THE GROUP OF UNITS IN A COMPACT RING

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If A is a ring with identity and G is the group of units of A, then G acts naturally on the additive group \( A^+ \) of A by left multiplication (the regular action) and by conjugation (the conjugate action). If X is the set of nonzero, nonunits of A, then X is invariant under both actions. It is shown that if A is a compact ring with identity and X is the union of finitely many orbits by the regular action or the conjugate action, then A is finite. Moreover, a characterization of those compact rings with identity for which either action is transitive on X is given.

In a compact ring A with identity, the structure of A is reflected by that of the group of units of A. It is shown that if 2 is a unit in A, then A is finite if and only if G is finite and A is commutative if and only if G is abelian. Moreover, if the conjugate action on X is trivial, then A is a commutative ring.

1. Introduction

Let A be a ring with identity, let G be the group of units of A and let X denote the set of nonzero, nonunits in A. We consider two natural group actions on \( A^+ \), the additive group of A, by G. We call the action, \((g, a) \mapsto g \cdot a\) from \( G \times A \) to A, the regular action and the action, \((g, a) \mapsto gag^{-1}\), the conjugate action. It is easy to see that X is invariant under both actions.

In Section 2, we prove that if A is a compact ring with identity such that X is the union of finitely many orbits under the regular action or the conjugate action, then A is a finite ring. Furthermore, the regular action is transitive on X if and only if A is a finite local ring such that its Jacobson radical J is square zero and J is a one-dimensional vector space over \( A/J \). It is also shown that the conjugate action of G on X is transitive if and only if A is a finite field, A is isomorphic to \( \mathbb{Z}_p \) or to \( \mathbb{Z}_2[x]/(x^2) \) or A is a finite local ring such that \( J^2 = (0) \), J is a one-dimensional vector space over \( A/J \) and for \( g \in G \setminus (1 + J) \) and \( x \in X \), \( gx \neq xg \).

In a compact ring A with identity, the structure of A is reflected by that of the group of units of A. In Section 3, we prove that if A is a compact ring with identity and G is finite, then \( \text{char} \ A \neq 0 \). Furthermore, if \( \text{char} \ A \) is odd, then G is finite if and only if A is finite. In particular, if A is a compact ring with identity such that
2 is a unit in $A$, then $G$ is finite if and only if $A$ is finite. We show as well that if $2$ is a unit in a compact ring $A$ with identity, then $G$ is abelian if and only if $A$ is a commutative ring. Finally, in Section 4, we prove that if $A$ is a compact ring with identity, then $A$ is a commutative ring if every orbit in $X$ by the conjugate action is trivial, that is, if $gxg^{-1} = x$ for all $g \in G$, $x \in X$.

Throughout this paper, unless stated otherwise, $A$ is a ring with identity, $G$ is the group of units of $A$ and $X$ is the set of nonzero, nonunits in $A$.

2. Regular and conjugate actions in a compact ring

We begin with some definitions.

A ring $A$ is a local ring provided that the set of nonunits in $A$ is an ideal of $A$, that is, $X \cup \{0\}$ is an ideal of $A$. In particular, if $A$ is a local ring, then $X \cup \{0\}$ is the unique maximal (right, left or two-sided) ideal of $A$.

If $A$ is a ring, let $J(A)$ denote the Jacobson radical of $A$. In case the ring involved is understood, we write $J$ in place of $J(A)$.

A ring $A$ is a topological ring if $A$ is a topological space and the mappings $(x, y) \to x - y$ and $(x, y) \to xy$ are continuous mappings from $A \times A$ to $A$. We will assume that each compact or locally compact topological ring is Hausdorff.

If $A$ is a topological ring, then $A$ is left bounded if, given any neighborhood $U$ of zero, there exists a neighborhood $V$ of zero such that $AV \subseteq U$. The definition of right bounded is analogous. $A$ is bounded if $A$ is both right and left bounded. Each compact ring is bounded [7, p. 161] but a bounded ring is not necessarily compact. Indeed, if $p$ is a prime, then the ring of integers equipped with the $p$-adic topology is a bounded ring that is not compact.

Lemma 2.1. If $A$ is a compact ring with identity and $G$ is the group of units in $A$, then $G$ is a compact topological group.

Proof. By [1, Exercise 12h, p. 119], $G$ is a closed subset of $A$ and hence is compact. By [2, Theorem], the mapping $g \to g^{-1}$ is continuous on $G$. Therefore $G$ is a topological group. ∎

Corollary. If $A$ is a compact ring with identity, then $X \cup \{0\}$ is an open subset of $A$ where $X$ is the set of nonzero, nonunits in $A$. ∎

If $f : G \times X \to X$ is a group action on $X$, for each $x$ in $X$ we define the orbit $O(x)$ of $x$ by $O(x) = \{ f(g, x) : g \in G \}$. If there exists an $x$ in $X$ such that $O(x) = X$, we say that the group action is transitive. For each $x$ in $X$, we define the stabilizer of $x$ by $\text{Stab}(x) = \{ g \in G : f(g, x) = x \}$.

Theorem 2.2. Let $A$ be a compact ring with identity, let $G$ be the group of units
in $A$ and let $X$ be the set of nonzero, nonunits in $A$. Suppose $f : G \times A \rightarrow A$ is a continuous group action. If $X$ is invariant under $f$ and $X$ is the union of finitely many orbits of $X$, then $A$ is a finite ring.

**Proof.** If $X = \emptyset$, then $A$ is a bounded division ring and hence discrete [7, Theorem 10]. Thus $A$ is finite. So we assume $X \neq \emptyset$. As $G$ is compact and $f$ is continuous, for each $x$ in $X$, $O(x)$ is a closed subset of $A$. Hence $X$ is a closed subset of $A$ and so $G \cup \{0\}$ is an open subset of $A$. But $X \cup \{0\}$ is open as well and thus $\{0\}$ is an open set. Therefore $A$ is a discrete compact space. Consequently, $A$ is a finite ring. □

**Corollary.** If $A$ is a compact ring with identity and $X$ is a finite union of orbits under the regular action or the conjugate action, then $A$ is a finite ring.

**Proof.** As $A$ is a topological ring, the mapping $(g, x) \mapsto gx$ is continuous from $G \times A$ to $A$. Moreover, by Lemma 2.1, $g \mapsto g^{-1}$ is continuous from $G$ to $G$. Hence the mapping $(g, x) \mapsto gxg^{-1}$ from $G \times A$ to $A$ is continuous as well. The corollary then follows from Theorem 2.2. □

**Theorem 2.3.** Let $A$ be a locally compact, bounded ring with identity such that $X$ is a finite union of orbits determined by the conjugate action. Then $A$ is discrete.

**Proof.** Once again, if $X = \emptyset$, then $A$ is discrete. Suppose $X \neq \emptyset$. As $A$ is both left and right bounded, for each neighborhood $U$ of zero, there exists a neighborhood $V$ of zero such that $AVA \subseteq U$.

Case 1. There exists a compact neighborhood $U$ of zero and a neighborhood $V$ of zero such that $AVA \subseteq U$ and $V \cap G \neq \emptyset$.

Then $A - A \cap U \subseteq U \subseteq A$. Hence $A$ is compact. Therefore by the preceding corollary, $A$ is discrete.

Case 2. For all compact neighborhoods $U$ of zero and for all neighborhoods $V$ of zero such that $AVA \subseteq U$, $V \cap G = 0$.

Let $x_1, x_2, \ldots, x_n \in X$ be such that $X = O(x_1) \cup O(x_2) \cup \cdots \cup O(x_n)$. Suppose $A$ is not discrete. Let $U'$ be any neighborhood of zero, let $U$ be a compact neighborhood of zero contained in $U'$ and let $V$ be a neighborhood of zero such that $AVA \subseteq U$. As $V \subseteq X \cup \{0\}$ and $A$ is not discrete, there exists an $x$ in $V \cap X$. Now, $x = gx_i g^{-1}$ for some $g \in G$, $i \in [1, n]$. Hence $x_i = g^{-1}xg \in A \cap U \subseteq U'$, Thus $0 \in \{x_1, x_2, \ldots, x_n\} = \{x_1, x_2, \ldots, x_n\}$, a contradiction. □

**Theorem 2.4.** Let $A$ be a ring with identity such that $X \neq \emptyset$. Then $G$ acts transitively on $X$ by the regular action if and only if $A$ is a local ring, $J^2 = \{0\}$ and $J$ is a one-dimensional left vector space over $A/J$.

**Proof.** Assume that $G$ acts transitively on $X$ by the regular action. Then there exists
x in X such that Gx = X. Note that for g_1 and g_2 in G, g_1 x g_2 x \in X \cup \{0\}. Now, let x_1, x_2 \in X. Let g_1, g_2 \in G be such that x_i = g_i x for i = 1, 2. Then x_1 + x_2 = (g_1 + g_2)x. If g_1 + g_2 \in G, then x_1 + x_2 \in X. If g_1 + g_2 \in X, let g_3 \in G be such that g_1 + g_2 = g_3 x. Then x_1 + x_2 = g_3 x^2 \in X \cup \{0\}. Hence X \cup \{0\} is closed under addition. By the above observation, AX and XA are subsets of X \cup \{0\}. Thus X \cup \{0\} is an ideal of A, that is, A is a local ring. Observe that x^2 = 0. Indeed, if x^2 \neq 0, then x^2 \in X and so there exists g in G with x^2 = gx. Consequently, (x - g)x = 0. But x - g \in G as A is a local ring and hence x = 0, a contradiction. Thus J \setminus \{0\} = X = Gx where x^2 = 0.

We next show that J^2 = \{0\}. Let a \in J. If a \neq 0, then a = gx for some g \in G. Consequently, ax = gx^2 = 0. Thus Jx = \{0\}. Since J is an ideal of A, JGx \subseteq Jx = \{0\}. Thus J^2 = \{0\}.

We finally note that if u \in A, then u is a unit in A if and only if u + J is a unit in A/J. Since A/J is a division ring, J is therefore a one-dimensional left vector space over A/J. The converse is obvious. □

Corollary. Let A be a compact ring with identity such that X \neq 0. Then G acts transitively on X by the regular action if and only if A is a finite local ring such that J^2 = \{0\} and J is a one-dimensional vector space over A/J.

Proof. The corollary follows from Theorem 2.4 and the corollary to Theorem 2.2. □

Theorem 2.5. Let A be a ring with identity such that the characteristic of A is p and |A| = p^2 for some prime p. Then A is commutative and A is isomorphic to Z_p \times Z_p, Z_p[x]/(x^2) or GF(p^2) where GF(p^2) is a finite field having p^2 elements.

Proof. By [9, Theorem 13], as A has an identity, A is commutative. If J = \{0\}, then A is isomorphic to Z_p \times Z_p or to GF(p^2) by [6, Theorem, p. 203; and 5, Theorem, p. 431]. If J \neq \{0\}, then |J| = p and so A is a local ring. Moreover, J^2 = \{0\} by Nakayama’s Lemma [6, Theorem, p. 412]. Since char A = p, A/J = \{1_A + J, \ldots, (p - 1) \cdot 1_A + J\} where 1_A is the identity of A. Let x_0 \in J \setminus \{0\}. By the above, Ax_0 = \{0, x_0, 2 \cdot 1_A x_0, \ldots, (p - 1) \cdot 1_A x_0\} = J. So if x \in A, then there exist unique integers m and n such that 0 \leq m, n \leq p - 1 and x = m \cdot 1_A + n \cdot 1_A x_0. Define \phi : Z_p[x]/(x^2) \to A by \phi(a + bx + (x^2)) = a \cdot 1_A + b \cdot 1_A x_0 where for a \in Z, a is the residue class of a modulo p. It is easy to verify that \phi is an isomorphism from Z_p[x]/(x^2) onto A. □

Corollary. Let A be a compact ring with identity such that X \neq 0 and the regular action on X by G is transitive. If |A/J| = p for some prime p, then A is isomorphic to Z_p[x]/(x^2) or to Z_p^2.

Proof. By the corollary to Theorem 2.4, A is a finite local ring such that |A/J| =
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Let $A$ be a compact ring with identity such that $X \neq 0$. If the conjugate action on $X$ by $G$ is transitive, then $A$ is a finite local ring such that $J^2 = (0)$ and $J$ is a one-dimensional vector space over $A/J$.

Proof. By the corollary to Theorem 2.2, $A$ is a finite ring. By [8, Proposition 1.3], every element of $X$ is a left and right zero divisor and $AX, AX \subseteq X \cup \{0\}$. Let $x'$ and $y'$ be elements of $X$ such that $x'y' = 0$. As $O(x') = X$, there exists $g \in G$ with $y' = gx'g^{-1}$. Then $0 = x'y' = x'gx'g^{-1}$ and hence $(gx')(gx') = 0$. Denote $gx'$ by $x_0$. Then $x_0 \in X$ and $x_0^2 = 0$. Now if $a$ is any element of $X$, then there exists $g_0 \in G$ with $a = g_0x_0g_0^{-1}$. So $a^2 = g_0x_0^2g_0^{-1} = 0$. Consequently for any $a$ and $b$ in $X$, $(a + b)^2 = 0$. So $X \cup \{0\}$ is closed under addition. Thus $A$ is a local ring and $J = X \cup \{0\}$.

To show that $J^2 = (0)$, let $x$ and $y$ be arbitrary elements of $J$. Then there exist $g_1, g_2 \in G$ such that $x = g_1x_0g_1^{-1}$ and $y = g_2x_0g_2^{-1}$. If $xy \neq 0$, then $x_0g_1^{-1}g_2x_0 \neq 0$. Hence $(g_1^{-1}g_2x_0)(g_1^{-1}g_2x_0) \neq 0$, a contradiction. Thus $J^2 = (0)$.

Since $A/J$ is a finite field [7, Theorem 16; 5, Theorem, p. 431; and 6, Theorem, p. 171], there exists $g \in G$ such that $A/J = \{j, 1 + J, g + J, \ldots, g^k + J\}$ where $k + 1$ is the order of $g + J$ in the multiplicative group of $A/J$. Observe that $J = \{0, x_0, gx_0g^{-1}, x_0g^{-1}, \ldots, g^kx_0g^{-1}\}$ and $|J| = |A/J|$. Therefore, $J$ is a one-dimensional vector space over $A/J$.

Corollary. If $A$ is a compact ring with identity such that $X \neq 0$ and the conjugate action on $X$ by $G$ is transitive, then the regular action on $X$ by $G$ is transitive.

Proof. This is an immediate consequence of Theorems 2.4 and 2.6.

Theorem 2.7. Let $A$ be a ring with identity. The following are equivalent:

(i) $A$ is a compact ring such that the conjugate action of $G$ on $X$ is transitive;
(ii) $A$ is a finite field, $A$ is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2[x]/(x^2)$ or $A$ is a finite local ring such that $J^2 = (0)$, $J$ is a one-dimensional vector space over $A/J$ and if $g \in G \setminus (1 + J)$ and $x \in X$, then $gx \neq xg$.

Proof. (i) $\Rightarrow$ (ii). If $X = 0$, then $A$ is a division ring and hence is discrete by [7, Theorem 10]. As $A$ is compact, $A$ is finite and therefore a field by Wedderburn’s Theorem [5, Theorem, p. 43]. If $X \neq 0$, then by Theorem 2.6, $A$ is a finite local ring such that $J^2 = (0)$ and $J$ is a one-dimensional vector space over $A/J$. Hence if $A$ is commutative, then $J = \{0, x\}$ for some $x \in X$. Furthermore, as $|A/J| = |J|$, $|A| = 4$. The Corollary to Theorem 2.5 yields immediately that $A$ is isomorphic to $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$.

Suppose that $A$ is not commutative. If $x, y \in J$, then $(1 + y)x(1 + y)^{-1} = x$ as $J^2 = (0)$. So for all $g \in 1 + J$, $x \in J$, $gx = xg$. Thus there exists $g_0$ in $G \setminus (1 + J)$ and $x \neq 0$.
in $X$ such that $g_0 x \neq x g_0$. By hypothesis, $O(x) = J \setminus \{0\}$. Define $\text{Stab}(x)$ for the conjugate action by $\text{Stab}(x) = \{g \in G : g x g^{-1} = x\}$. As $A/J$ is a finite field, $|A/J| = p^m$ for some prime $p$ and some positive integer $m$. As $|J| = |A/J|$, $|A| = p^{2m}$ and $|G| = p^{2m} - p^m$. Now, $|O(x)| = |G/\text{Stab}(x)|$ by [5, Theorem 1.10, p. 74]. Thus $|\text{Stab}(x)| = (p^{2m} - p^m)/(p^m - 1) = p^m$. As $1 + J \subseteq \text{Stab}(x)$, $\text{Stab}(x) = 1 + J$ for all $x$ in $X$. Thus for any $x \in X$, $g \in G \setminus (1 + J)$, $g x \neq x g$.

(ii) $\Rightarrow$ (i). Clearly, if $A$ is isomorphic to $Z_4$ or to $Z_2[x]/(x^2)$ or if $A$ is a finite field, then $A$ is compact for the discrete topology and the conjugate action of $G$ on $X$ is transitive. Suppose $A$ is a finite local ring such that $J^2 = (0)$, $J$ is a one-dimensional vector space over $A/J$ and for $g \in G \setminus (1 + J)$ and $x \in X$, $g x \neq x g$. Then $A$ is compact for the discrete topology. Moreover, if $x \in X$, then as in the above, $|G| = p^{2m} - p^m$, $|O(x)| = |G/\text{Stab}(x)|$ and $1 + J \subseteq \text{Stab}(x)$. As $g x \neq x g$ for $g \in G \setminus (1 + J)$, $\text{Stab}(x) = 1 + J$. Thus $|O(x)| = p^m - 1$ and hence $X = O(x)$.

Theorem 2.8. Let $A$ be a compact ring with identity such that the regular action or the conjugate action on $X$ by $G$ is transitive. The following statements are equivalent:

(i) $G$ is simple;

(ii) $A$ is a finite field of characteristic 2 such that $|A| = 2^m$ for some positive integer $m$ and $2^m - 1$ is prime or $A$ is isomorphic to $Z_4$, $Z_2[x]/(x^2)$ or to $Z_3$.

Proof. (ii) implies (i) is obvious. Assume (i) holds. If $X = \emptyset$, then $A \setminus \{0\}$ is a group under multiplication and so $A$ is a division ring. By Corollary 2.1, $A$ is discrete, and so finite. Thus $A$ is a finite field. So $|A| = p^m$ for some prime $p$ and some positive integer $m$. If $p$ is odd, then as $G$ is simple, $p = 3$ and $m = 1$, that is, $A$ is isomorphic to $Z_3$. If $p = 2$, then as $G$ is simple, $2^m - 1$ is prime.

If $X \neq \emptyset$, by Theorems 2.4 and 2.6, $A$ is a finite local ring, $J^2 = (0)$ and $|A/J| = |J|$. As $1 + J$ is a normal subgroup of $G$ and as $X \neq \emptyset$, $G = 1 + J$. As $J^2 = (0)$, for any $x$ in $X$, $G x = \{x\}$ and $\{g x g^{-1} : g \in G\} = \{x\}$. Thus for either group action, if $x \in X$, then $O(x) = \{x\}$. So $J = \{0, x\}$ for some $x$ in $A$ and $|A/J| = 2$. Thus $A$ is a 4 element local ring with identity. Consequently, $A$ is isomorphic to $Z_4$ or to $Z_2[x]/(x^2)$. (The proof is the same as that used to establish Theorem 2.7.)

Theorem 2.9. Let $A$ be a compact ring with identity. For each $x$ in $X$, define the stabilizer of $x$ for the regular action by $\text{Stab}(x) = \{g \in G : g x = x\}$. The following are equivalent:

(i) For each $x$ in $X$, $\text{Stab}(x)$ is a trivial subgroup of $G$;

(ii) $A$ has no divisors of zero or $A$ is isomorphic to $\prod_{a \in A} Z_2$ for some index set $A$.

Proof. Clearly (ii) implies (i). Suppose (i) holds. If $A$ has a divisor of zero, then there exist $x$ and $y$ in $X$ with $xy = 0$. We first show that $J = (0)$. Indeed, if $J \neq (0)$, then $J x \neq (0)$ since otherwise $\text{Stab}(x) \supseteq 1 + J$. But then $1 + J x$ is a nontrivial subgroup
of $G$ and $\text{Stab}(y) \supseteq 1 + Jx$, a contradiction. So $J = (0)$. Hence by [7, Theorem 16; 6, Theorem, p. 171; and 5, Theorem, p. 431], $A$ is isomorphic to $\prod_{\alpha \in A} M_{\alpha}$ where each $M_{\alpha}$ is the set of $n_{\alpha} \times n_{\alpha}$ matrices over a finite field $F_{\alpha}$. As $A$ contains divisors of zero, $|A| \geq 2$ or $A$ is isomorphic to the set of $n \times n$ matrices over a finite field $F$ for some $n \geq 2$. In the latter case, there exists $x \in X$ with $x^{2} = 0$. As $(1 + x)(1 - x) = 1$, $1 + x \in G \setminus \{1\}$. So $\text{Stab}(x) \supseteq \{1, 1 + x\}$, a contradiction. In the former case, the above argument yields that each $M_{\alpha}$ is a finite field $F_{\alpha}$. If there exists an $\alpha$ in $A$ such that $|F_{\alpha}| \neq 2$, then the stabilizer of the element $\prod_{\beta \in A} x_{\beta}$ where $x_{\beta} = 1_{\beta}$ for $\beta \neq \alpha$ and $x_{\alpha} = 0_{\alpha}$ contains the set $\prod_{\beta \in A} H_{\beta}$ where $H_{\beta} = \{1_{\beta}\}$ for $\beta \neq \alpha$ and $H_{\alpha} = F_{\alpha}^{*}$, the set of nonzero elements of $F_{\alpha}$, a contradiction. So $A$ is isomorphic to $\prod_{\alpha \in A} Z_{2}$. □

3. Properties of $A$ when $G$ is finite or abelian

We first observe that if $A$ is a ring with identity and $\phi$ is the canonical homomorphism of $A$ onto $A/J$, then the group of units of $A/J$ is the image under $\phi$ of the group of units of $A$. Indeed, if $x, y \in A$ are such that $xy - 1, yx - 1 \in J$, then $x, y \in 1 + J \subseteq G$, i.e., $x$ has both a right and a left inverse in $A$. Hence $x \in G$. So each unit in $A/J$ is the image of some element in $G$. The converse is obvious.

**Theorem 3.1.** Let $A$ be a compact ring with identity.

(i) $G$ is finite if and only if $J$ is finite and there exists a nonnegative integer $n$ and an index set $\Lambda$ such that $A/J$ is isomorphic to $\prod_{i=1}^{n} M_{i} \times \prod_{\alpha \in A} Z_{2}$, where for each $i \in [1, n]$, $M_{i}$ is the set of all $n_{i} \times n_{i}$ matrices over a finite field $F_{i}$.

(ii) If $G$ is cyclic, then $J$ is finite and there exists a nonnegative integer $n$ and an index set $\Lambda$ such that $A/J$ is isomorphic to $\prod_{i=1}^{n} F_{i} \times \prod_{\alpha \in A} Z_{2}$ where each $F_{i}$ is a finite field and for $i \neq j$,

$$|F_{i}| - 1, |F_{j}| - 1 = 1.$$ 

(iii) $X$ is finite if and only if $A$ is finite.

**Proof.** (i) If $G$ is finite, then $J$ is finite as $1 + J \subseteq G$ and by the previous observation, the group of units of $A/J$ is finite. By [7, Theorem 16; 5, Theorem, p. 431; and 6, Theorem, p. 171], $A/J$ is isomorphic to $\prod_{\alpha \in A} A_{\alpha}$ where each $A_{\alpha}$ is the ring of $n_{\alpha} \times n_{\alpha}$ matrices over a finite field $F_{\alpha}$. As the group of units of $A/J$ is finite, $A_{\alpha}$ is isomorphic to $Z_{2}$ for all but finitely many $\alpha \in A$. To prove the converse, we note that the group $G'$ of units of $A/J$ is finite and by the previous observation, $g \in G$ if and only if $g + J \in G'$.

Since $J$ and $G'$ are finite, $G$ is therefore finite.

(ii) By Lemma 2.1, $G$ is a compact topological group. Thus if $G$ is cyclic, then $G$ is finite [3, Lemma 5.28, p. 42]. So the group $G'$ of units of $A/J$ is a finite cyclic group as $G' = \phi(G)$ where $\phi$ is the canonical homomorphism of $A$ onto $A/J$. By (i), $J$ is finite and $A/J$ is isomorphic to $\prod_{i=1}^{n} M_{i} \times \prod_{\alpha} Z_{2}$ for some nonnegative integer $n$. As $G'$ is abelian, each $M_{i}$ is a finite field $F_{i}$ and since $G'$ is cyclic, for $i \neq j$,

$$|F_{i}| - 1, |F_{j}| - 1 = 1.$$
For each subgroup $H$ of $G$, define $A^H$ by $A^H = \{ x \in A : gx = x \text{ for all } g \in H \}$.

**Theorem 3.2.** Let $A$ be a compact ring with identity and let $H = 1 + J$. If $A^H = \{0\}$ or $A^H = A$, then $G$ is abelian if and only if $A$ is commutative.

**Proof.** Suppose $G$ is abelian. If $x, y \in J$, then $(1 + x)(1 + y) = (1 + y)(1 + x)$; so $xy = yx$. Now, if $J = (0)$, then $A$ is isomorphic to a product of finite fields [7, Theorem 16; 5, Theorem, p. 431; and 6, Theorem, p. 171]. Hence $A$ is commutative. Suppose $J \neq (0)$. Then there exists $g$ in $(1 + J) \setminus \{1\}$. So $g^2 \neq g$. Hence $A^H = \{0\}$. Let $a, b \in A$. Suppose $x, y \in J$. Then $axyb = a(xy)(yb) = a(yb)x = a(y)(bx) = a(bx)y = abxy$ and so $x(ab - ba)y = xaby - xbay = abyx - ayx - abxy - abyx = 0$. Thus $J(ab - ba)J = \{0\}$. Since $A^H = \{0\}$, $(ab - ba)J = \{0\}$. If $ab - ba \neq 0$, then $J(ab - ba) \neq \{0\}$ since $A^H = \{0\}$. But if $g \in G, x \in J$, then $gx = xg$. Indeed, let $u = g + x \in G$. Then $gx = g(u - g) = gu - g^2 = ug - g^2 = (u - g)g = xg$. Thus $(1 + J)(ab - ba) = J(ab - ba)(1 + J) = J(ab - ba)$. Hence $J(ab - ba) \subseteq A^H = \{0\}$, a contradiction. So $ab - ba$ for all $a, b \in A$. \qed

Recall that a ring $A$ is prime if whenever $xAy = \{0\}$, $x$ or $y$ is 0.

**Corollary 1.** Let $A$ be a compact prime ring with identity. Then $G$ is abelian if and only if $A$ is an integral domain.

**Proof.** $A^H = \{ x \in A : (1 + J)x = \{x\} \} = \{ x \in A : Jx = \{0\} \}$. If $J = (0)$, then $A^H = A$. If $J \neq (0)$ and $x \in A^H$, then $JAx = \{0\}$ and so $x = 0$. The corollary then follows from Theorem 3.2. \qed

**Corollary 2.** Let $A$ be a compact prime ring with identity. Then $A$ is a finite field if and only if $G$ is a finite abelian group.

**Proof.** If $G$ is finite, then $J$ is finite by Theorem 3.1. Hence by [6, Theorem, p. 412], there exists a positive integer $n$ such that $J^n = (0)$. Now, if $G$ is abelian, then $A/J$ is an integral domain. Indeed, if $(x + J)(y + J) = J$, then $(xy)^n = 0$ and hence $x$ or $y$ is zero by Corollary 1. Consequently, $A/J$ is a finite field by Theorem 3.1. Thus $A$ is finite as well. So $A$ is a finite integral domain and hence a field. \qed

**Theorem 3.3.** If $A$ is a compact ring with identity such that $G$ is finite, then the characteristic of $A$ is nonzero.

**Proof.** By Theorem 3.1, the characteristic of $A/J$ is nonzero. Hence there exists a positive integer $m$ such that $m \cdot 1 \in J$. By a previous argument, $J' = (0)$ for some positive integer $l$. So char $A \neq 0$. \qed
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Theorem 3.4. Suppose $A$ is a compact ring with identity such that the characteristic of $A$ is odd. Then $G$ is finite if and only if $A$ is finite.

Proof. If $G$ is finite and the characteristic of $A$ is odd, then $A/J$ is isomorphic to $\prod_{i=1}^{n} M_i$ where each $M_i$ is the ring of $n_i \times n_i$ matrices over a finite field $F_i$ of odd characteristic by Theorem 3.1. Thus $A/J$ is finite. But $J$ is finite as well. Consequently, $A$ is a finite ring. □

Corollary. Let $A$ be a compact ring with identity such that 2 is a unit in $A$. The following statements are equivalent:

(i) $G$ has a finite number of conjugate classes;
(ii) There are a finite number of orbits in $X$ by the conjugate action of $G$ on $X$;
(iii) $A$ is a finite ring.

Proof. If (i) holds, then as $G$ is a compact topological group (Lemma 2.1), each conjugate class of $G$ is a closed subset of $G$. Since $G$ has a finite number of conjugate classes, $\{1\}$ is open in $G$. Thus $G$ is a finite group. As 2 is a unit in $A$, Theorems 3.3 and 3.4 yield that $A$ is a finite ring. Consequently, (i) and (iii) are equivalent. Obviously (iii) implies (ii). By the corollary to Theorem 2.2, (ii) implies (iii). □

Recall that an element $a$ in $A$ is quasi-regular if there exists $b$ in $A$ such that $a + b + ab - a + ba = 0$. If $A$ has an identity, then $a$ is a quasi-regular if and only if $1 + a$ is a unit in $A$. In particular, if $A$ is a division ring, then every element $a$ of $A \setminus \{-1\}$ is quasi-regular. Moreover, $a + J$ is quasi-regular in $A/J$ if and only if $a$ is quasi-regular in $A$ since $g + x$ is a unit in $A$ for each $g \in G$ and $x \in J$, and each unit in $A/J$ is of the form $g + J$ for some $g$ in $G$.

If $A$ is compact and $G$ is abelian, then $A/J$ is isomorphic to $\prod_{\alpha \in A} F_\alpha$ where each $F_\alpha$ is a finite field by Theorem 3.1. For simplicity we assume that $A/J = \prod_{\alpha \in A} F_\alpha$. Let $\phi : A \to A/J$ denote the canonical epimorphism and for each $\alpha$, let $A_\alpha = \phi^{-1}(H_\beta)$ where $H_\beta = \{0\}$ for $\beta \neq \alpha$ and $H_\alpha = F_\alpha$. Let $\phi_\alpha = \phi | A_\alpha$. Then $\ker \phi_\alpha = \{x \in A : \text{proj}_\alpha(\phi_\alpha(x)) = 0_{\alpha}\}$ where $\text{proj}_\alpha$ is the projection of $\prod_{\beta \in A} F_\beta$ to $F_\alpha$. Therefore, $\ker \phi_\alpha = J$ for each $\alpha$ in $A$. Note also that each $A_\alpha$ is an ideal of $A$. If $1_\alpha$ is the identity of $F_\alpha$, let $\bar{1}_\alpha$ denote the identity of $\prod_{\beta \in A} H_\beta$, that is $\bar{1}_\alpha = \prod_{\beta \in A} x_\beta$ where $x_\beta = 0_{\beta}$ for $\beta \neq \alpha$ and $x_\alpha = 1_\alpha$. Observe that $\phi_\alpha^{-1}(\{\bar{1}_\alpha\})$ is contained in the center of $A_\alpha$ if and only if $\phi_\alpha^{-1}(\{-\bar{1}_\alpha\})$ is contained in the center of $A_\alpha$.

Lemma 3.5. Let $A$ be a ring with identity such that $G$ is abelian. If $x$ and $y$ are quasi-regular elements of $A$, then $xy = yx$. In particular, the radical of $A$ is commutative.

Proof. As $G$ is abelian, $(1 + x)(1 + y) = (1 + y)(1 + x)$. Hence $xy = yx$. Since each element of $J$ is quasi-regular, $J$ is commutative. □
Lemma 3.6. Let $A$ be a ring with identity such that $G$ is abelian and $A/J = \prod_{a \in A} F_a$ where each $F_a$ is a finite field. If $\phi_a^{-1}({\overline{1}_a})$ is contained in the center of $A_a$, then $A_a$ is commutative. Consequently, if $\phi_a^{-1}({\overline{1}_a})$ is contained in the center of $A_a$ for each $a$ in $A$, then $J$ is contained in the center of $\Sigma_{a \in A} A_a$.

Proof. First observe that if $B$ is any ring, $I$ an ideal of $B$ and $x \in I$, then $x$ is a quasi-regular element of the ring $I$ if and only if $x$ is a quasi-regular element of the ring $B$. Indeed, if $b \in B$ is such that $b + x \mid bx = b + x - xb = 0$, then $b \in I$ as $I$ is an ideal of $B$. In particular, by the preceding comments, if $x \in A_a$, then $x$ is quasi-regular if and only if $\phi(x)$ is quasi-regular in $A/J$, that is, $\phi_a(x)$ is quasi-regular in $\overline{F}_a = \prod_{\beta \in A} H_{\beta}$ where $H_{\beta} = \{0,\}$ for $\beta \neq a$ and $H_a = F_a$. Thus for $x \in A_a$, $x$ is quasi-regular if and only if $\text{proj}_a(\phi_a(x)) + 1_a \neq 0_a$.

Now let $x, y \in A_a$. If $x$ and $y$ are quasi-regular, then $xy = yx$ by Lemma 3.5. If $x$ is not quasi-regular, then $\text{proj}_a(\phi_a(x)) + 1_a = 0_a$, that is, $x \in \phi_a^{-1}({\overline{1}_a})$. Thus $x$ is in the center of $A_a$, and so $xy = yx$. Similarly, if $y$ is not quasi-regular, then $yx = xy$.

The last statement of the lemma follows from the above and the observation that $J \subseteq A_a$ for each $a \in A$.

Lemma 3.7. Let $A$ be a ring with identity such that $G$ is abelian and $A/J = \prod_{a \in A} F_a$ where each $F_a$ is a finite field. If $\phi_a^{-1}({\overline{1}_a})$ is contained in the center of $A_a$ for all $a$ in $A$, then $\Sigma_{a \in A} A_a$ is commutative.

Proof. Let $a, \beta \in A$, $a \neq \beta$ and let $x \in A_a$, $y \in A_\beta$. By Lemma 3.6, it suffices to show that $xy = yx$. By Lemma 3.5, we may assume that not both $x$ and $y$ are quasi-regular. Without loss of generality, assume that $x$ is not quasi-regular. Then $\text{proj}_a(\phi_a(x)) = -1_a$. As $xy - yx$ if and only if $(-x)y - y(-x)$, we may assume that $\text{proj}_a(\phi_a(x)) = 1_a$. Now $xy$ and $yx$ are in $A_a \cap A_\beta$ as $A_a$ and $A_\beta$ are ideals of $A$. But for $a \neq \beta$, $A_a \cap A_\beta = J$. So $xy, yx \in J$. By Lemma 3.6, $xy$ and $yx$ are in the center of $A_a$ for each $a$. Hence $x(xy) = (xy)x = x^2y = (yx)x$, that is, $x^2y = yx^2$. Moreover, as $\text{proj}_a(\phi_a(x^2 - x)) = 0_a$, $x^2 - x \in J$. So $(x^2 - x)y = y(x^2 - x)$. Hence $-xy = -yx$, that is, $xy = yx$.

Lemma 3.8. Let $A$ be a ring with identity such that $G$ is abelian and $A/J = \prod_{a \in A} F_a$ where each $F_a$ is a finite field. If $|F_a| \geq 3$ for some $a \in A$, then $\phi_a^{-1}({\overline{1}_a})$ is contained in the center of $A_a$.

Proof. Let $u_a \in F_a \setminus \{0_a, -1_a\}$. Then there exists $w_a$ in $F_a$ such that $u_a w_a = 1_a$. If $w_a = -1_a$, then $u_a = -1_a$, a contradiction. So $w_a + 1_a \neq 0_a$ and hence $u_a$ and $w_a$ are quasi-regular elements of $F_a$. Let $u - \prod_{\beta \in A} x_\beta$ where $x_\beta = 0_\beta$ for $\beta \neq a$ and $x_a = u_a$. Let $w = \prod_{\beta \in A} y_\beta$ where $y_\beta = 0_\beta$ for $\beta \neq a$ and $y_a = w_a$. Then $u$ and $w$ are quasi-regular elements of $\overline{F}_a$ where $\overline{F}_a$ is defined as in Lemma 3.6. Let $x, y, e \in A_a$ be such that $\phi_a(x) = u$, $\phi_a(y) = w$ and $\phi_a(e) = \overline{1}_a$. Then $\text{proj}_a(\phi_a(e - xy)) = 0_a$. So $e - xy \in \text{Ker} \phi_a = J$. As $\phi(x)$ and $\phi(y)$ are quasi-regular elements of $A/J$, $x$ and $y$ are quasi-regular elements of $A$. 


Let \( a \in A \). We will show that \( ae = ea \). If \( a \) is quasi-regular, then \( a(e-xy) = (e-xy)a \) as \( e-xy \) is quasi-regular. Hence \( ae - axy = ea - xya \). But \( x \) and \( y \) are quasi-regular as well and so \( axy = xya \). Thus \( ae = ea \). If \( a \) is not quasi-regular, then \( \text{proj}_a(\phi_a(a)) = 1_a \). Thus \( a + e \in \text{Ker} \phi_a = J \). So \( a = -e + z \) for some \( z \in J \). By the above, as \( z \) is quasi-regular in \( A_a \), \( ze = ez \). So \( ae = -e^2 + ze = -e^2 + ez = ea \). Thus \( \phi_a^{-1}(\{1_a\}) \) is contained in the center of \( A_a \).

**Lemma 3.9.** Let \( A \) be a ring with identity such that \( G \) is abelian and \( A/J = \prod_{a \in A} F_a \) where each \( F_a \) is a finite field. If \( \phi_a^{-1}(\{1_a\}) \) is contained in the center of \( A_a \) for each \( a \in A \), then \( A \) is commutative.

**Proof.** We first show that \( \sum_{a \in A} \overline{A_a} = A \). Suppose that there exists \( x \) in \( A \setminus \sum_{a \in A} A_a \). Then there exists a neighborhood \( U \) of \( x \) such that \( U \cap \sum_{a \in A} A_a \) is empty. As \( x \notin \sum_{a \in A} A_a \), \( \text{proj}_a(\phi(x)) \neq 0_a \) for infinitely many \( a \in A \). Now, \( \phi(U) \) is a neighborhood of \( \phi(x) \) in \( A/J \). As the topology on \( \prod_{a \in A} F_a \) is the product topology, \( \phi(U) \cap \prod_{a \in A} U_a \) where each \( U_a \) is a neighborhood of \( \text{proj}_a(\phi(x)) \) and \( U_a = F_a \) for all but finitely many \( a \in A \). Let \( A' = \{a \in A: U_a \neq F_a\} \). Then \( U \cap \sum_{a \in A} A_a \) is nonempty, a contradiction. Thus \( \sum_{a \in A} A_a = A \). As the continuous mappings \( (x, y) \to xy \) and \( (x, y) \to yx \) agree on \( \sum_{a \in A} A_a \) by Lemma 3.7, they agree on \( A \). Thus \( A \) is commutative.

**Theorem 3.10.** Let \( A \) be a compact ring with identity such that 2 is a unit in \( A \). Then \( G \) is abelian if and only if \( A \) is a commutative ring.

**Proof.** If \( A \) is a compact ring with identity such that \( G \) is abelian, then \( A/J \) is isomorphic to \( \prod_{a \in A} F_a \) where each \( F_a \) is a finite field. Without loss of generality, we may assume that \( A/J = \prod_{a \in A} F_a \). Since 2 is a unit in \( A \), \( 2 + J \) is a unit in \( A/J \). Hence \( \text{char} F_a \neq 2 \) for \( a \in A \). Thus \( |F_a| \geq 3 \) for each \( a \in A \). The theorem then follows from Lemmas 3.7, 3.8 and 3.9.

**Corollary.** If \( A \) is a compact ring with identity such that 2 is a unit in \( A \) and \( G \) is cyclic, then \( A \) is a finite commutative ring.

**Proof.** As \( G \) is cyclic, \( G \) is abelian. Moreover, as in the proof of Theorem 3.1, \( G \) is finite. Thus by Theorems 3.3, 3.4 and 3.10, \( A \) is a finite commutative ring.

**Remark.** If \( A = \{(x, y, z) \in \mathbb{Z}^3: x, y, z \in \mathbb{Z}_2\} \), then \( A \) is a compact ring under the discrete topology and \( G \) is abelian. However, \( A \) is not commutative.

**Theorem 3.11.** Let \( A \) be a compact ring with identity such that 2 is a unit in \( A \). Then \( G \) is finite if and only if \( A \) is finite.

**Proof.** As 2 is a unit, the characteristic of \( A \) is odd. Hence the assertion follows from Theorem 3.4.
We note that $\prod_{i=1}^{\infty} Z_2$ is an infinite ring, compact for the product topology when $Z_2$ is given the discrete topology, and the group of units of $A$ is finite.

4. Trivial conjugate action on $X$

Lemma 4.1. Let $A$ be a compact ring with identity. If the conjugate action on $X$ by $G$ is trivial, that is, if $gxg^{-1} = x$ for all $x$ in $X$ and $g$ in $G$, then $G$ is abelian.

Proof. Suppose $G$ is not abelian. Then there exist $a$ and $b$ in $G$ such that $1-aba^{-1}b^{-1} \neq 0$. Let $x \in X$. Then $(1-aba^{-1}b^{-1})x = x - aba^{-1}(b^{-1}x) = x - ab(b^{-1}x)a^{-1} = x - axa^{-1} = 0$. Thus $X \cup \{0\} = \{y \in A : (1-aba^{-1}b^{-1})y = 0\}$. Therefore, $X \cup \{0\}$ is closed under addition and multiplication. Moreover, for all $g$ in $G$, $g(X \cup \{0\}) \subset X \cup \{0\}$. So $X \cup \{0\}$ is an ideal of $A$, that is, $A$ is a local ring. Since $A$ is a compact ring, $A/J$ is therefore a finite field [7, Theorem 16; 5, Theorem, p. 431; and 6, Theorem p. 171]. Let $g \in G$ be such that $g + J$ is the cyclic generator of the multiplicative group of $A/J$. Let $(A/J)^*$ denote $A/J \setminus \{0\}$. Then $G = (1+J) \cup (g+J) \cup \cdots \cup (g^k+J)$ where $k+1$ is the order of $g+J$ in $(A/J)^*$. If $x, y \in J$, then $(1+x)y = y(1+x)$ as $1+x \in G$. So $xy = yx$ for all $x$ and $y$ in $J$. Thus $G$ is abelian.

Theorem 4.2. Let $A$ be a compact ring with identity such that $gxg^{-1}=x$ for all $x$ in $X$ and $g$ in $G$. Then $A$ is a commutative ring.

Proof. As $G$ is abelian, we may assume that $A/J = \prod_{a \in A} F_a$ where each $F_a$ is a finite field. Using the terminology in Section 3, it suffices to show that $\phi_a^{-1}(\bar{1}_a)$ is contained in the center of $A_a$ for each $a$ in $A$ by Lemma 3.9. By our remarks preceding Lemma 3.5, we need only show that $\phi_a^{-1}(\bar{-1}_a)$ is contained in the center of $A_a$.

Let $a, b \in A_a$ be such that $\phi_a(a) = -\bar{1}_a$. Then proj$_a(\phi_a(a)) = -1_a$. As in the proof of Lemma 3.6, if $x \in A_a$, then $x$ is quasi-regular if and only if proj$_a(\phi_a(a)) \neq -1_a$. So $a$ is not quasi-regular and hence $1+a \in X \cup \{0\}$. If $b$ is quasi-regular, then $1+b \in G$. So $(1+b)(1+a) = (1+a)(1+b)$ by hypothesis. Thus $ba = ab$. Suppose $b$ is not quasi-regular. Then proj$_a(\phi_a(b)) = -1_a$. Let $e_a \in A_a$ be such that $\phi_a(e_a) = -\bar{1}_a$. Then proj$_a(\phi_a(a-e_a)) = 0_a$ and so $a-e_a \in \text{Ker} \phi_a - J$. Similarly, $b-e_a \in J$. Let $x, y \in J$ be such that $a = e_a + x$ and $b = e_a + y$. Note that $x+1, y+1 \in G$ and $e_a + 1 \in X \cup \{0\}$ as $e_a$ is not quasi-regular. So $(x+1)(e_a+1) = (e_a+1)(x+1)$ and consequently $xe_a = e_x x$. A similar argument establishes that $ye_a = e_y y$. By the above and Lemma 3.5, $ab = e_a^2 + e_a y + xe_a + xy = e_a^2 + ye_a + e_x x + yx = ba$. Hence $A$ is a commutative ring.  \[
\]

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References