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The ANTI-order for caccc posets — Part II

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Abstract

In Part I we defined the ANTI-order, ANTI-good subsets, ANTI-perfect sequences and ANTI-cores for caccc posets. In this part we prove the main result: If $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ is an ANTI-perfect sequence of a connected caccc poset P which does not contain a one-way infinite fence, then P_ξ is a retract of P for all $\xi \leqslant \lambda$.

Keywords: Caccc posets; Retracts; ANTI-order

1. Introduction

This is a continuation of [1], where we defined the ANTI-order, ANTI-good subsets, ANTI-perfect sequences and ANTI-cores for caccc posets. We refer the reader to [1] for the definitions of these and other special notation. In this part we prove the main result.

Theorem 1.1. *Let $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ be an ANTI-perfect sequence of a connected caccc poset P which contains no one-way infinite fence. Then P_ξ is a retract of P for every $\xi \leqslant \lambda$; in particular, the ANTI-core P_λ is a retract of P.*

By Theorem 3.4(2) of [1] an ANTI-good subset of a caccc poset is a retract, and so the conclusion of the theorem is obvious if the length λ of Π is finite. Before proving the theorem, we give an example to show that the length of an ANTI-perfect sequence of a connected caccc poset with no one-way infinite fence may be infinite and so it is necessary to consider limit steps when we prove the theorem. (This is different from the case for a PT-perfect sequence in a cc poset with no infinite antichain which is always finite — see (1.4) of [1].)

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The example is a modification of the poset shown in Fig. 5 in [1] which does contain a one-way infinite fence. For $n < \omega$, let $(A_n, \leqslant_n)$ be the poset shown in Fig. 1, in which

$$\{a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,2n-2}, a_{n,2n-1}, x_n, z_n\}$$

is a finite fence,

$$\{y_{n,1}, y_{n,3}, y_{n,5}, \ldots, y_{n,2n-1}, y_n\}$$

is a finite decreasing chain, $a_{n,2k-1} < y_{n,2k-1}$ for $1 \leqslant k \leqslant n$, $z_n < y_n$ and there are no other comparabilities except for those demanded by transitivity.

The poset $(P, \leqslant)$ shown in Fig. 2 is obtained in the following way. Let $P = \cup\{A_n : n < \omega\} \cup \{y\}$ and define the order on P so that $\leqslant$ is the same as $\leqslant_n$ on A_n, $y_n > y_{n+1,1}$

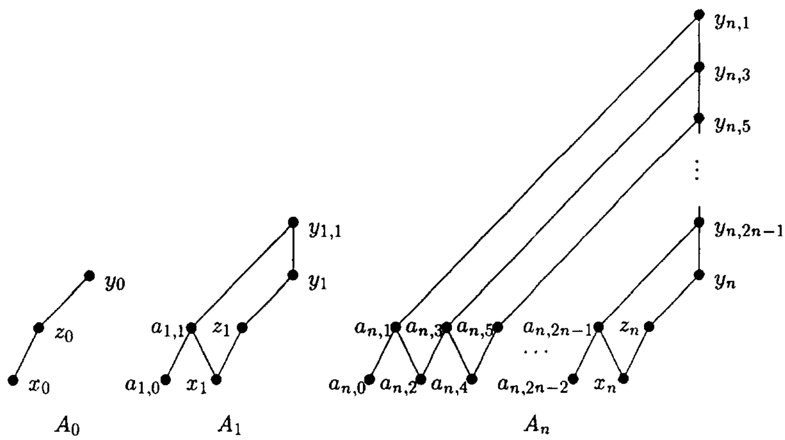

Fig. 1.

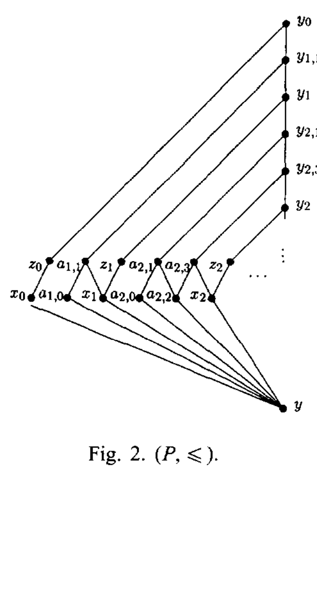

Fig. 2. $(P, \leqslant)$.

and y is the smallest element of P, there are no other comparabilities except for those required for transitivity. Using the same argument for the poset shown in Fig. 5 in [1], we easily see that it is a connected caccc poset with no one-way infinite fence, and that $\Pi = \langle P_\xi : \xi \leqslant \omega \rangle$ is an ANTI-perfect sequence of P, where $P = P_0$, $P_n - P_{n+1} = \{a_{i,n} : [n/2] + 1 \leqslant i < \omega\}$ $(n < \omega)$ and $P_\omega = \cap\{P_n : n < \omega\}$. In other words, P_{n+1} is obtained from P_n, by removing all a's having n as the second subscript, and $P_\omega = \{x_i : i < \omega\} \cup \{y_i : i < \omega\} \cup \{z_i : i < \omega\}$.

2. Some additional lemmas

In this section we introduce some new definitions and prove two easy lemmas needed for the proof of the main theorem. Let $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ be an ANTI-perfect sequence for a caccc poset P. For each $x \in P$ we define the *index of* x, denoted by $i(x)$, to be λ if $x \in P_\lambda$, and $i(x) = \xi$ if $x \in P_\xi - P_{\xi+1}$ for some $\xi < \lambda$. We also define

$$I(>x) = \{i(y) : y > x \wedge i(y) \geqslant i(x)\},$$

$$I(<x) = \{i(y) : y < x \wedge i(y) \geqslant i(x)\},$$

$$I(x) = I(<x) \cup I(>x).$$

Lemma 2.1. *Let* $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ *be an ANTI-perfect sequence of a caccc-poset* P, $X \subseteq P$ *and* $\alpha = \min\{i(y) : y \in X\}$. *If* $x = \inf X$ $(x = \sup X)$ *exists, then* $x \in P_\alpha$. *Furthermore,* $x \in P_{\alpha+1}$ *if* $x \notin X$.

Proof. When $x \in X$, the conclusion is obvious. Suppose that $x \notin X$. We have that $X \subseteq P_\alpha$ since $\alpha \leqslant i(y)$ for all $y \in X$. By induction on $\eta \leqslant \alpha + 1$, we show that $x \in P_\eta$. If η is a limit this is clear since, in this case, $P_\eta = \cap_{\zeta<\eta} P_\zeta$. If $\eta = \zeta + 1$ and $x \in P_\zeta$, then $x = \inf_{P_\zeta} X$ and therefore, by [1, Lemma 3.2] x belongs to any $\lll$-good subset of P_ζ, and in particular to $P_{\zeta+1}$. Hence $x \in P_{\alpha+1}$. □

Corollary 2.2. *Let* $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ *be an ANTI-perfect sequence of a caccc poset* P *and let* $\xi \leqslant \lambda$. *If* $X \subseteq P_\xi$ *and* $x = \inf X$ $(\sup X)$ *exists, then* $x \in P_\xi$ *and hence* $\inf_{P_\xi} X$ $(\sup_{P_\xi} X)$ *also exists and is equal to* x.

Lemma 2.3. *Let* $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ *be an ANTI-perfect sequence of a caccc poset* P, *and let* $\xi \leqslant \lambda$ *be a limit ordinal. If* C *is a chain and* $C \cap P_\eta$ *is coinitial* (*cofinal*) *in* C *for all* $\eta < \xi$, *then* $x = \inf C \in P_\xi$ $(x = \sup C \in P_\xi)$.

Proof. For each $\eta < \xi$, since $C \cap P_\eta$ is coinitial in C, $x = \inf C \cap P_\eta$ and therefore, by Lemma 2.1, $x \in P_\eta$. Thus, $x \in P_\xi = \cap_{\eta<\xi} P_\eta$. □

3. Proof of the main theorem

Let $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ be an ANTI-perfect sequence of a connected caccc poset P with no one-way infinite fence. Let $\lll_\xi$ be the ANTI-order on P_ξ, i.e. $\lll_\xi = \lll_{P_\xi}$, and let $g_\xi : P_\xi \to P_{\xi+1}$ be an ANTI-good retraction for all $\xi \leqslant \lambda$ (see [1, Theorem 3.4]). We shall inductively define maps $f_\xi : P \to P_\xi$ for each $\xi \leqslant \lambda$ so that the conditions $(1)_\xi$–$(19)_\xi$ below are satisfied. We start with $f_0 = \mathrm{id}_P$, the identity mapping on P, so that all these conditions are trivially satisfied for $\xi = 0$. We assume that $\xi > 0$ and that f_η has been defined for all $\eta < \xi$ so that the corresponding conditions are satisfied.

For any $x \in P$, the sequence $\mathrm{orb}_\xi(x) = \langle f_\eta(x) : \eta < \xi \rangle$ is called the *ξ-orbit of x*; for $A \subseteq \xi$ we also define $\mathrm{Orb}(A,x) = \{ f_\eta(x) : \eta \in A \}$. The ξ-orbit of x is *eventually constant*, if there is $\alpha < \xi$ such that $f_\alpha(x) = f_\eta(x)$ for $\alpha \leqslant \eta < \xi$, and in this case, x is called a *ξ-stable point*. We say that x and y are *ξ-distinct* if their ξ-orbits are disjoint, i.e. $f_\eta(x) \neq f_\eta(y)$ for all $\eta < \xi$. We say that $x \in P$ is a *ξ-regular point* if there exists $y \in P$ which is ξ-distinct from x and such that $y \| f_\alpha(x)$ for some $\alpha < \xi$. The point $x \in P$ is *ξ-bad* if it is neither ξ-stable nor ξ-regular. If $\xi = \eta + 1$ is a successor, then every $x \in P$ is ξ-stable and so there are no ξ-bad points.

For any $x, y \in P$ and $\xi < \lambda$, we define

$$M_{\xi,x}(y) = \{ \eta : \eta < \xi \wedge f_\eta(x) \geqslant y \}$$

and

$$N_{\xi,x}(y) = \{ \eta : \eta < \xi \wedge f_\eta(x) \leqslant y \}.$$

If ξ is a limit ordinal, a subset $S \subseteq \xi$ is a CA-, a CI-, or a CD-set for $x \in P$ if S is cofinal in ξ and the set $\mathrm{Orb}(A,x)$ is respectively an antichain, an increasing chain or a decreasing chain in P.

Let ξ be a limit ordinal, $x \in P$ and let A be a CA-set for x such that $\mathrm{Orb}(A,x)$ is bounded below. Since P is caccc, for any $\alpha \in A$,

$$x_\alpha = \inf \{ f_\eta(x) : \eta \in A \wedge \eta \geqslant \alpha \}$$

exists and, by Lemma 2.1 $x_\alpha \in P_\alpha$. Since $\langle x_\alpha : \alpha \in A \rangle$ is increasing,

$$z = \sup \{ x_\alpha : \alpha \in A \}$$

exists and belongs to P_ξ by Lemma 2.3. We call this supremum the *down-up limit* of $\mathrm{Orb}(A,x)$ and write

$$z = \text{du-lim}\, \mathrm{Orb}(A,x).$$

In a similar way, if $\mathrm{Orb}(A,x)$ is bounded above we define the *up-down limit* ud-limOrb (A,x).

3.1. Statements of the 19 conditions

$(1)_\xi$ If ξ is a limit ordinal, $x \in P$, $i(x) < \xi$, and if $P(>x) \cap P_\eta \neq \emptyset$ $(P(<x) \cap P_\eta \neq \emptyset)$ for all $\eta < \xi$, then $P(>x) \cap P_\xi \neq \emptyset$ $(P(<x) \cap P_\xi \neq \emptyset)$.

$(2)_\xi$ If ξ is a limit, $i(x) < \xi$, if $I(>x) \neq \emptyset$ and $P(>x) \cap P_\xi = \emptyset$ $(I(<x) \neq \emptyset$ and $P(<x) \cap P_\xi = \emptyset)$, then $\max I(>x)$ $(\max I(<x))$ exists and is less than ξ.

$(3)_\xi$ $P_\xi \neq \emptyset$, and there is no ξ-bad point.

$(4)_\xi$ If ξ is a limit, $x < y$ and x, y are ξ-distinct, then either there exists $\alpha < \xi$ such that $M_{\xi,y}(f_\alpha(x))$ is cofinal in ξ or there exists $\beta < \xi$ such that $N_{\xi,x}(f_\beta(y))$ is cofinal in ξ.

$(5)_\xi$ If ξ is a limit, $x \in P$, and A is a cofinal subset of ξ, then A contains either a CA-set, a CD-set or a CI-set for x.

$(6)_\xi$ If ξ is a limit ordinal and $x \in P$, then there is a cofinal subset $A \subseteq \xi$ such that $\mathrm{Orb}(A,x)$ is either bounded above or bounded below in P.

$(7)_\xi$ If ξ is a limit ordinal and $x \in P$ and if A and B are both CD-sets (CI-sets) in ξ for x, then

$$\inf \mathrm{Orb}(A,x) = \inf \mathrm{Orb}(B,x) \quad (\sup \mathrm{Orb}(A,x) = \sup \mathrm{Orb}(B,x))$$

$(8)_\xi$ If ξ is a limit ordinal, $x \in P$, and A is a cofinal subset of ξ, then either (i) every maximal subset $B \subseteq A$ such that $\mathrm{Orb}(B,x)$ is an antichain is cofinal in ξ or (ii) no such B is cofinal in ξ.

$(9)_\xi$ If ξ is a limit ordinal, $x, y \in P$, and if $M_{\xi,x}(y)$ $(N_{\xi,x}(y))$ is cofinal in ξ and there is a CA-set in $M_{\xi,x}(y)$ $(N_{\xi,x}(y))$ for x, then $M_{\xi,x}(y)$ $(N_{\xi,x}(y))$ is a final segment of ξ, i.e. it is a set of form $\{\eta : \alpha \leqslant \eta < \xi\}$ for some $\alpha < \xi$.

$(10)_\xi$ If ξ is a limit ordinal, $x \in P$, and if A and B are CA-sets in ξ for x such that $\mathrm{Orb}(A,x)$ and $\mathrm{Orb}(B,x)$ are both bounded below (above), then

$$\text{du-}\lim \mathrm{Orb}(A,x) = \text{du-}\lim \mathrm{Orb}(B,x) \quad (\text{ud-}\mathrm{Orb}(A,x) = \text{ud-}\lim \mathrm{Orb}(B,x)).$$

$(11)_\xi$ If ξ is a limit ordinal, $x \in P$, and if A is a CD-set (CI-set) in ξ for x, and B is a CA-set in ξ for x such that $\mathrm{Orb}(B,x)$ is bounded below (above), then

$$\inf \mathrm{Orb}(A,x) = \text{du-}\lim \mathrm{Orb}(B,x) \quad (\sup \mathrm{Orb}(A,x) = \text{ud-}\lim \mathrm{Orb}(B,x)).$$

If ξ is a limit ordinal, $x \in P$, and if ξ contains either a CD-set (CI-set) A for x or a CA-set B for x such that $\mathrm{Orb}(B,x)$ is bounded below (above), then we define the *down-limit* (*up-limit*) of $\mathrm{Orb}(\xi,x)$, which we denote by $s_\xi(x)$ $(t_\xi(x))$ to be either the infimum (supremum) of $\mathrm{Orb}(A,x)$ or the du-limit (ud-limit) of $\mathrm{Orb}(B,x)$. By $(7)_\xi$, $(10)_\xi$ and $(11)_\xi$, we see that these definitions of $s_\xi(x)$ and $t_\xi(x)$, when they exist, do not depend upon the choices of A or B.

$(12)_\xi$ If ξ is a limit ordinal and $x \in P$, then
(a) either $s_\xi(x)$ or $t_\xi(x)$ exists;
(b) if it exists, then $s_\xi(x)$ $(t_\xi(x))$ belongs to P_ξ.

$(13)_\xi$ If ξ is a limit ordinal, $x, y \in P$, and if $M_{\xi,x}(y)$ ($N_{\xi,x}(y)$) contains a final segment of ξ, then $y \leqslant s_\xi(x)$ and $y \leqslant t_\xi(x)$ ($y \geqslant s_\xi(x)$ and $y \geqslant t_\xi(x)$), whenever these limits exist.

$(14)_\xi$ If ξ is a limit ordinal and $x \in P$ then $s_\xi(x) \leqslant t_\xi(x)$ if both these limits exist.

$(15)_\xi$ If $\xi = \eta + 1$ is a successor ordinal, define $f_\xi = g_\eta \circ f_\eta$. If ξ is a limit ordinal and $x \in P$, then we define $f_\xi(x)$ as follows: if ξ contains a CD-set for x, then $f_\xi(x) = s_\xi(x)$ (which exists); if ξ contains no CD-set but contains a CI-set for x, then $f_\xi(x) = t_\xi(x)$ (which exists); if ξ contains no CD- or CI-set, then we define $f_\xi(x) = s_\xi(x)$ if $s_\xi(x)$ exists, and $f_\xi(x) = t_\xi(x)$ otherwise (thus $f_\xi(x)$ is well defined by $(12)_\xi(a)$).

$(16)_\xi$ If ξ is a limit ordinal and $x, y \in P$, then

(a) $y \geqslant f_\xi(x)$ if $N_{\xi,x}(y)$ is cofinal in ξ;

(b) $y \leqslant f_\xi(x)$ if $M_{\xi,x}(y)$ contains a final segment of ξ;

(c) if $M_{\xi,x}(y)$ is cofinal in ξ and $y \nleqslant f_\xi(x)$, then there exists a CD-set D in ξ for x such that $t_\xi(x) = \inf \mathrm{Orb}(D,x)$ and $M_{\xi,x}(y) \cap M_{\xi,x}(f_\xi(x))$ is cofinal in ξ.

$(17)_\xi$ f_ξ is a retraction and $P_\xi = f_\xi[P]$ is a retract (and hence P_ξ is connected).

$(18)_\xi$ For any $x, y \in P$,

$$\alpha \leqslant \beta \leqslant \xi \wedge f_\alpha(x) \leqslant y \wedge y \in P_\xi \rightarrow f_\beta(x) \leqslant y$$

and

$$\alpha \leqslant \beta \leqslant \xi \wedge f_\alpha(x) \geqslant y \wedge y \in P_\xi \rightarrow f_\beta(x) \geqslant y.$$

$(19)_\xi$ $f_\xi \circ f_\eta = f_\eta \circ f_\xi = f_\xi$ for any $\eta \leqslant \xi$.

When the induction is completed, $(17)_\xi$ implies the desired conclusion that P_ξ is a retract of P for all $\xi \leqslant \lambda$.

3.2. *Proof of $(1)_\xi$*

Suppose that $P(>x) \cap P_\eta \neq \emptyset$ for all $\eta < \xi$. We want to show that there exists $z \in P_\xi$ with $x < z$. Fix a cofinal increasing sequence $\langle \eta_\alpha : \alpha < \mathrm{cf}(\xi) \rangle$ in ξ with $\eta_0 > i(x)$. For each $\alpha < \mathrm{cf}(\xi)$, since P_{η_α} is chain complete there is a maximal element y_α of P_{η_α} such that $x < y_\alpha$. Then $y_\alpha \nless y_\beta$ for $\alpha < \beta < \mathrm{cf}(\xi)$. We consider two cases.

Case 1. There is a cofinal subset A of $\mathrm{cf}(\xi)$ such that $\{y_\alpha : \alpha \in A\}$ is an antichain.

Since x is a lower bound of $Y_\alpha = \{y_\beta : \beta \in A \wedge \beta \geqslant \alpha\}$, $z_\alpha = \inf Y_\alpha$ exists and it belongs to P_{η_α} by Lemma 2.1. Obviously, $\langle z_\alpha : \alpha \in A \rangle$ is increasing and so $z = \sup\{z_\alpha : \alpha \in A\}$ exists and belongs to P_ξ by Lemma 2.3. Note that $z_0 \in P_{\eta_0}$ and $i(x) < \eta_0$, hence $x \neq z_0$ and so $x < z_0 \leqslant z$.

Case 2. Whenever $A \subseteq \mathrm{cf}(\xi)$ and $\{y_\alpha : \alpha \in A\}$ is an antichain, then A is not cofinal in ξ.

Let A be a maximal subset of $\mathrm{cf}(\xi)$ such that $\{y_\alpha : \alpha \in A\}$ is an antichain. Then A is not cofinal in $\mathrm{cf}(\xi)$ and so there is $\mu < \mathrm{cf}(\xi)$ such that $\alpha < \mu$ for all $\alpha \in A$. For $\mu \leqslant \beta < \mathrm{cf}(\xi)$, there is $\alpha \in A$ such that $y_\alpha \| y_\beta$ and hence $y_\beta \leqslant y_\alpha$. We have

$$\{\beta : \mu \leqslant \beta < \mathrm{cf}(\xi)\} = \cup\{B_\alpha : \alpha \in A\},$$

where $B_\alpha = \{\beta : \mu \leqslant \beta < \mathrm{cf}(\xi) \wedge y_\beta \leqslant y_\alpha\}$. Since $\mathrm{cf}(\xi)$ is a regular cardinal and $|A| < \mathrm{cf}(\xi)$, B_α is cofinal in $\mathrm{cf}(\xi)$ for some $\alpha \in A$. For any $\beta \in B_\alpha$, $y_\alpha \geqslant y_\beta \in P_{\eta_\beta}$ and therefore $y_\beta \leqslant f_{\eta_\beta}(y_\alpha) \in P_{\eta_\beta}$ since f_{η_β} is a retraction by $(17)_{\eta_\beta}$. Since y_β is a maximal element of P_{η_β} it follows that $y_\beta = f_{\eta_\beta}(y_\alpha)$. If $\beta, \gamma \in B_\alpha$ and $\gamma \leqslant \beta$, then $\eta_\alpha \leqslant \eta_\gamma \leqslant \eta_\beta$, $f_{\eta_\gamma}(y_\alpha) = y_\alpha \geqslant y_\beta$ and $y_\beta \in P_{\eta_\beta}$, it follows from $(18)_{\eta_\beta}$ that $y_\beta \leqslant y_\gamma = f_{\eta_\gamma}(y_\alpha) \leqslant y_\alpha$. Thus, $\langle y_\beta : \beta \in B_\alpha \rangle$ is a decreasing sequence with x as a lower bound. Let $z = \inf\{y_\beta : \beta \in B_\alpha\}$. Then $x \leqslant z$ and, by Lemma 2.3, $z \in P_\xi$. Since $i(x) < \xi$, it follows that $x < z$. □

3.3. *Proof of* $(2)_\xi$

It follows from $(1)_\xi$ that $I(>x)$ is not cofinal in ξ so that $\eta = \sup I(>x) < \xi$. If $\eta \notin I(>x)$, then η is a limit ordinal and $P(>x) \cap P_\zeta \neq \emptyset$ for all $\zeta < \eta$. Therefore, $P(>x) \cap P_\eta \neq \emptyset$ by $(1)_\eta$ and hence $\eta \in I(>x)$. This contradiction shows that $\eta \in I(>x)$ and hence that $\max I(>x) = \eta < \xi$. □

3.4. *Proof of* $(3)_\xi$

We first show $P_\xi \neq \emptyset$. If $\xi = \eta + 1$ is a successor, then $P_\eta \neq \emptyset$ by $(3)_\eta$. Since P_ξ is a $\leqslant_\eta$-good subset of P_η it is a retract of P_η and therefore non-empty by Theorem 3.4(2) of [1]. Now assume that ξ is a limit and suppose for a contradiction that $P_\xi = \emptyset$. We shall inductively define a sequence $\langle a_n : n < \omega \rangle$ in P and a sequence $\langle \eta_n : n < \omega \rangle$ in ξ satisfying conditions (i)–(iv). The conditions (iii) and (iv) immediately give the desired contradiction since these imply that $\langle a_n : n < \omega \rangle$ is a one-way infinite fence in P.

(i) $\eta_m < \eta_n$ for $m < n < \omega$.

(ii) $I(>a_n) \neq \emptyset$ if n is even; $I(<a_n) \neq \emptyset$ if n is odd.

(iii) $i(a_n) = \eta_n$. If $n > 0$ is even $\eta_n = \max I(<a_{n-1})$ and a_n is a minimal element of P_{η_n} such that $a_n < a_{n-1}$, and if n is odd $\eta_n = \max I(>a_{n-1})$ and a_n is a maximal element of P_{η_n} such that $a_n > a_{n-1}$.

(iv) $a_n \perp a_m$ if $m \leqslant n-2$.

To start, choose a_0 to be a minimal element of P and let $\eta_0 = i(a_0)$. By assumption, $\eta_0 < \xi$. Also $P(>a_0) \neq \emptyset$ since $\xi > 0$ and P is connected. Suppose that $n > 0$ and that the a_i and η_i have been suitably defined for $i < n$. If n is odd, then by the induction hypothesis $I(>a_{n-1}) \neq \emptyset$ and so by $(2)_\xi$, $\eta_n = \max I(>a_{n-1})$ exists and is less than ξ. There is $y > a_{n-1}$ such that $i(y) \geqslant i(a_{n-1}) = \eta_{n-1}$. By Theorem 3.4(1) in [1], $g_{\eta_{n-1}}(y) \geqslant a_{n-1}$. But since $g_{\eta_{n-1}}(y) \in P_{\eta_{n-1}+1}$ and $a_{n-1} \notin P_{\eta_{n-1}+1}$, it follows that $g_{\eta_{n-1}}(y) > a_{n-1}$. Therefore, $\eta_n = \max I(>a_{n-1}) > \eta_{n-1}$ and (i) holds for n. There is $z > a_{n-1}$ such that $i(z) = \eta_n$ and since P_{η_n} is chain complete, there is a maximal element a_n of P_{η_n} such that $z \leqslant a_n$ and so (iii) also holds for n. Since the ANTI-perfect sequence Π is strictly decreasing, the set P_{η_n} contains elements other than a_n, and since it is connected by $(17)_{\eta_n}$ and a_n is maximal there is an element of P_{η_n} strictly less than a_n. Therefore, $P_{\eta_n}(< a_n) \neq \emptyset$ and (ii) holds. Let $m \leqslant n-2$. If m is even, then a_m is a minimal element of P_{η_m} and hence $a_n \not< a_m$ since $a_n \in P_{\eta_n} \subseteq P_{\eta_m}$. Also, $a_n \not> a_m$ since $i(a_n) = \eta_n > \eta_{m+1} = \max I(> a_m)$. Thus (iv) holds in this case since $i(a_n) = \eta_n > \eta_m = i(a_m)$, and so $a_n \neq a_m$. Similarly, (iv) holds for odd $m \leqslant n-2$. The inductive step when n is even is similar and we omit the details.

Suppose there is a ξ-bad point $x \in P$. Then ξ is a limit. Since x is not ξ-regular, if $y \in P$ is comparable with x, then it is not ξ-distinct from x, and so y is not ξ-stable. If y is ξ-regular, then there is $z \| y$ such that z is ξ-distinct from y. But, for large enough $\alpha < \xi$, we have that $f_\alpha(z) \| f_\alpha(y) = f_\alpha(x)$ and $f_\alpha(z)$ is ξ-distinct from x, which is a contradiction. Therefore, y is also ξ-bad. Since P is connected, it follows that every point of P is ξ-bad. But $P_\xi \neq \emptyset$ and points of P_ξ are ξ-stable. This contradiction shows that there are no ξ-bad points in P. □

3.5. *Proof of* $(4)_\xi$

We assume the hypothesis of $(4)_\xi$ and that the conclusion is false; we will obtain the contradiction that P contains a one-way infinite fence. By assumption, for any $\alpha, \beta < \xi$, neither $M_{\xi,y}(f_\alpha(x))$ nor $N_{\xi,x}(f_\beta(y))$ is cofinal in ξ. Therefore, there are mappings $u, v: \xi \to \xi$ where

$$u(\alpha) = \sup M_{\xi,y}(f_\alpha(x)), \qquad v(\beta) = \sup N_{\xi,x}(f_\beta(y)).$$

We begin by proving the following five claims.

Claim 1. *For any* $\alpha, \beta < \xi$, $f_\alpha(x) \not\geqslant f_\beta(y)$.

If $f_\alpha(x) \geqslant f_\beta(y)$, then, by $(17)_\gamma$ and $(19)_\gamma$, $f_\gamma(x) \geqslant f_\gamma(y)$, for $\max\{\alpha, \beta\} < \gamma < \xi$. On the other hand $f_\gamma(x) \leqslant f_\gamma(y)$ since $x < y$, and so $f_\gamma(x) = f_\gamma(y)$. This is a contradiction since x and y are ξ-distinct.

Claim 2. $\alpha < u(\alpha)$ $(\beta < v(\beta))$ *for all* $\alpha < \xi$ $(\beta < \xi)$.

Since x, y are ξ-distinct, it follows from $x < y$ and $(17)_\alpha$ that $f_\alpha(x) < f_\alpha(y)$. Then by $(15)_{\alpha+1}$ and Theorem 3.4(1) in [1], $f_\alpha(x) \leqslant f_{\alpha+1}(y)$ and so $\alpha + 1 \in M_{\xi,y}(f_\alpha(x))$. Therefore, $\alpha < \alpha + 1 \leqslant u(\alpha)$.

Claim 3. *For any* $\alpha < \xi$, $f_\eta(x) \perp f_\zeta(y)$ *for* $\eta \leqslant \alpha$ *and* $u(\alpha) < \zeta$. (*For any* $\beta < \xi$, $f_\zeta(y) \perp f_\eta(x)$ *for* $\zeta \leqslant \beta$ *and* $v(\beta) < \eta$.)

By Claim 2, $\eta \leqslant \alpha < \zeta$, and by Claim 1, if $f_\eta(x) \| f_\zeta(y)$, then $f_\eta(x) < f_\zeta(y)$. Therefore, by $(18)_\zeta$, $f_\alpha(x) \leqslant f_\zeta(y)$ and so $\zeta \in M_{\xi,y}(f_\alpha(x))$. This implies the contradiction that $\zeta \leqslant u(\alpha)$.

Claim 4. *For any $\alpha < \xi$, $f_\zeta(y) \perp f_{\zeta'}(y)$ whenever $\zeta \in M_{\xi,y}(f_\alpha(x))$ and $v \circ u(\alpha) < \zeta'$. (For any $\beta < \xi$, $f_\eta(x) \perp f_{\eta'}(x)$ whenever $\eta \in N_{\xi,x}(f_\beta(y))$ and $u \circ v(\beta) < \eta'$.)*

By Claim 2, $\zeta \leqslant u(\alpha) < v \circ u(\alpha) < \zeta'$. Since $f_\alpha(x) \leqslant f_\zeta(y)$, and $\zeta' > u(\alpha)$ it follows that $f_\zeta(y) \nleqslant f_{\zeta'}(y)$. If $f_{\zeta'}(y) < f_\zeta(y)$, then by $(17)_{\zeta'}$, $f_{\zeta'}(x) \leqslant f_{\zeta'}(y) < f_\zeta(y)$, and this contradicts Claim 3.

Claim 5. *For any $\alpha < \xi$, if $u(\alpha) \notin M_{\xi,y}(f_\alpha(x))$, then there is $\zeta \in M_{\xi,y}(f_\alpha(x))$ such that $\alpha < \zeta$ and $f_\zeta(y) \geqslant f_{u(\alpha)}(y)$. For any $\beta < \xi$, $v(\beta) \in N_{\xi,x}(f_\beta(y))$.*

Since $u(\alpha) \notin M_{\xi,y}(f_\alpha(x))$, it follows that $u(\alpha)$ is a limit. Therefore, by $(16)_{u(\alpha)}(c)$ $M_{u(\alpha),y}(f_\alpha(x)) \cap M_{u(\alpha),y}(f_{u(\alpha)}(y))$ is cofinal in $u(\alpha)$. By Claim 2, $u(\alpha) > \alpha$ and so there is ζ such that $\alpha < \zeta < u(\alpha)$, $f_\zeta(y) \geqslant f_\alpha(x)$ and $f_\zeta(y) \geqslant f_{u(\alpha)}(y)$. For the last part, suppose $\beta < \xi$ and $v(\beta) \notin N_{\xi,x}(f_\beta(y))$. Then $v(\beta)$ is a limit and $N_{v(\beta),x}(f_\beta(y))$ is cofinal in $v(\beta)$. Therefore, by $(16)_{v(\beta)}(a)$, $f_\beta(y) \geqslant f_{v(\beta)}(x)$, and so $v(\beta) \in N_{\xi,x}(f_\beta(y))$ after all.

We now obtain the desired contradiction by constructing a one-way infinite fence in P. Inductively define ordinals $\alpha_n, \beta_n < \xi$ for $n < \omega$ so that $\alpha_0 = 0$, $\beta_0 = u(0)$, $\alpha_{n+1} = v(\beta_n)$, and $\beta_{n+1} = u(\alpha_{n+1})$. By Claim 2

$$\alpha_0 < \beta_0 < \cdots < \alpha_n < \beta_n < \cdots.$$

Define $a_n = f_{\alpha_n}(x)$ $(n < \omega)$. If $f_{\beta_n}(y) \geqslant f_{\alpha_n}(x)$, then we define $\gamma_n = \beta_n$; otherwise, by Claim 5, there is an ordinal γ_n such that $\alpha_n < \gamma_n < \beta_n$ and

$$a_n = f_{\alpha_n}(x) \leqslant f_{\gamma_n}(y) \geqslant f_{\beta_n}(y) \geqslant f_{\alpha_{n+1}}(x).$$

Now define $b_n = f_{\gamma_n}(y)$. From these definitions, we have

$$\alpha_0 < \gamma_0 \leqslant \beta_0 < \cdots < \alpha_n < \gamma_n \leqslant \beta_n < \cdots$$

and

$$a_0 \leqslant b_0 \geqslant \cdots \geqslant a_n \leqslant b_n \geqslant a_{n+1} \leqslant \cdots.$$

Therefore, by Claim 1, it follows that

$$a_0 < b_0 > \cdots > a_n < b_n > a_{n+1} < \cdots.$$

For $m < n < \omega$,

$$\alpha_m \leqslant \alpha_{n-1} < \beta_{n-1} = u(\alpha_{n-1}) < \gamma_n,$$

and then by Claim 3,

$$a_m = f_{\alpha_m}(x) \perp f_{\gamma_n}(y) = b_n.$$

For $m < n - 1 < \omega$,

$$\gamma_m \leqslant \beta_{n-2} < \alpha_{n-1} = v(\beta_{n-2}) < \alpha_n,$$

and so by Claim 3,

$$b_m = f_{\gamma_m}(y) \perp f_{\alpha_n}(x) = a_n.$$

By Claim 5, for $2 \leqslant n < \omega$,

$$\alpha_{n-1} \in N_{\xi,x}(f_{\beta_{n-2}}(y)) \text{ and } \alpha_n > \beta_{n-1} = u \circ v(\beta_{n-2})$$

and so, by Claim 4,

$$a_{n-1} = f_{\alpha_{n-1}}(x) \perp f_{\alpha_n}(x) = a_n.$$

It follows from $(18)_{\alpha_n}$ that $a_m \perp a_n$ for all $m < n$. In the same way, we have $b_m \perp b_n$ for $m < n < \omega$. This shows that

$$\langle a_1, b_1, \ldots, a_n, b_n, \ldots \rangle$$

is indeed a one-way infinite fence. □

3.6. *Proof of* $(5)_\xi$

Let $B \subseteq A$ be maximal such that $\mathrm{Orb}(B,x)$ is an antichain. If B is cofinal in A, it is a CA-set in A for x. So, we assume that B is not cofinal in A and that there is $\alpha \in A$ such that $\beta < \alpha$ for all $\beta \in B$. By the maximality of B, for $\alpha \leqslant \eta < \xi$, $f_\eta(x) \| f_\beta(x)$ for some $\beta \in B$ and then, by $(18)_\eta$, $f_\eta(x) \| f_\alpha(x)$. Let

$$D = \{\eta : \eta \in A \wedge \eta \geqslant \alpha \wedge f_\eta(x) \leqslant f_\alpha(x)\}$$

and

$$I = \{\eta : \eta \in A \wedge \eta \geqslant \alpha \wedge f_\eta(x) > f_\alpha(x)\}.$$

By $(18)_\eta$, $f_{\eta'}(x) \geqslant f_\eta(x)$ for $\eta', \eta \in D$ and $\eta' < \eta$, and so $\langle f_\eta(x) : \eta \in D \rangle$ is decreasing. Similarly, $\langle f_\eta(x) : \eta \in I \rangle$ is increasing. Since $D \cup I$ is a final segment of A, either D or I is cofinal in A, and we conclude that A either contains a CD-set or a CI-set for x. □

3.7. *Proof of* $(6)_\xi$

This is obvious when x is a ξ-stable point. So, by $(3)_\xi$, we may assume that x is ξ-regular, in other words there is $y \in P$ such that the ξ-orbits of x and y are ξ-distinct and $y \| f_\alpha(x)$, say $f_\alpha(x) \leqslant y$, for some $\alpha < \xi$. Then, by $(17)_\alpha$, $f_\alpha(x) \leqslant f_\alpha(y)$; in fact, $f_\alpha(x) < f_\alpha(y)$ since the ξ-orbits of x and y are ξ-distinct. By $(4)_\xi$, either (i) there is β ($\alpha \leqslant \beta < \xi$) such that $M = M_{\xi,y}(f_\beta(x))$ is cofinal in ξ or (ii) there is γ ($\alpha \leqslant \gamma < \xi$) such that $N = N_{\xi,x}(f_\gamma(y))$ is cofinal in ξ.

If (ii) holds, then we are done since $\mathrm{Orb}(N,x)$ is bounded above by $f_\gamma(y)$. Suppose (i) holds. Then $\mathrm{Orb}(M,y)$ is bounded below by $f_\beta(x)$. By $(5)_\xi$ it follows that M contains either a CA-set or a CI-set or a CD-set for y. Suppose M contains the CA-set A. Since $f_\beta(x)$ is a lower bound of $\mathrm{Orb}(A,y)$ it follows that

$$z = \text{du-}\lim \mathrm{Orb}(A,y)$$

exists, $f_\beta(x) \leqslant z$ and $z \in P_\xi$. Therefore, if $\beta \leqslant \eta < \xi$, then $(18)_\eta$ implies that $f_\eta(x) \leqslant z$ and hence $(6)_\xi$ holds. Suppose that M contains the CI-set I. Then $z = \sup \mathrm{Orb}(I,y)$ exists and again $(6)_\xi$ holds. Similarly, if M contains a CD-set. □

3.8. *Proof of $(7)_\xi$*

Let $\alpha = \min A$. For each $\zeta \geqslant \alpha$ in B, there is $\gamma \in A$ such that $\gamma \geqslant \zeta$. Since $f_\alpha(x) \geqslant f_\gamma(x)$, it follows by $(18)_\gamma$ that $f_\gamma(x) \leqslant f_\zeta(x)$. Hence

$$\inf \mathrm{Orb}(A,x) \leqslant f_\zeta(x),$$

for all $\zeta \in B$ with $\zeta \geqslant \alpha$. Therefore,

$$\inf \mathrm{Orb}(A,x) \leqslant \inf \mathrm{Orb}(B,x).$$

The opposite inequality holds by symmetry. □

3.9. *Proof of $(8)_\xi$*

Suppose that B is a CA-set in A for x. We need to show that, if C is a maximal subset of A such that $\mathrm{Orb}(C,x)$ is an antichain, then C is cofinal in A. Suppose not. Then there is $\alpha \in A$ such that $\gamma < \alpha$ for all $\gamma \in C$. Since B is cofinal in A, there are $\eta, \zeta \in B$ such that $\alpha < \eta < \zeta$. By the maximality of C there is some $\gamma \in C$ such that $f_\zeta(x) \| f_\gamma(x)$ and it follows from $(18)_\zeta$ that $f_\zeta(x) \| f_\eta(x)$. This is a contradiction, since $\mathrm{Orb}(B,x)$ is an antichain. □

3.10. *Proof of $(9)_\xi$*

Let $M_{\xi,x}(y)$ be cofinal in ξ and suppose it contains a CA-set for x. We inductively show that $\beta \in M_{\xi,x}(y)$ for all $\alpha \leqslant \beta < \xi$, where $\alpha = \min M_{\xi,x}(y)$. When β is a limit ordinal, then by the induction hypotheses and $(16)_\beta(b)$, $y \leqslant f_\beta(x)$ and hence $\beta \in M_{\xi,x}(y)$. Suppose that $\beta = \gamma + 1$ is a successor ordinal. In this case, $\gamma \in M_{\xi,x}(y)$. Let A be a maximal subset of $M_{\xi,x}(y)$ which contains γ and is such that $\mathrm{Orb}(A,x)$ is an antichain. By $(8)_\xi$, A is a cofinal subset of $M_{\xi,x}(y)$ and so $B = \{\eta \colon \eta \in A \wedge \gamma \leqslant \eta < \xi\}$

is also a CA-set in $M_{\xi,x}(y)$ for x. B is infinite and since P is caccc Orb(B,x) has an infimum $z \geqslant y$. Moreover, Lemma 2.1 implies $z \in P_\beta$. By Theorem 3.4(2) in [1], g_γ is a retraction from P_γ onto P_β. Therefore, since $z \leqslant f_\gamma(x)$ it follows that $y \leqslant z = g_\gamma(z) \leqslant g_\gamma(f_\gamma(x)) = f_\beta(x)$, and hence $\beta \in M_{\xi,x}(y)$. □

3.11. Proof of $(10)_\xi$

Suppose that A, B are cofinal subsets of ξ such that Orb(A,x) and Orb(B,x) are both bounded below. Then

$$a = \text{du-}\lim \text{Orb}(A,x), \quad b = \text{du-}\lim \text{Orb}(B,x),$$

both exist and $a = \sup\{a_\alpha : \alpha \in A\}$, where $a_\alpha = \inf\{f_\eta(x) : \eta \in A \wedge \eta \geqslant \alpha\}$, and $b = \sup\{b_\beta : \beta \in B\}$, where $b_\beta = \inf\{f_\zeta(x) : \zeta \in B \wedge \zeta \geqslant \beta\}$. Let $\alpha \in A$, $\beta \in B$, $\alpha \leqslant \beta$. $M_{\xi,x}(a_\alpha)$ contains $\{\eta : \eta \in A \wedge \alpha \leqslant \eta\}$, which is a CA-set in ξ for x. Therefore, $M_{\xi,x}(a_\alpha)$ is a final segment of ξ by $(9)_\xi$. It follows that $a_\alpha \leqslant f_\zeta(x)$ for all $\zeta \in B$ with $\beta \leqslant \zeta$ and hence $a_\alpha \leqslant b_\beta \leqslant b$. Since $\alpha \in A$ was arbitrary, we have that $a \leqslant b$. By symmetry, we also have that $b \leqslant a$. □

3.12. Proof of $(11)_\xi$

Suppose that A is a CD-set and B is a CA-set in ξ for x such that Orb(B,x) is bounded below. Then

$$a = \inf \text{Orb}(A,x) \quad \text{and} \quad b = \text{du-}\lim \text{Orb}(B,x)$$

both exist. By definition, $b = \sup\{b_\beta : \beta \in B\}$, where $b_\beta = \inf\{f_\zeta(x) : \zeta \in B \wedge \beta \leqslant \zeta\}$. Let $\alpha = \min A$, $\beta \in B$ and $\alpha < \beta$. Note that $a \leqslant f_\alpha(x)$ and, by Lemma 2.3, $a \in P_\xi \subseteq P_\zeta$ for all $\zeta < \xi$. Then, $(18)_\zeta$ ensures that $a \leqslant f_\zeta(x)$ for all $\zeta \in B$ with $\zeta \geqslant \beta$. Hence, $a \leqslant b_\beta \leqslant b$. On the other hand, for each $\beta \in B$, $M_{\xi,x}(b_\beta)$ contains the CA-set $\{\zeta : \zeta \in B \wedge \beta \leqslant \zeta\}$ for x and so, by $(9)_\xi$, $M_{\xi,x}(b_\beta)$ is a final segment of ξ. Therefore, $b_\beta \leqslant f_\eta(x)$ for all $\eta \in A$ with $\beta \leqslant \eta$ and hence $b_\beta \leqslant a$. Since this holds for all $\beta \in B$, it follows that $b \leqslant a$. □

3.13. Proof of $(12)_\xi$

By $(6)_\xi$, we may assume that there is a cofinal subset $A \subseteq \xi$ such that Orb(A,x) is bounded below in P. It follows from $(5)_\xi$ that A either contains a CA-set B for x or a CD-set D for x or a CI-set I for x. If such a B exists then $s_\xi(x) = \text{du-lim}\,\text{Orb}(B,x)$ exists; if such a D exists, then again $s_\xi(x) = \inf \text{Orb}(D,x)$ exists; similarly, if such an I exists, then $t_\xi(x) = \sup \text{Orb}(I,x)$ exists. This proves (a).

We now show that, if it exists, $s_\xi(x) \in P_\xi$. If $s_\xi(x) = \inf \text{Orb}(D,x)$ for some CD-set D in ξ for x, then $s_\xi(x) \in P_\xi$ by Lemma 2.3. If $s_\xi(x) = \sup\{a_\alpha : \alpha \in B\}$, where B is a CA-set in ξ for x, and $a_\alpha = \inf\{f_\eta(x) : \eta \in B \wedge \alpha \leqslant \eta\}$ then $a_\alpha \in P_\alpha$ for all $\alpha \in B$

by Lemma 2.1, and therefore $s_\xi(x) \in P_\xi$ by Lemma 2.3. Similarly, $t_\xi(x) \in P_\xi$ if it exists. □

3.14. Proof of $(13)_\xi$

Suppose $M_{\xi,x}(y)$ contains a final segment of ξ and that $s_\xi(x)$ exists. Then there is $\gamma < \xi$ such that $y \leqslant f_\eta(x)$ for $\gamma \leqslant \eta < \xi$. If there is a CD-set D in ξ for x, then $s_\xi(x)$ is defined as $\inf \mathrm{Orb}(D,x)$. We can assume that $\eta \geqslant \gamma$ for $\eta \in D$ and therefore, by $(7)_\xi$, $s_\xi(x) \geqslant y$. Otherwise, $s_\xi(x) = \text{du-lim}\, \mathrm{Orb}(A,x)$, where A is a CA-set in ξ for x. Again we can assume that $\eta \geqslant \gamma$ for $\eta \in A$. Then by definition, $s_\xi(x) = \sup\{a_\alpha : \alpha \in A\}$, where $a_\alpha = \inf\{f_\eta(x) : \eta \in A \wedge \alpha \leqslant \eta\}$ and so $y \leqslant a_\alpha \leqslant s_\xi(x)$. Similarly, if $t_\xi(x)$ exists, then $y \leqslant t_\xi(x)$.

3.15. Proof of $(14)_\xi$

Assume that both $s_\xi(x)$ and $t_\xi(x)$ exist. If there is a CD-set D in ξ for x, then $s_\xi(x) = \inf \mathrm{Orb}(D,x)$ and so $s_\xi(x) \leqslant f_\alpha(x)$ for some $\alpha \in D$. By $(12)_\xi(b)$, $s_\xi(x) \in P_\xi \subseteq P_\eta$ for all $\eta < \xi$ and then, by $(18)_\eta$, $s_\xi(x) \leqslant f_\eta(x)$ for all $\alpha \leqslant \eta < \xi$, and hence $M_{\xi,x}(s_\xi(x))$ contains a final segment of ξ. Then $s_\xi(x) \leqslant t_\xi(x)$ follows from $(13)_\xi$. Otherwise,

$$s_\xi(x) = \text{du-}\lim \mathrm{Orb}(A,x)$$

for some CA-set A in ξ for x, and in this case $s_\xi(x) = \sup\{a_\alpha : \alpha \in A\}$, where $a_\alpha = \inf\{f_\eta(x) : \eta \in A \wedge \alpha \leqslant \eta\}$. For fixed $\alpha \in A$, $M_{\xi,x}(a_\alpha)$ contains the CA-set $\{\eta : \eta \in A \wedge \alpha \leqslant \eta\}$ in ξ for x. Then by $(9)_\xi$ it follows that $M_{\xi,x}(a_\alpha)$ contains a final segment of ξ and hence $a_\alpha \leqslant t_\xi(x)$ by $(13)_\xi$. Since α is arbitrarily chosen, it follows that $s_\xi(x) \leqslant t_\xi(x)$. □

3.16. Proof of $(16)_\xi$

(a) Assume that $N_{\xi,x}(y) = \{\eta < \xi : f_\eta(x) \leqslant y\}$ is cofinal in ξ. By $(15)_\xi$, $f_\xi(x) = s_\xi(x)$ or $t_\xi(x)$. If $N_{\xi,x}(y)$ contains a final segment of ξ, then $f_\xi(x) \leqslant y$ by $(13)_\xi$. Therefore, by $(9)_\xi$ we can assume that $N_{\xi,x}(y)$ contains no CA-set. Therefore, by $(5)_\xi$, $N_{\xi,x}(y)$ either contains a CD-set, D, or a CI-set, I. Then by $(15)_\xi$, $f_\xi(x)$ is either $\inf \mathrm{Orb}(D,x)$ or $\sup \mathrm{Orb}(A,x)$. But in either case, $f_\xi(x) \leqslant y$ since D (or I) is a subset of $N_{\xi,x}(y)$.

(b) This is an immediate consequence of $(13)_\xi$ and $(15)_\xi$.

(c) Suppose that $M_{\xi,x}(y)$ is cofinal in ξ but $y \nleqslant f_\xi(x)$. We prove the following two claims.

Claim 1. *$M_{\xi,x}(y)$ contains a CI-set I for x.*

If $M_{\xi,x}(y)$ contains a CD-set D for x, then, by $(7)_\xi$, $(11)_\xi$ and $(15)_\xi$,

$$y \leqslant \inf \mathrm{Orb}(D,x) = f_\xi(x),$$

which is a contradiction. If $M_{\xi,x}(y)$ contains a CA-set for x, then, by $(9)_\xi$, it is a final segment of ξ and so by $(13)_\xi$ we have the same contradiction that $y \leqslant f_\xi(x)$. Therefore, by $(5)_\xi$, $M_{\xi,x}(y)$ contains a CI-set I for x.

Claim 2. *ξ contains a CD-set D for x and $f_\xi(x) = \inf \mathrm{Orb}(D,x)$.*

Assume that no such CD-set exists. Then by $(15)_\xi$,

$$f_\xi(x) = \sup \mathrm{Orb}(I,x) \geqslant y$$

since I is a CI-set in ξ for x, and this is a contradiction. Hence there is a CD-set D and $f_\xi(x) = \inf \mathrm{Orb}(D,x)$ by $(15)_\xi$.

Claim 3. *$M_{\xi,x}(y) \cap M_{\xi,x}(f_\xi(x))$ is cofinal in ξ.*

Let $\alpha = \min D$. For $\alpha \leqslant \eta < \xi$, by $(15)_\xi$ and $(12)_\xi$(b) we have that $f_\xi(x) \in P_\xi \subseteq P_\eta$. By $(18)_\eta$, $f_\xi(x) \leqslant f_\alpha(x)$ implies $f_\xi(x) \leqslant f_\eta(x)$. Therefore, $M_{\xi,x}(f_\xi(x))$ contains a final segment of ξ and so $M_{\xi,x}(y) \cap M_{\xi,x}(f_\xi(x))$ is cofinal in ξ. □

3.17. Proof of $(17)_\xi$

If $\xi = \eta + 1$ is a successor ordinal then $f_\xi = g_\eta \circ f_\eta$ by $(15)_\xi$. f_η is a retraction by $(17)_\eta$ and g_η is a retraction by Theorem 3.4(2) in [1]. Hence, f_ξ is a retraction. We now assume that ξ is a limit ordinal.

By $(12)_\xi$(b) and $(15)_\xi$ it follows that $f_\xi(x) \in P_\xi$ for any $x \in P$ and so it is enough to verify that f_ξ is order preserving and that $x = f_\xi(x)$ for $x \in P_\xi$. If $x \in P_\xi$, then $x \in P_\eta$ for $\eta < \xi$, and so $x = f_\eta(x)$ by $(17)_\eta$. Therefore, ξ itself is a CD-set for x, so $f_\xi(x) = \inf \mathrm{Orb}(\xi,x) = x$.

Suppose that $x, y \in P$ and $x < y$, we want to show that $f_\xi(x) \leqslant f_\xi(y)$. If the ξ-orbits of x and y are not ξ-disjoint, then $f_\xi(x) = f_\xi(y)$. Therefore, by $(4)_\xi$, we may assume that either (a) there exists $\alpha < \xi$ such that $M_{\xi,y}(f_\alpha(x))$ is cofinal in ξ or (b) there exists $\beta < \xi$ such that $N_{\xi,x}(f_\beta(y))$ is cofinal in ξ.

Suppose (a) holds. If $f_\alpha(x) \leqslant f_\xi(y)$, by $(18)_\eta$, $f_\eta(x) \leqslant f_\xi(y)$ for all $\alpha \leqslant \eta < \xi$, and so $N_{\xi,x}(f_\xi(y))$ contains a final segment of ξ. Then, by $(16)_\xi$(a), $f_\xi(x) \leqslant f_\xi(y)$. On the other hand, if $f_\alpha(x) \nleqslant f_\xi(y)$, then by $(16)_\xi$(c), there is a CD-set D in ξ for y such that $f_\xi(y) = \inf \mathrm{Orb}(D, y)$. If $\zeta, \eta \in D$, and $\zeta \leqslant \eta$, then by $(17)_\eta$, $f_\zeta(y) \geqslant f_\eta(y) \geqslant f_\eta(x)$. Therefore, $N_{\xi,x}(f_\zeta(y))$ is cofinal in ξ, and hence $f_\zeta(y) \geqslant f_\xi(x)$ by $(16)_\xi$(a). Thus $f_\xi(x)$ is a lower bound of $\mathrm{Orb}(D, y)$ and so $f_\xi(x) \leqslant f_\xi(y)$.

If (b) holds, by $(16)_\xi$(a), $f_\xi(x) \leqslant f_\beta(y)$ and then, by $(18)_\zeta$, $f_\xi(x) \leqslant f_\zeta(y)$ for $\beta \leqslant \zeta < \xi$. Therefore, $M_{\xi,y}(f_\xi(x))$ contains a final segment of ξ and so $f_\xi(x) \leqslant f_\xi(y)$ by $(16)_\xi$(b).

3.18. Proof of $(18)_\xi$

Let $\alpha \leqslant \beta \leqslant \xi$, $f_\alpha(x) \leqslant y$ and $y \in P_\xi$. For $\beta < \xi$, we see that $f_\beta(x) \leqslant y$ by $(18)_\beta$. So, we need only consider the case when $\alpha < \beta = \xi$. If $\xi = \eta + 1$ is a successor ordinal, then $f_\eta(x) \leqslant y$ by $(18)_\eta$. Then $(15)_\xi$ and Theorem 3.4(2) of [1] imply $f_\xi(x) \leqslant y$. If ξ is a limit ordinal, using $(18)_\eta$, we have $f_\eta(x) \leqslant y$ for all $\alpha \leqslant \eta < \xi$. Therefore, $N_{\xi,x}(y)$ contains a final segment of ξ and then, by $(16)_\xi$(a), $f_\xi(x) \leqslant y$. The second implication of $(18)_\xi$ follows in essentially the same way, the only difference is that we use $(16)_\xi$(b) instead of $(16)_\xi$(a). □

3.19. Proof of $(19)_\xi$

If $\eta \leqslant \xi$ and $x \in P$, then by $(17)_\xi$ $f_\xi(x) \in P_\xi \subseteq P_\eta$, and so by $(17)_\eta$, $f_\eta(f_\xi(x)) = f_\xi(x)$, i.e. $f_\eta \circ f_\xi = f_\xi$. Suppose $\eta < \xi$. If $\xi = \zeta + 1$, then by $(19)_\zeta$ and $(15)_\xi$,

$$f_\xi(f_\eta(x)) = g_\zeta(f_\zeta(f_\eta(x)) = g_\zeta(f_\zeta(x)) = f_\xi(x).$$

Also, if ξ is a limit and $\eta \leqslant \zeta < \xi$, then by $(19)_\zeta$, $f_\zeta(x) = f_\zeta(f_\eta(x))$ and so the ξ-orbits of x and $f_\eta(x)$ are eventually the same, and therefore, $f_\xi(x) = f_\xi(f_\eta(x))$ by $(15)_\xi$. In either case, $f_\xi \circ f_\eta = f_\xi$. □

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