Comparison of attitude determination approaches using multiple Global Positioning System (GPS) antennas

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Abstract: GPS-based attitude system is an important research field, since it is a valuable technique for the attitude determination of platforms. There exist two classes approaches for attitude determination using the GPS. The one determines attitude via baseline estimates in two frames, the other one solves for attitude by incorporating the attitude parameters directly into the GPS measurements. However, comparisons between these two classes approaches have been unexplored. First of all, two algorithms are introduced in detail which on behalf of these two kinds of approaches. Then we present numerical simulations demonstrating the performance of our algorithms and provide a comparison evaluating.

Key words: GPS; attitude determination; baseline; attitude matrix; constrained LAMBDA; multivariate constrained LAMBDA

1 Introduction

Attitude determination (AD) is the estimation of the orientation of a platform relative to a reference frame. Many sensors and technologies are used to estimate the attitude of a platform, and there is a growing attention in attitude determination based on GPS. Although the accuracy of a stand-alone GPS attitude system might not be comparable with some modern attitude sensors, it has several advantages: driftless, minor maintenance, less expensive than other high-precision systems, such as INS and Star Trackers [1]. According to the analyses about the related literature, GPS-based attitude determination algorithms can be divided into two main categories; the one determines the attitude rotation using baseline vectors in two frames [2] (body frame and reference frame), the other one solves for attitude by incorporating the attitude parameters, in the form of either attitude matrix or quaternion, directly into the GPS phase measurements equations [3]. We will refer to the two approaches as the baseline and attitude approaches respectively.

For the first category, we must find the precise estimation of the baseline vectors. In other words, we have to find the correct carrier phase integer ambiguity values. Many integer ambiguity resolution methods have been proposed in GPS positioning and navigation [4], but these methods are not necessarily optimal for the GPS-based attitude determination problem, for which the baseline length is often known. This baseline-length information has been mostly used for setting up or reducing the searching space of the integer ambiguities, or validating the estimated integers [5–10]. Although these methods improve the efficiency and accuracy of ambiguity resolution, they do not make the best of the given information. Teunissen developed an algorithm called Constrained LAMBDA method [11], it rigorously incorporates length constraints into the integer estimation process, thus it directly aids the ambiguity and baselines fixing. Once the baseline vectors are estimated, we can use an algorithm, TRIAD/QUEST [12–13], or nonlinear least-square fit (NLLSFit)
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method\(^{[14]}\), which could find the rotation matrix transforming the ECEF baselines to body-level baselines.

Now a few researches focus on attitude approaches. Axelrad and Ward\(^{[15]}\) presented an Extended Kalman Filter (EKF) algorithm, which simultaneously determines the attitude quaternion and integer ambiguities. The biggest difficulty in this method is that if the initial is too far off to be correctly modeled by the linear approximation, then it is possible that the solution will not converge to the right answer. Teunissen also developed a multivariate Constrained LAMBDA method (MC-LAMBDA)\(^{[16]}\), which solves for the GPS integer ambiguities and the attitude matrix in an integral manner. It does not require any a priori information about the attitude or the dynamics of the platform, and the observation equation is linear in terms of the unknown ambiguity and attitude matrix. It belongs to point estimation algorithm.

In this work, we use Teunissen’s baseline and attitude matrix version in order to determine whether the two kinds of methods are equivalent in performance, or conclude which one is favorable, and then expound their advantages and disadvantages.

2 The GPS attitude observables model

At least three antennas (non collinear) are necessary to estimate the full orientation of a platform. Due to the short distances of baselines on the same platform, we can mitigate most errors of carrier phase measurements through the double differences. The double difference (DD) code and phase observables for each baseline were taken at the same time, the different satellites \(s_1\) and \(s_2\) can be written as

\[
\begin{align*}
\nabla \Delta \varphi &= s_{1z}^T \delta^E + \lambda \varphi + e \\
\nabla \Delta \rho &= s_{2z}^T \delta^E + e
\end{align*}
\]

where \(\delta^E\) is the baseline vector in ECEF (Earth-Centered, Earth-Fixed) coordinates, \(s\) is the DD line-of-sight vector in ECEF coordinates, \(\lambda\) is the wavelength of the L1 carrier, \(n\) is the DD integer ambiguity, \(e\) and \(\varepsilon\) are the unmodeled errors of code and phase measurements, respectively.

Suppose that \(m + 1\) antennas simultaneously track \(n + 1\) GPS satellites, taking the satellite \(r\) as a reference, the DD code and phase data observed by baseline \(i\) are collected in the vector \(y_i\):

\[
y_i = [\nabla \Delta \rho_i, \cdots, \nabla \Delta \rho_i, \nabla \Delta \varphi_i, \cdots, \nabla \Delta \varphi_i]^T
\]

The DD GPS code and phase observations are then cast into the model

\[
y_i = A z_i + S b_i^E + v \\
z_i \in Z^n, \ b_i^E \in R^3, \ D(y_i) = Q_y
\]

where \(z_i\) contains the unknown integer ambiguities \([z_{1z}^E, \cdots, z_{mz}^E]^T\), \(A\) contains the carrier wavelength, while \(S\) is the matrix of the DD line-of-sight vectors:

\[
A = \begin{bmatrix} 0 \\ \lambda I_n \end{bmatrix}, \ S = \begin{bmatrix} (-s_{1z}^E)^T \\ \vdots \\ (-s_{mz}^E)^T \end{bmatrix}
\]

For multiple baselines, the measurement equations for all the baselines can be combined:

\[
Y = AZ + SB^E + V \\
Z \in Z^{nxn}, \ B^E \in R^{3xn}, \ D(Y) = Q_T
\]

where \(Y\) is the matrix whose columns are the DD code and phase observations of each baseline, \(Z\) is the matrix whose columns are the DD integer-valued ambiguities of each baseline, and \(B^E\) is the matrix whose columns are the baseline vectors. In order to define the variance-covariance matrix of the observables \(Y\), we introduce the vec operator, it stacks \(2n \times m\) matrix \(Y\) into \(2nm \times 1\) matrix vec\((Y)\)\(^{[17]}\).

In the observation equation (5), we can incorporate attitude matrix \(R\) directly into the measurement equation. Suppose \(B^n\) is a known baseline matrix in the body frame, using the attitude relation \(R B^3 = B^E\), the GPS multiple antenna model can also be expressed as:

\[
Y = AZ + SRB^n + V \\
D(Y) = Q_T
\]

In the following section, we present two previous attitude determination algorithms which are pertinent to
the research.

3 Two GPS-based attitude determination methods

Various methods of ambiguity resolution for attitude determination have been proposed in the literature. In order to make such a possible comparison, first we need to present two important integer estimation principles, the CLAMBDA method and the MC-LAMBDA method, which are both proposed by Teunissen.

3.1 The constrained LAMBDA method

The LAMBDA method has become the standard AR-method for the majority of current GPS models. The method is numerically efficient, and it has been proven to provide the highest possible success rate. Due to the nonlinear constraint \( \| b \| = l \) in attitude determination, the LAMBDA method can not be used here. Therefore the constrained LAMBDA method has been developed, it solves the nonlinear constrained GNSS model in a strict integer least-squares sense, so it includes the following three consecutive steps like the LAMBDA method:

(1) First we disregard the integer nature of the ambiguities, adopt the least-squares (4) and obtain the so-called float solutions:

\[
\begin{align*}
\begin{bmatrix}
\tilde{\mathbf{z}} \\
\mathbf{b}^f
\end{bmatrix} &= N^{-1} \begin{bmatrix}
A^T \mathbf{Q}_r^{-1} \mathbf{y} \\
S^T \mathbf{Q}_r^{-1} \mathbf{y}
\end{bmatrix} \\
N &= \begin{bmatrix}
A^T \mathbf{Q}_r^{-1} A & A^T \mathbf{Q}_r^{-1} S \\
S^T \mathbf{Q}_r^{-1} A & S^T \mathbf{Q}_r^{-1} S
\end{bmatrix}
\end{align*}
\]

(7)

(8)

The \( \nu - c \) matrix of the float solutions is obtained by the inversion of the normal matrix \( N \).

(2) The sum-of-squares expression that has to be minimized in the constrained case reads

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, |\mathbf{h}| = l, \mathbf{b}^f \in \mathbb{R}^m} & \| \mathbf{y} - A \mathbf{z} - S \mathbf{b}^f \|_2^2 + \\
& \| \mathbf{h} \|_{Q_h} + \min_{x \in \mathbb{R}^n} \| \tilde{\mathbf{z}} - \mathbf{z} \|_2^2 + \\
& \min_{|\mathbf{b}^f| = l, \mathbf{b}^f \in \mathbb{R}^m} \| \mathbf{b}^f(z) - \mathbf{b}^f \|_2^2 \left[ Q_{\mathbf{b}^f(\mathbf{z})} \right]
\end{align*}
\]

(9)

where \( Q_{\mathbf{b}^f(\mathbf{z})} \) is the \( \nu - c \) matrix of \( \mathbf{b}^f(z) \).

The integer ambiguities are now estimated as the solution of the minimization problem.

\[
\tilde{\mathbf{z}} = \arg \min_{\mathbf{z} \in \mathbb{Z}^n, \mathbf{b}^f \in \mathbb{R}^m} \| \tilde{\mathbf{z}} - \mathbf{z} \|_2^2 + \\
\min_{|\mathbf{b}^f| = l, \mathbf{b}^f \in \mathbb{R}^m} \| \mathbf{b}^f(z) - \mathbf{b}^f \|_2^2 \left[ Q_{\mathbf{b}^f(\mathbf{z})} \right]
\]

(10)

with

\[
\tilde{\mathbf{b}}^f(z) = \arg \min_{|\mathbf{b}^f| = l, \mathbf{b}^f \in \mathbb{R}^m} \| \tilde{\mathbf{b}}^f(z) - \mathbf{b}^f \|_2^2 \left[ Q_{\mathbf{b}^f(\mathbf{z})} \right]
\]

(11)

The search for the integer minimizer (10) is more complex than in the unconstrained case for two reasons: firstly, the search space is no longer ellipsoidal; secondly, the evaluation of the objective function implies the solution of a nonlinear least squares problem to extract the vector \( \tilde{\mathbf{b}}^f(z) \), and this has to be done for each integer candidate. Two search algorithms are employed to search ambiguities in an efficient way, namely the Search and Shrink approach\[14\], and the Search and Expansion approach\[19\].

(3) When the integer minimizer \( \tilde{\mathbf{z}} \) is found, the constrained fixed baseline solution is obtained as

\[
\tilde{\mathbf{b}}^f(\tilde{\mathbf{z}}) = \arg \min_{|\mathbf{b}^f| = l, \mathbf{b}^f \in \mathbb{R}^m} \| \tilde{\mathbf{b}}^f(\tilde{\mathbf{z}}) - \mathbf{b}^f \|_2^2 \left[ Q_{\mathbf{b}^f(\tilde{\mathbf{z}})} \right]
\]

(12)

The constrained method achieves a higher performance due to the rigorous inclusion into the integer estimation process of the nonlinear constraint. Once baselines are determined in both coordinate systems, the sets of vectors are fed to TRIAD/Quest, or NLLSFIT method which return the attitude rotation matrix.

3.2 The MC-LAMBDA method

We introduce an algorithm which solves for attitude directly using the carrier-phase measurements while applying the same optimization techniques from the baseline method. This method parametrizes the attitude using attitude matrix.

Application of the least-squares (LS) estimation principle to model(6), taking the constraints on \( Z \) and \( R \) into account, gives the minimization problem re-
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specifying the constraints posed on the unknowns:

$$\min_{Z \in Z^{**}, R \in O^3} \| \text{vec}(Y - AZ - SRB^b) \|_{\mathcal{Q}}^2$$

(13)

with $$\| \cdot \|_{\mathcal{Q}}^2 = (\cdot)^T \mathcal{Q}^{-1} (\cdot)$$.

According to the equations (1) and (13), the norm (13) can be decomposed into a sum of squares:

$$\| \text{vec}(Y - AZ - SRB^b) \|_{\mathcal{Q}}^2 = \| \text{vec}(Y) - (I_n \otimes A) \text{vec}(Z) - (B^b \otimes S) \text{vec}(R) \|_{\mathcal{Q}}^2 = \| \text{vec}(E) \|_{\mathcal{Q}}^2 + \| \text{vec}(Z - Z) \|_{\mathcal{Q}}^2 + \| \text{vec}(\mathcal{R}(Z) - R) \|_{\mathcal{Q}}^2$$

(14)

where $$\otimes$$ denotes the Kronecker product. The following property of the vec operator $$\text{vec}(M_1 \otimes M_2 \otimes M_3) = (M_1^T \otimes M_2) \text{vec}(M_3)$$ has been used. $$\hat{E}$$ is the matrix of least-squares residuals, $$E = Y - AZ - SRB^b$$.

The decomposition (14) makes use of the float solution, which is the least-squares solution of the equation (13) obtained disregarding both the integer constraint on $$Z$$ and the orthonormality constraint on $$R$$:

$$N = \begin{bmatrix} I_n \otimes A^T & B^b \otimes S^T \end{bmatrix} Q_T^{-1} Q_{GR}^{-1} \text{vec}(Y)$$

(15)

and

$$N = \begin{bmatrix} I_n \otimes A^T & B^b \otimes S^T \end{bmatrix} Q_T^{-1} [I_n \otimes A \ (B^b \otimes S)]$$

(16)

Matrices $$\mathcal{Z}$$ and $$\mathcal{R}$$ are referred as the float solutions of model (13), which do not generally respect the constraints; $$\mathcal{Z}$$ is integer-valued and $$\mathcal{R}$$ is orthogonal. The $$v - c$$ matrices of the float solutions are obtained by inverting the normal matrix:

$$\begin{bmatrix} Q_2 & Q_{2k} \\ Q_{2k} & Q_h \end{bmatrix} = N^{-1}$$

(17)

Once the integer ambiguity matrix $$\mathcal{Z}$$ was computed, the conditional rotation matrix $$\mathcal{R}$$ solution can be obtained as:

$$\text{vec}(\mathcal{R}(\mathcal{Z})) = \text{vec}(\hat{R}) - Q_{2k} Q_h^{-1} \text{vec}(Z - Z)$$

(18)

Application of the variance propagation law to expression (18) gives the $$v - c$$ matrix of $$\mathcal{R}(\mathcal{Z})$$ as:

$$\mathcal{Q}_R(\mathcal{Z}) = \mathcal{Q}_R - Q_{2k} Q_h^{-1} Q_{2k}$$

(19)

The inverse of this matrix is used as the weight matrix in the last term of equation (14). Though the precision of $$\mathcal{R}(\mathcal{Z})$$ is higher than that of $$\hat{R}$$, it is generally not orthogonal.

From the orthogonal decomposition (14), it is clear that the minimization problem that has to be solved is:

$$\hat{Z} = \arg \min_{Z \in Z^{**}} C(Z)$$

(20)

$$C(Z) = \| \text{vec}(Z - Z) \|_{\mathcal{Q}}^2 + \| \text{vec}(\mathcal{R}(Z) - \tilde{R}) \|_{\mathcal{Q}}^2$$

(21)

with

$$\tilde{R} = \arg \min_{R \in O^3} \| \text{vec}(\mathcal{R}(Z) - R) \|_{\mathcal{Q}}^2$$

(22)

A closed-form solution for the minimizer (20) is not known, so a direct search in the space of $$\Omega(\chi^2)$$ must be employed:

$$\Omega(\chi^2) = \{ Z \in Z^{**} | C(Z) \leq \chi^2 \}$$

(23)

where $$\chi^2$$ is a suitably chosen positive constant.

It is clear that the search process of equation (23) is complicated by the tight coupling between the integer term and the attitude term in equation (21). Like the CLAMBDA method, in order to overcome this problem, two search strategies have been also developed: the Search and Shrink approach, and the Expansion approach. Three steps are involved in the MC-LAMBDA method: first, the float estimates of the unknowns are derived as equation (15); then the search for the integer minimizer $$\hat{Z}$$ is performed inside the set $$\Omega(\chi^2)$$; finally, the attitude matrix is extracted by solving the nonlinear constrained problem (22).

The MC-LAMBDA method firstly solves for the ambiguities and then estimates the attitude matrix by solving equation (22). This is different from these methods, where the attitude is determined based on an estimation of the baseline vectors.
4 Experimental results and discussion

4.1 Data

A static experiment is carried out to test the performance of the two algorithms. The GPS design matrices needed for the simulations were constructed by means of the VISUAL software\cite{20}, using the assumed receiver locations and the actual GPS satellite constellation information in YUMA ephemeris format on January 22, 2008. The further details are given the simulation inputs in table 1. Suppose that three baselines (four antennas) are arranged in a platform, which can be expressed in the antenna coordinate system (the body frame system) as, $b_1 = [2 \ 0 \ 0]^T$, $b_2 = [1 \ 1.7 \ 0]^T$, $b_3 = [1 \ 0.5 \ 1.6]^T$ in centimeters. The true values of Euler angles are yaw $49^\circ$, roll $27^\circ$ and pitch $-34^\circ$ respectively.

The simulated data was assumed to be uncorrelated and normally distributed. We simulated two sets of data, which the noise levels are data1 (code:15cm/phase;3mm) and data2 (code:9cm/phase;3mm) respectively. No multipath effect was introduced in the simulated observations.

4.2 Results and analysis

The two sets of data were processed with both the CLAMBDA method and the MC-LAMBDA method. We convert attitude determination results to Euler angles in order to compare their precision. Three aspects of the proposed methods were carefully investigated: the correctness of ambiguity resolution, the precision of the Euler angle estimation and the computational times. The summary results are shown in table 2.

Figures 1 and 2 display the Euler angle estimate results using the CLAMBDA method (baseline method) and the MC-LAMBDA method (attitude matrix method).

Table 1 The simulated conditions

<table>
<thead>
<tr>
<th>Time</th>
<th>09:00–10:00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>Lat: 50°, Lon: 3°, Height: 100m</td>
</tr>
<tr>
<td>Frequency</td>
<td>L1</td>
</tr>
<tr>
<td>Number of satellites</td>
<td>6</td>
</tr>
<tr>
<td>Epoch interval(second)</td>
<td>1</td>
</tr>
<tr>
<td>Elevation mask(degree)</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 2 The two sets of data solution results using the two methods

<table>
<thead>
<tr>
<th>Data</th>
<th>Method name</th>
<th>The success rate of ambiguity resolution (%)</th>
<th>Standard deviation (degree)</th>
<th>Average time per epoch (second)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Yaw</td>
<td>Roll</td>
</tr>
<tr>
<td>Data 1</td>
<td>CLAMBDA</td>
<td>98.9</td>
<td>0.067</td>
<td>0.228</td>
</tr>
<tr>
<td></td>
<td>MC-LAMBDA</td>
<td>100</td>
<td>0.034</td>
<td>0.059</td>
</tr>
<tr>
<td>Data 2</td>
<td>CLAMBDA</td>
<td>100</td>
<td>0.057</td>
<td>0.059</td>
</tr>
<tr>
<td></td>
<td>MC-LAMBDA</td>
<td>100</td>
<td>0.035</td>
<td>0.058</td>
</tr>
</tbody>
</table>

Figure 1 Comparison between two Euler angle estimation results of data 1 using the two methods
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From the above tables and figures, the following conclusions can be drawn:

1. From Table 2 and Figures 1–2, the MC-LAMBDA method shows a large robustness, obtaining a successful fixing (success rates are both 100%) in the two different conditions, and providing a higher success rate and precise attitude solution than the CLAMBDA method in the conditions (data 1). As shown in Figure 1(a), the accuracy of the Euler angle solution is lower at five epochs than other epochs, the reason is that the ambiguities are not fixed to the correct values. As expected, we conducted a statistical study about the epochs which are shown in Table 3. The superior success rate performance compared to the one of the CLAMBDA method is due to the full incorporation of the nonlinear constraints into the integer estimation process, which are not only the baseline lengths, but also the relative orientations between the antennas. And all of these constraints about baselines are converted to orthogonal attitude matrix.

2. In the condition with 3mm phase noise and 9cm code noise, which is maybe better than practice, the CLAMBDA method could obtain the same success rate (100%) as the MC-LAMBDA method, however the precision of attitude solution is little lower than that of the MC-LAMBDA method. The reason is that the MC-LAMBDA solves for attitude angles based on the orthogonality constraint of attitude matrix, but the CLAMBDA method only based on the baselines length constraints.

3. For each method, we compare the average CPU time per epoch. From Table 2, the CLAMBDA method is faster than the MC-LAMBDA method at each epoch. We know that the CLAMBDA method estimates the baselines separately, and then find the attitude, but the MC-LAMBDA method solves for all GPS integer ambiguities and the attitude matrix in an integral manner, so there are more unknowns to be solved, more computations are required.

5 Conclusion

We compared two kinds of approaches for GPS-based attitude determination using the CLAMBDA method and the MC-LAMBDA method. The two approaches were shown to perform more or less equivalently in terms of accuracy, once ambiguities are fixed. The
MC-LAMBDA method rigorously incorporates the non-linear constraints into the integer ambiguities estimation and attitude matrix estimation process; these constraints are given by orthogonal attitude matrix. The strengthening of the model makes the MC-LAMBDA method a very robust method, capable of providing precise attitude estimation in a wider range of conditions. The main disadvantage of the attitude matrix method is that its process speed is lower than the baseline method, which need to be improved in the future.

References