-p} = hd\eta_{q-p+1}$, which implies that the form $hd\eta_{q-p+1}$ is $\pi^{r+1,r}$-projectable.

5. For $p = 1$, formula (21) gives

$$\eta_{q-1} = \eta_q + \nu_q + d\tau_q$$

in which η_q has the form

$$\eta_q = A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q}.$$

Since the form

$$\begin{aligned} hd\eta_q &= dA_{i_1 i_2 \dots i_q} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q} \\ &= \left(hd' A_{i_1 i_2 \dots i_q} + \frac{\partial A_{i_1 i_2 \dots i_q}}{\partial y^\sigma_I} y^\sigma_{I i_0}\, dx^{i_0} \right) \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q} \end{aligned}$$

is again $\pi^{r+1,r}$-projectable, we obtain

$$\frac{\partial A_{i\,i_2\ldots i_q}}{\partial y_I^\sigma}\delta^j_{i_0} = 0 \quad \mathrm{alt}(i_0 i_1 i_2 \ldots i_q), \quad \mathrm{sym}(Ij), \quad |I| = r.$$

Thus, it is evident that the form η_q is $\pi^{r,r-1}$-projectable by Lemma 3. Denoting by χ the form for which $\eta_q = (\pi^{r,r-1})^*\chi$ we complete the proof.

4. Variational sequence

In this section, we briefly recall basic concepts of the calculus of variations, related to the Euler–Lagrange mapping (see, e.g., [17]).

Let (Y, π, X) be a fibered manifold, $\dim Y = n + m$, $\dim X = n$, and J^rY its r-jet prolongation, $\dim J^rY = N$, $\pi^r : J^rY \to X$, $\pi^{r,s} : J^rY \to J^sY$ the canonical projections (see Section 2). Let $\gamma : X \to Y$ be a section of the manifold (Y, π, X) and $J^r\gamma : X \to Y$ its r-jet prolongation. Any π^r-horizontal n-form $\lambda \in \Omega^r_n Y$ is called a *lagrangian* of the order r; (π, λ) is called a *Lagrange structure*. In a fibered chart (V, ψ) on Y and the associated fibered chart (V^r, ψ^r) on J^rY we have

$$\lambda = \mathcal{L}(x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n. \tag{25}$$

For a given λ and any compact, n-dimensional submanifold $\Omega \subset X$ with boundary, we get a real-valued function

$$\gamma \to \int_\Omega J^r\gamma^*\lambda \tag{26}$$

defined on the set of sections of (Y, π, X), called the *action function* associated with λ and Ω. The *first variational formula* for (26) can be derived in an intrinsic way by means of the Lepage forms, the Lie derivative, and the exterior derivative d. This involves the introducing the global *Euler–Lagrange form* associated with λ

$$E_\lambda = E_\sigma(\mathcal{L})\, dy^\sigma \wedge dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \tag{27}$$

$$E_\sigma(\mathcal{L}) = \frac{\partial\mathcal{L}}{\partial y^\sigma} + \sum_{k=1}^{r}(-1)^k d_{i_1} d_{i_2} \ldots d_{i_k}\left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k}}\right). \tag{28}$$

$E_\sigma(\mathcal{L})$ being the *Euler–Lagrange expressions*. The mapping $\lambda \to E_\lambda$ is called the *Euler–Lagrange mapping*. The kernel of this mapping describes *trivial* or *null* lagrangians, its image describes *variational* $(n+1)$*-forms*.

Let us now take into account the notation introduced in Section 2; Ω^r_q, $q \geqslant 0$, is the *direct image* of the sheaf of smooth q-forms over J^rY by the jet projection $\pi^{r,0}$. For $1 \leqslant q \leqslant n$, resp. for $q \geqslant n + 1$, denote by $\Omega^r_{q,c}$ the subsheaf of contact, resp, strongly contact, forms ($\Omega^r_{0,c} = \{0\}$). Define $\Theta^r_q = \Omega^r_{q,c} + d\Omega^r_{q-1,c}$, $q \geqslant 1$, where $d\Omega^r_{q-1,c}$ is the *image sheaf* of $\Omega^r_{q-1,c}$ by d. We get the subsequence

$$0 \to \Theta^r_1 \to \Theta^r_2 \to \cdots \to \Theta^r_P \to 0, \qquad P = m\binom{n+r-1}{r-1}$$

of the deRham sequence

$$0 \to \mathbb{R}_Y \to \Omega_1^r \to \Omega_2^r \to \cdots \to \Omega_N^r \to 0$$

and the corresponding quotient sequence

$$0 \to \mathbb{R}_Y \to \Omega_1^r/\Theta_1^r \to \Omega_2^r/\Theta_2^r \to \cdots \to \Omega_P^r/\Theta_P^r \to \Omega_{P+1}^r \to \cdots \to \Omega_N^r \to 0 \quad (29)$$

which is called the *variational sequence*. The *quotient mapping*, denoted by E, is defined by putting $E([\varrho]) = [d\varrho]$, where $[\eta]$ denotes the class of the form η. It can be shown that the variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}_Y$. Consequently, by the abstract deRham theorem, the complex of global sections of the variational sequence has the same cohomology as the manifold Y. Denoting the sequence (29) by $0 \to \mathbb{R}_Y \to \mathcal{V}^r$, and the corresponding cochain complex of global sections by $0 \to \Gamma(Y, \mathbb{R}_Y) \to \Gamma(Y, \Omega_1^r/\Theta_1^r) \to \Gamma(Y, \Omega_2^r/\Theta_2^r) \to \cdots$, or simply by $\Gamma(Y, \mathcal{V}^r)$, we have $H^k(\Gamma(\mathbb{R}_Y, \mathcal{V}^r)) = H^k(Y, \mathbb{R})$.

A basic observation connecting the variational sequence with the calculus of variations comes from the analysis of the $(n-1)$-, n- and $(n+1)$-terms in (29), so called *variational terms*. Describing the sheaf Ω_n^r/Θ_n^r, resp. $\Omega_{n+1}^r/\Theta_{n+1}^r$, as a certain subsheaf the sheaf of forms Ω_n^{r+1}, resp. Ω_{n+1}^{2r+1}, one can easily see that the corresponding representation of the quotient mapping $E : \Omega_n^r/\Theta_n^r \to \Omega_{n+1}^r/\Theta_{n+1}^r$ concides with the Euler–Lagrange mapping $\lambda \to E_\lambda$.

5. Trivial lagrangians

Now we are in a position to describe the local structure of trivial lagrangians of order r. Recall that the rth order lagrangian $\lambda \in \Omega_n^r Y$ over a fibered manifold (Y, π, X) is called *trivial* if its Euler–Lagrange form is the identically zero, i.e., $E_\lambda = 0$.

Theorem 2. *A lagrangian λ of order r over (Y, π, X) is trivial if and only if to each point $y \in Y$ there exist a fibered chart (V, ψ) on Y and an $(n-1)$-form χ on $V^{r-1} \subset J^{r-1}Y$ such that $\lambda = hd\chi$ on V^r.*

Proof. It is immediately given by the variational sequence of order r that a lagrangian λ is trivial if and only if it can be locally expressed in the form $\lambda = hd\eta$, up to a projection, where η is an $(n-1)$-form. Thus, only the $\pi^{r,r-1}$-projectability of η needs proof. Assume that $hd\eta = (\pi^{r+1,r})^*\lambda$. Since $hd\eta$ is $\pi^{r+1,r}$-projectable, it follows from Theorem 1 that η is of the form $\eta = (\pi^{r,r-1})^*\chi + \nu + d\tau$, where ν is a contact and χ is defined on V^{r-1}. Consequently, $hd\eta = h(\pi^{r,r-1})^*d\chi = (\pi^{r,r-1})^*hd\chi$ which completes the proof.

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