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Families Satisfying the Haar Condition

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1. LINEAR FAMILIES AND THE HAAR CONDITION

Let X be a compact normal space and V a closed subset of X. Let $C(V, X)$ be the family of continuous real functions on X which vanish on V. In most cases of interest V will be empty but there exist many cases of interest in which V consists of a single point.

DEFINITION. An n-dimensional linear subspace of $C(V, X)$ is said to be a *Haar subspace* (*on* X) *with null space* V if only the zero element vanishes on n points of $X \sim V$.

DEFINITION. The functions $\phi_1, \ldots, \phi_n$ of $C(V, X)$ form a *Chebyshev set* on X with null set V if no nontrivial linear combination of them vanishes on n points of $X \sim V$.

Therefore, a basis $\{\phi_1, \ldots, \phi_n\}$ of a linear subspace of $C(V, X)$ is a Chebyshev set if and only if the linear subspace is a Haar subspace.

2. THE CHEBYSHEV APPROXIMATION PROBLEM

For $g \in C(X)$ define

$$\| g \| = \sup\{| g(x)| : x \in X\}.$$

Let F be an approximating function with parameter space P such that $F(A, \cdot) \in C(V, X)$ for all $A \in P$. In particular F may be a linear approximating function

$$F(A, x) = \sum_{k=1}^{n} a_k \phi_k(x), \qquad P = E_n .$$

The Approximation problem is: Given $f \in C(V, X)$ to find a parameter $A^* \in P$ for which $\| f - F(A, \cdot)\|$ is minimal. Such a parameter A^* is called *best* and $F(A^*, \cdot)$ is called a *best Chebyshev approximation* to f.

The following theorem is a generalization of the Haar Theorem.

THEOREM 1. *A necessary and sufficient condition that each $f \in C(V, X)$ have a unique best Chebyshev approximation by an n-dimensional linear subspace L is that L be a Haar subspace whose null space is contained in V.*

Proof. Sufficiency is proven in [7]. We now prove necessity. Let L not be a Haar subspace; then there exists $p \in L$ not vanishing identically which vanishes on n distinct points $x_1, ..., x_n$ of $X \sim V$. From L being of dimension at most $n - 1$ on $\{x_1, ..., x_n\}$ it can be seen that there exist signs $\sigma_1, ..., \sigma_n$ such that there is no $q \in L$ such that

$$\operatorname{sgn}(q(x_i)) = \sigma_i, \qquad i = 1, ..., n. \tag{0}$$

Define $g(x_i) = \sigma_i$, $i = 1, ..., n$, and $g(x) = 0$ for $x \in V$, then g is continuous on $\{x_1, ..., x_n\} \cup V$. By the Tietze extension theorem there is a continuous extension of g to X such that $\| g \| = 1$. Let

$$f(x) = g(x)[\| p \| - | p(x) |],$$

then $| f(x_i) | = \| p \|$, $i = 1, ..., n$ and by choice of f, g, we have $\| f \| = \| p \|$. If $\| f - q \| < \| f \|$ then q would satisfy (0), which is impossible for $q \in L$. Hence 0 is best and since

$$| f(x) - p(x) | \leqslant | f(x) | + | p(x) | \leqslant \| p \| - | p(x) | + | p(x) |,$$

p is also best to f.

In the case of linear Chebyshev approximation on an interval $[\alpha, \beta]$ we have as a special case of the theory of [5]

THEOREM 2. *Let L be a Haar subspace of dimension n on $[\alpha, \beta]$ with null space V, which consists of at most α and β. A necessary and sufficient condition that $F(A, \cdot) \in L$ be a best linear Chebyshev approximation to $f \in C(V, [\alpha, \beta])$ is that $f - F(A, \cdot)$ alternate n times. Best approximations are unique.*

The above characterization and uniqueness result is valid for approximation on a compact subset Y of $[\alpha, \beta]$ such that $Y \sim V$ contains n or more points, where alternation is on Y.

3. L_p NORMS

Let us consider L_p norms. An inspection of the proof of Cheney [1, 220] gives a result stated as a problem in [1, 223, Problem 11].

THEOREM 3. *Let L be a Haar subspace on (α, β), then each $f \in C[\alpha, \beta]$ has a unique best L_1 approximation from L.*

In [4] is given a theorem concerning oscillation of the error curve of a best linear approximation with respect to a generalized integral norm on an interval in terms of Haar subspaces. In the case of approximation with respect to an L_p norm, $1 \leqslant p < \infty$, the theorem reduces to the following:

THEOREM 4. *Let $L(A, \cdot)$ be a best linear approximation to $f \neq L(A, \cdot)$ on $[\alpha, \beta]$. If the linear family contains a Haar subspace of dimension n on (α, β), then*

(i) *$f - L(A, \cdot)$ has n sign changes, or*

(ii) *$p = 1$ and $\mu\{x: f(x) - L(A, x) = 0\} > 0$.*

In [8] is given a theorem concerning oscillation of the error curve of a best linear approximation with respect to a generalized integral norm on a finite subset of an interval in terms of Haar subspaces. In the case of approximation with respect to an L_p norm, $1 < p < \infty$, the theorem reduces to the following:

THEOREM 5. *Let $L(A, \cdot)$ be a best linear approximation to $f \neq L(A, \cdot)$ on a finite subset of $[\alpha, \beta]$. If the linear family contains a Haar subspace of dimension n on $[\alpha, \beta]$, then $f - L(A, \cdot)$ has n sign changes.*

THEOREM 6. *Let $L(A, \cdot)$ be a best approximation with respect to a weighted L_p "norm," $0 < p < 1$, on finite X. Let the linear family contain a Haar subspace of dimension n on X. Then $f - L(A, \cdot)$ has at least n zeros on X.*

We use the arguments of Rice [13, 289–290]. The theorem also holds for $L(A, \cdot)$ locally best.

Study of L_1 approximation by constants on a set X of two points shows that best L_1 approximations on finite X need not oscillate (as in Theorem 5) or interpolate (as in Theorem 6). However, best L_1 approximations must weakly interpolate [13, 278, Theorem 13-7; 8, Theorem 3]. The results of Rice [12, pp. 114–116] also apply.

4. SPACES OF DEFINITION

We consider on what spaces X can a Chebyshev set exist. Consider first the case when the Chebyshev set consists of one element. If V is empty, then the constant function 1 forms a Chebyshev set. Let X be a perfectly normal space (which is true if it is a metric space) and V be a closed subset of X, then there exists $\phi \in C(X)$ such that $\phi(x) = 0$ if and only if $x \in V$ [3, 148]. We see, therefore, that the existence of a Chebyshev set of one element does not restrict X significantly.

The existence of a Chebyshev set of two or more elements does restrict X greatly [2, p. 1028; 9].

THEOREM 7. *Let X be a compact Hausdorff space. If a Chebyshev set of two or more elements exists on X with no null points, then X is homeomorphic to a closed subset of the circumference of the unit circle in 2-space.*

If there are null points, X can be slightly more complex. For example X can be a figure eight, consisting of two circles X_1 and X_2 touching at a single point x_0. $\{1, x\}$ is a Chebyshev set on $[-1, 1]$. Hence by the following lemma $T = \{(x - 1)\, x(x + 1), (x - 1)\, x^2(x + 1)\}$ is a Chebyshev set with null set $\{-1, 0, 1\}$. We can map $[-1, 1]$ continuously onto the figure eight so that $(-1, 0)$ is mapped 1:1 onto X_1, $(0, 1)$ is mapped 1:1 onto X_2, and $\{-1, 0, 1\}$ are mapped onto x_0. With the corresponding change of variable, T is a Chebyshev set on the figure eight with $\{x_0\}$ the null set.

5. THEORY CONCERNING CHEBYSHEV SETS

In the following $*$ denotes multiplication.

LEMMA 1. *Let s be a continuous nonnegative function on X and V_1 be its set of zeros. Let $\{\phi_1, ..., \phi_n\}$ be a Chebyshev set with null set V_2. Then $\{s * \phi_1, ..., s * \phi_n\}$ is a Chebyshev set with null set $V_1 \cup V_2$.*

The following lemma (proved by Rolle's Theorem) is useful in finding Chebyshev sets.

LEMMA 2. *Let $\{\psi_1, ..., \psi_n\}$ be a Chebyshev set on $[0, \alpha]$ whose null set contains at most $\{0, \alpha\}$. Let $\phi_i' = \psi_i$, $i = 1, ..., n$, then $\{1, \phi_1, ..., \phi_n\}$ is a Chebyshev set on $[0, \alpha]$.*

LEMMA 3. *Let $\{\psi_1, ..., \psi_n\}$ be a Chebyshev set on $[0, \alpha]$ whose null set contains at most $\{0, \alpha\}$. Let $\phi_i' = \psi_i$ and $\phi_i(0) = 0$, $i = 1, ..., n$; then $\{\phi_1, ..., \phi_n\}$ is a Chebyshev set on $[0, \alpha]$ with null set $\{0\}$.*

The proof is similar to that for Lemma 2.

LEMMA 4. *Let $\phi^{(n)}$ be continuous and nonvanishing on (α, β); then $\{1, x, ..., x^{n-1}, \phi\}$ is a Chebyshev set on $[\alpha, \beta]$.*

This is problem 8 of [1, 77] and is proved by Rolle's Theorem.

LEMMA 5. *Let* $\{\phi, \psi\}$ *be in* $C^n[\alpha, \beta]$. *Let* $\psi^{(n)}$ *not vanish on* (α, β) *and* $\phi^{(n)}$ *have at most one zero in* (α, β). *Let* $\phi^{(n)}/\psi^{(n)}$ *be strictly monotonic on* (α, β). *Then* $\{1,..., x^{n-1}, \phi, \psi\}$ *is a Chebyshev set on* $[\alpha, \beta]$.

Proof. Suppose

$$L(A, x) = a_1 + \cdots + a_n x^{n-1} + a_{n+1}\phi(x) + a_{n+2}\psi(x)$$

has $n + 2$ zeros on $[\alpha, \beta]$. The first possibility is that $a_{n+1} = a_{n+2} = 0$. Since $\{1,..., x^{n-1}\}$ is a Chebyshev set, this implies $A = 0$. We can, therefore, suppose that one of a_{n+1}, a_{n+2} is nonzero. Suppose first that $a_{n+2} = 0$. In this case $L^{(n)}(A, x) = a_{n+1}\phi^{(n)}(x)$ has two zeros in (α, β), contrary to hypothesis. Next let $a_{n+2} \neq 0$. In this case $L^{(n)}(A, x) = a_{n+1}\phi^{(n)}(x) + a_{n+2}\psi^{(n)}(x)$ has two zeros in (α, β), hence $a_{n+1}\phi^{(n)}(x)/\psi^{(n)}(x) + a_{n+2}$ has two zeros, contrary to strict monotonicity of $\phi^{(n)}/\psi^{(n)}$.

We consider the case where $\phi = x\psi'$. If $x\psi^{(n+1)}/\psi^{(n)}$ is strictly monotonic on (α, β) and $\psi^{(n)}$ does not vanish on (α, β), (i) $\phi^{(n)} = x\psi^{(n+1)} + n\psi^{(n)}$ is strictly monotonic and hence has at most one zero on (α, β), and (ii) $\phi^{(n)}/\psi^{(n)} = [x\psi^{(n+1)}/\psi^{(n)}] + n$ is strictly monotonic.

COROLLARY. *Let* ψ *be in* $C^{n+1}[\alpha, \beta]$. *Let* $\psi^{(n)}$ *not vanish on* (α, β). *Let* $x\psi^{(n+1)}/\psi^{(n)}$ *be strictly monotonic on* (α, β). *Then* $\{1,..., x^{n-1}, \psi, x\psi'\}$ *is a Chebyshev set on* $[\alpha, \beta]$.

THEOREM 8. *Let* T *be a Chebyshev set on* $[-\alpha, \alpha]$ *and be composed of even functions* $\{\phi_1, ..., \phi_n\}$ *and odd functions* $\{\psi_1, ..., \psi_m\}$. *The even set is a Chebyshev set on* $[0, \alpha]$. *The odd set is a Chebyshev set with null point* 0 *on* $[0, \alpha]$.

Proof. Let f be a given element of $C[0, \alpha]$ then by defining

$$f(-x) = f(x) \qquad 0 \leqslant x \leqslant \alpha,$$

f becomes an even element of $C[-\alpha, \alpha]$. Let $E + O$ be a best approximation by T to f on $[-\alpha, \alpha]$, E a sum of even functions and O a sum of odd functions. The error curve of this best approximation is

$$f(x) - E(x) - O(x) = f(-x) - E(-x) + O(-x),$$

and so it follows that

$$\begin{aligned} &\max\{| f(x) - E(x) - O(x)|: -\alpha \leqslant x \leqslant \alpha\} \\ &\quad = \max\{| f(x) - E(x) + O(x): -\alpha \leqslant x \leqslant \alpha\}. \end{aligned}$$

Hence $E - O$ is also best. But since T is a Chebyshev system, a best Chebyshev approximation is unique, hence $O \equiv 0$. Consider now the problem of best approximation of f on $[0, \alpha]$ by a linear combination of $\{\phi_1, ..., \phi_n\}$. Suppose F was a best approximation, then by evenness of $f - F$,

$$\max\{|f(x) - F(x)|: 0 \leqslant x \leqslant \alpha\} = \max\{|f(x) - F(x)|: -\alpha \leqslant x \leqslant \alpha\}.$$

It follows that F is a best approximation on $[-\alpha, \alpha]$ by a linear combination of $\{\phi_1, ..., \phi_n\}$. But there is a unique solution to the problem, namely E. Since this is true for all f, $\{\phi_1, ..., \phi_n\}$ is a Haar subspace on $[0, \alpha]$.

Let f be a given element of the continuous functions on $[0, \alpha]$ vanishing at 0. By defining

$$f(-x) = -f(-x) \qquad 0 \leqslant x \leqslant \alpha$$

f becomes an odd element of $C[-\alpha, \alpha]$. Let $E + O$ be a best Chebyshev approximation by T to f on $[-\alpha, \alpha]$, E a sum of even functions and O a sum of odd functions. The error curve of this best approximation is

$$f(x) - E(x) - O(x) = -(f(-x) + E(-x) - O(-x))$$

and so it follows that

$$\max\{|f(x) - E(x) - O(x)|: -\alpha \leqslant x \leqslant \alpha\} = \max\{|f(x) + E(x) - O(x)|: -\alpha \leqslant x \leqslant \alpha\}.$$

Hence $-E + O$ is also best. But since T is a Chebyshev system, a best approximation is unique and so $E \equiv 0$. Consider now the problem of best approximation of f on $[0, \alpha]$ by a linear combination of $\{\psi_1, ..., \psi_m\}$. Suppose F was a best approximation, then by oddness of $f - F$,

$$\max\{|f(x) - F(x)|: 0 \leqslant x \leqslant \alpha\} = \max\{|f(x) - F(x)|: -\alpha \leqslant x \leqslant \alpha\}.$$

It follows that F is a best approximation on $[-\alpha, \alpha]$ by a linear combination of $\{\psi_1, ..., \psi_m\}$. But there is a unique solution to this problem, namely O. Since this is true for all f, $\{\psi_1, ..., \psi_m\}$ is a Haar subspace with null point 0.

6. Examples of Chebyshev Sets

Example 1. $\{1, x, ..., x^n\}$ is a Chebyshev set on any finite interval. Let us consider arbitrary sets of nonnegative powers. Consideration of the case $\{1, x^2\}$ shows that these need not be Chebyshev sets on an interval containing zero as an interior point. Let α be positive and consider the interval $[0, \alpha]$. Let $0 < \gamma(1) < \cdots < \gamma(n)$. Since $x^\gamma = \exp(\gamma \log(x))$ for $\gamma > 0$ and $x \geqslant 0$, it can be deduced from the remarks following Example 6 that $\{x^{\gamma(1)}, ..., x^{\gamma(n)}\}$

is a Chebyshev set on $[0, \alpha]$ with null set $\{0\}$. If the function 1 is added, it is a Chebyshev set on $[0, \alpha]$.

EXAMPLE 2. $\{1, \cos x, \sin x,..., \cos(nx), \sin(nx)\}$ is a Chebyshev set on $[-\pi, \pi]$ with endpoints identified [11, p. 84].

EXAMPLE 3. $\{1, \cos x,..., \cos(nx)\}$ is a Chebyshev set on $[0, \pi]$. An argument in terms of change of variable is used by Remez [11, p. 82], but we can also use Example 2 and Theorem 8.

EXAMPLE 4. $\{\sin x,..., \sin(nx)\}$ is a Chebyshev set on $[0, \pi]$ with null set 0 by Example 2 and Theorem 8, or by Example 3 and Lemma 4.

EXAMPLE 5. $\{\exp(\gamma_1 x),..., \exp(\gamma_n x)\}$, $\gamma_1 < \cdots < \gamma_n$, is a Chebyshev set on any finite interval. An argument using a special case of Lemma 2 and induction is given by Remez [11, 80]. This example is a special case of Example 6 with $m(1) = m(1) = \cdots = m(n) = 0$.

EXAMPLE 6. Let $\gamma_1 < \cdots < \gamma_n$. $\{\exp(\gamma_1 x), x\exp(\gamma_1 x),..., x^{m(1)}\exp(\gamma_1 x), \exp(\gamma_2 x),..., x^{m(2)}\exp(\gamma_2 x),..., \exp(\gamma_n x),..., x^{m(n)}\exp(\gamma_n x)\}$ forms a Chebyshev set on any finite interval [10, p. 313; 11, p. 81].

Let α be finite. The sets of Examples 5 and 6 are Chebyshev sets on $[\alpha, \infty]$ with null set $\{\infty\}$ when $\gamma_1 < \cdots < \gamma_n < 0$. The sets of Examples 5 and 6 are Chebyshev sets on $[\alpha, \infty]$ when $\gamma_1 < \cdots < \gamma_n = 0$ and $m(n) = 0$.

In the following three examples we still have a Chebyshev set if all powers of x indicated are deleted.

EXAMPLE 7. Let $0 < \gamma_1 < \cdots < \gamma_n$. The set $\{\sinh(\gamma_i x), x\sinh(\gamma_i x), \cosh(\gamma_i x), x\cosh(\gamma_i x): i = 1,..., n\} \cup \{1, x,..., x^m\}$ is a Chebyshev set on any finite interval. The basis is equivalent to a basis of Example 6 with $m(i) = 1$ and powers of x added. It remains a Chebyshev set if any pairs $\{x\sinh(\gamma_i x), x\cosh(\gamma_i x)\}$ are deleted.

EXAMPLE 8. Let $0 < \gamma_1 < \cdots < \gamma_n$. The set $\{\sinh(\gamma_i x), x\cosh(\gamma_i x): i = 1,..., n\} \cup \{x, x^3,..., x^{2m+1}\}$ is a Chebyshev set on $[0, \beta]$ with null set $\{0\}$ by Example 7 and Theorem 8. It remains a Chebyshev set if any elements $x\cosh(\gamma_i x)$ are deleted.

EXAMPLE 9. Let $0 < \gamma_1 < \cdots < \gamma_n$. The set $\{\cosh(\gamma_i x), x\sinh(\gamma_i x): i = 1,..., n\} \cup \{1, x^2,..., x^{2m}\}$ is a Chebyshev set on $[0, \beta]$ by Example 7 and Theorem 8. It remains a Chebyshev set if any elements $x\sinh(\gamma_i x)$ are deleted.

EXAMPLE 10. Let α be a positive number and $-1/\alpha < \gamma_1 < \cdots < \gamma_n$. $\{1/(1+\gamma_1 x),..., 1/(1+\gamma_n x)\}$ is a Chebyshev set on $[0, \alpha]$ by Cauchy's Lemma [1, p. 195]. If $0 < \gamma_1 < \cdots < \gamma_n$, it is a Chebyshev set on $[0, \infty]$ with null set $\{\infty\}$. If $0 = \gamma_1 < \cdots < \gamma_n$, it is a Chebyshev set on $[0, \infty]$.

EXAMPLE 11. Let $\alpha > 0$, $-1/\alpha < \gamma_1 < \cdots < \gamma_n$, and none of $\gamma_1,..., \gamma_n$ be zero. $\{\log(1+\gamma_1 x),..., \log(1+\gamma_n x)\}$ is a Chebyshev set on $[0, \alpha]$ with null point 0 by Example 10 and Lemma 3. If the function 1 is added, we have a Chebyshev set on $[0, \alpha]$ by Example 10 and Lemma 2.

EXAMPLE 12. Let $\alpha > 0$ and $n, m \geqslant 0$. Let $0 < \gamma_1 < \cdots < \gamma_m < 1/\alpha^2$ and $0 < \delta_1 < \cdots < \delta_n$. $\{\text{arctanh}(\gamma_1 x),..., \text{arctanh}(\gamma_m x), \arctan(\delta_1 x),..., \arctan(\delta_n x)\}$ is a Chebyshev set on $[0, \alpha]$ with null set $\{0\}$ by Example 10 (with change of variable to x^2) and Lemma 3. If the function 1 is added to the basis, we have a Chebyshev set on $[0, \alpha]$ by Example 10 and Lemma 2.

The corollary to Lemma 5 gives sufficient conditions for $\{1, \psi, x\psi'\}$ to be a Chebyshev set. In [6] are given many ψ and intervals (μ, ν) (sometimes $[0, \nu]$) on which $\{1, \psi, x\psi'\}$ is a Chebyshev set.

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