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Groups with many abelian subgroups

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It is known that a (generalized) soluble group whose proper subgroups are abelian is either abelian or finite, and finite minimal non-abelian groups are classified. Here we describe the structure of groups in which every subgroup of infinite index is abelian.

1. Introduction

The structure of groups in which every member of a given system $\mathfrak{X}$ of subgroups is abelian has been studied for several choices of the collection $\mathfrak{X}$. The first step of such investigation is of course the case of the system consisting of all proper subgroups: it is well known that any locally graded group whose proper subgroups are abelian is either finite or abelian, and finite minimal non-abelian groups are completely described. Here a group G is called *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; the class of locally graded groups contains all locally (soluble-by-finite) groups and all residually finite groups, and so it is a large class of generalized soluble groups. On the other hand, the consideration of Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) shows that there exist infinite minimal non-abelian groups. A further progress in this research area was due to G.M. Romalis and N.F. Sesekin,

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who considered groups in which all non-normal subgroups are abelian, and proved that (generalized) soluble groups with such property have finite commutator subgroups (see [6–8]). Recently, groups in which every non-abelian subgroup satisfies a given generalized normality condition have also been studied (see [1]).

The aim of this paper is to describe the class $\mathfrak{A}^\infty$ consisting of all groups in which every subgroup of infinite index is abelian. Of course, the class $\mathfrak{A}^\infty$ is closed with respect to subgroups and homomorphic images and contains all minimal non-abelian groups. Moreover, it is also clear that non-abelian groups in the class $\mathfrak{A}^\infty$ are finitely generated and that every periodic subgroup of a non-periodic $\mathfrak{A}^\infty$-group is abelian. Note also that any $\mathfrak{A}^\infty$-group satisfies the maximal condition on non-abelian subgroups; information on the structure of these latter groups has been obtained by L.A. Kurdachenko and D.I. Zaicev [3].

Our main result on $\mathfrak{A}^\infty$- groups is the following.

Theorem. *Let G be an infinite non-abelian $\mathfrak{A}^\infty$-group, and let T be its largest periodic normal subgroup. Then G satisfies one of the following conditions*:

(a) *G is finitely generated, $G/Z(G)$ is a just-infinite group with no abelian subgroups of finite index and any two maximal abelian subgroups of $G/Z(G)$ have trivial intersection.*

(b) *G is soluble with derived length at most 3, T is finite abelian and one of the following statements holds*:
- (b1) *$G = \langle x\rangle \ltimes T$, where x is an element of infinite order such that $[T, x] \neq \{1\}$.*
- (b2) *$G = \langle x\rangle \ltimes (T \times \langle y\rangle \times \langle z\rangle)$, where x, y and z are elements of infinite order such that $[T, x] = \{1\}$, $[x, z] = 1$ and $[x, y] = tz^n$ for some $t \in T$ and some positive integer n.*
- (b3) *$G = (\langle x\rangle \ltimes (T \times \langle y\rangle)\langle g\rangle$, where x and y are elements of infinite order such that $[T, x] = [T, g] = \{1\}$, $g^3 \in T$, $1 \neq [x, y] \in T$, $x^g = y$ and $y^g = zy^{-1}x^{-1}$ for some $z \in T$.*
- (b4) *$G = (\langle x\rangle \ltimes (T \times \langle z\rangle \times \langle y\rangle)\langle g\rangle$, where $[T, x] = [T, g] = \{1\}$, $[z, x] = [z, g] = 1$, $[x, y] = tz^n$ for some element t of T and some positive integer n, $g^3 \in T\langle z\rangle$, $x^g = y$ and $y^g = uz^m y^{-1}x^{-1}$ for some element u of T and some non-negative integer m.*
- (b5) *G is isomorphic to a subgroup of a direct product $T \times K$, where T is a finite abelian group and $K = \langle g\rangle \ltimes A$ is a semidirect product of a torsion-free abelian normal subgroup A and a finite cyclic subgroup $\langle g\rangle$ whose non-trivial elements act rationally irreducibly on A.*
- (b6) *G is isomorphic to a subgroup of a direct product $T \times K$, where T is a finite abelian group and $K = \langle g\rangle \ltimes A$ is a semidirect product of a torsion-free abelian normal subgroup A and an infinite cyclic subgroup $\langle g\rangle$ such that $C_{\langle g\rangle}(A) = \langle g^m\rangle$ for some integer $m > 1$ and g^n acts rationally irreducibly on A for each proper divisor n of m.*

Most of our notation is standard and can be found in [4].

2. Preliminary results

The first lemma is an obvious property of groups containing a cyclic subgroup of finite index; as a consequence, cyclic-by-finite $\mathfrak{A}^\infty$-groups are easily described.

Lemma 2.1. *Let G be a cyclic-by-finite group. Then G contains a finite normal subgroup N such that the factor group G/N either is cyclic or infinite dihedral. In particular, if G is not finite-by-cyclic, then it is generated by its finite subgroups.*

Corollary 2.2. *Let G be a cyclic-by-finite infinite group and let T be its largest periodic normal subgroup. Then G belongs to the class $\mathfrak{A}^\infty$ if and only if either T is abelian and G/T is infinite cyclic or $T = Z(G)$ and G/T is infinite dihedral.*

In the following, we shall denote by $Z_\alpha(G)$ the α-th term of the upper central series of the group G, and by $\bar{Z}(G)$ its hypercentre. Our next two results describes the behaviour of the upper central series of $\mathfrak{A}^\infty$-groups.

Lemma 2.3. *Let G be an $\mathfrak{A}^\infty$-group and let N be a normal subgroup of G. Then either N is contained in $Z(G)$ or G/N is finite-by-cyclic.*

Proof. Suppose that N is not contained in $Z(G)$. Of course, it can be assumed that G/N is infinite, so that in particular N is abelian. Consider an element x of G such that $\langle x, N\rangle$ is not abelian; then the index $|G : \langle x, N\rangle|$ is finite and hence G/N is cyclic-by-finite. If G/N is generated by its finite subgroups, we have that G is generated by its abelian subgroups containing N, and so $N \leqslant Z(G)$, a contradiction. Thus G/N is not generated by its finite subgroups, and hence G/N is finite-by-cyclic by Lemma 2.1. □

Corollary 2.4. *Let G be an $\mathfrak{A}^\infty$-group. Then either $Z(G) = Z_2(G)$ or the factor group $G/\bar{Z}(G)$ is finite.*

Proof. Suppose that $Z(G)$ is properly contained in $\bar{Z}(G)$. Then $G/\bar{Z}(G)$ is finite-by-cyclic by Lemma 2.3, and so $G/\bar{Z}(G)$ is finite since it has trivial centre. □

It is now possible to describe torsion-free nilpotent $\mathfrak{A}^\infty$-groups. The proof of our next lemma makes use of the well-known result by A.I. Mal'cev stating that if G is a torsion-free nilpotent group, then the factor group $G/Z(G)$ is likewise torsion-free (see [4, Part 1, p. 53]).

Lemma 2.5. *Let G be a torsion-free nilpotent non-abelian $\mathfrak{A}^\infty$-group. Then*

$$G = \langle x\rangle \ltimes \big(\langle y\rangle \times \langle z\rangle\big),$$

where $[x, z] = 1$ and $[x, y] = z^n$ for some positive integer n.

Proof. Since G is not abelian, there exist elements x and y such that $xy \neq yx$ and $\langle x, y\rangle$ has class 2. Then $\langle x, y\rangle$ has finite index in G, so that G itself has class 2 and $[x, y]$ belongs to $Z(G)$. As $G/Z(G)$ is torsion-free, it follows that the index $|Z(G) : \langle [x, y]\rangle|$ is finite and hence $Z(G)$ is cyclic; let z be the generator of $Z(G)$ such that $[x, y] = z^n$ for some integer $n > 0$. Clearly, the factor group $G/Z(G)$ is free abelian of rank 2, and so the elements x and y can be chosen in such a way that $G = \langle x, y\rangle Z(G)$. Therefore G has the structure described in the statement. □

Recall that a group is said to be *minimax* if it has a series of finite length whose factors either satisfy the minimal or the maximal condition on subgroups. It was proved by P.H. Kropholler [2] that a finitely generated soluble group either is minimax or has a section which is isomorphic to the standard restricted wreath product of a group of prime order by an infinite cyclic group. The following lemma shows that Kropholler's theorem applies to soluble-by-finite $\mathfrak{A}^\infty$-groups.

Lemma 2.6. *Let G be an $\mathfrak{A}^\infty$-group. Then G has no sections isomorphic to the standard restricted wreath product of a group of prime order by an infinite cyclic group.*

Proof. Since all sections of G are likewise $\mathfrak{A}^\infty$-groups, it is enough to show that G cannot be isomorphic to such a wreath product. Assume for a contradiction that

$$G = \langle x\rangle \ltimes \big(\mathrm{Dr}_{n\in\mathbb{Z}}\langle a_n\rangle\big),$$

where each a_n has the same prime order p and $a_n^x = a_{n+1}$ for every integer n. Then the non-abelian subgroup

$$\langle a_0, x^2\rangle = \langle x^2\rangle \ltimes \big(\mathrm{Dr}_{n\in\mathbb{Z}}\langle a_{2n}\rangle\big)$$

has infinite index in G, and this contradiction proves the statement. □

Corollary 2.7. *Let G be a soluble-by-finite non-abelian $\mathfrak{A}^\infty$-group. Then G is minimax.*

Proof. Since G is finitely generated, the statement follows from Lemma 2.6 and the above quoted result of Kropholler. □

Next step is the description of $\mathfrak{A}^\infty$-groups which are finite over their hypercentre.

Lemma 2.8. *Let G be an infinite non-abelian $\mathfrak{A}^\infty$-group whose hypercentre has finite index. Then the set T of all elements of finite order of G is a finite abelian subgroup and one of the following conditions holds*:

(a) $G = \langle x\rangle \ltimes T$, *where* $[T, x] \neq \{1\}$;
(b) $G = \langle x\rangle \ltimes (T \times \langle y\rangle \times \langle z\rangle)$, *where* $[T, x] = \{1\}$, $[x, z] = 1$ *and* $[x, y] = tz^n$ *for some* $t \in T$ *and some positive integer* n.

Proof. As the group G is finitely generated, also its hypercentre is finitely generated. Thus G is nilpotent-by-finite and there exists a positive integer n such that $G/Z_n(G)$ is finite; it follows that the $(n+1)$-th term $\gamma_{n+1}(G)$ of the lower central series of G is finite (see [4, Part 1, p. 113]). Therefore T is a finite abelian subgroup of G and the factor group G/T is torsion-free and nilpotent. Moreover, G/T has class at most 2 by Lemma 2.5.

It is clear that condition (a) holds, provided that G/T is cyclic. Assume now that G/T is an abelian non-cyclic group. As G is not abelian, it contains a non-abelian 2-generator subgroup E and the index $|G : E|$ must be finite, so that G/T has rank 2. Thus there exist elements of infinite order x and y of G such that

$$G = \langle x\rangle \ltimes \big(\langle y\rangle \ltimes T\big).$$

The subgroup $\langle x, T\rangle$ and $\langle y, T\rangle$ have infinite index in G, so that they are abelian and hence $\langle y, T\rangle = \langle y\rangle \times T$ and $[T, x] = \{1\}$. It is also clear that in this case $[x, y]$ is a non-trivial element of T, and condition (b) is satisfied with $z = 1$ and $t \neq 1$. Suppose finally that G/T has class 2. Then by Lemma 2.5 the centre Z/T of G/T is infinite cyclic and G/Z is free abelian of rank 2, so that G has the structure described in (b) also in this case. □

The following easy result will be relevant in the study of abelian-by-finite $\mathfrak{A}^\infty$-groups.

Lemma 2.9. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group. Then either G is metacyclic-by-finite or in every subgroup of G the centre and the hypercentre coincide.*

Proof. Assume that G is infinite and contains a subgroup H whose centre $Z(H)$ is a proper subgroup of the hypercentre $\bar{Z}(H)$. Clearly, H is infinite, since it has finite index in G, and $H/\bar{Z}(H)$ is finite by Corollary 2.4. Therefore H satisfies one of the conditions (a), (b) of the statement of Lemma 2.8. On the other hand, groups satisfying condition (b) are not abelian-by-finite, and hence H must be finite-by-metacyclic. Therefore the group G is finite-by-metacyclic-by-finite and so also metacyclic-by-finite. □

Recall that a group G is said to be *just-infinite* if it is infinite but all its proper homomorphic images are finite; in particular, the infinite cyclic group is the unique abelian just-infinite group. The structure of just-infinite groups has been described by J.S. Wilson [9]. It turns out that just-infinite groups occur as sections of $\mathfrak{A}^\infty$-groups which are not soluble-by-finite.

Lemma 2.10. *Let G be an $\mathfrak{A}^\infty$-group which is not soluble-by-finite. Then the factor group $G/Z(G)$ is just-infinite and any two maximal abelian subgroups of $G/Z(G)$ have trivial intersection.*

Proof. Since G is finitely generated, it contains a normal subgroup N of infinite index such that all normal subgroups of G properly containing N have finite index. Thus N is abelian and the factor group G/N is just-infinite. As G is not soluble-by-finite, it follows from Lemma 2.3 that N lies in $Z(G)$, so that $N = Z(G)$ and $G/Z(G)$ is just-infinite. Assume that the group $\bar{G} = G/Z(G)$ has two different maximal abelian subgroups $\bar{A}$ and $\bar{B}$ such that $\bar{A} \cap \bar{B} \neq \{1\}$. The centralizer $C_{\bar{G}}(\bar{A} \cap \bar{B})$ contains $\langle \bar{A}, \bar{B} \rangle$, so that $C_{\bar{G}}(\bar{A} \cap \bar{B})$ is not abelian and hence it has finite index in $\bar{G}$. It follows that the normal closure $(\bar{A} \cap \bar{B})^{\bar{G}}$ is central-by-finite. On the other hand, $(\bar{A} \cap \bar{B})^{\bar{G}}$ has finite index in $\bar{G}$ and so $\bar{G}$ is soluble-by-finite. This contradiction shows that any two maximal abelian subgroups of $G/Z(G)$ have trivial intersection. □

The latter lemma of this section yields that if G is any $\mathfrak{A}^{\infty}$-group which is not soluble-by-finite, then $G/Z(G)$ is a just-infinite $\mathfrak{A}^{\infty}$-group, and the last result of this section deals with the structure of just-infinite $\mathfrak{A}^{\infty}$-groups.

Lemma 2.11. *Let G be a just-infinite $\mathfrak{A}^{\infty}$-group which is not residually finite. Then G contains a simple minimal non-abelian normal subgroup of finite index.*

Proof. As the group G is just-infinite, but not residually finite, it is non-abelian and contains a unique minimal normal subgroup N; clearly, G/N is finite and hence N is finitely generated. Moreover, N has no proper subgroup of finite index, and so it follows from the property $\mathfrak{A}^{\infty}$ that N is a minimal non-abelian group. Assume for a contradiction that N is not simple, so that $N = M^G$, where M is a proper non-trivial normal subgroup of N. It follows that N is locally nilpotent and so even nilpotent, contradicting the assumption that G is not residually finite. Therefore N is simple. □

3. Soluble-by-finite groups

The description of soluble-by-finite $\mathfrak{A}^{\infty}$-groups will be accomplished through a series of lemmas. The first of these essentially allows to reduce to the case of nilpotent-by-finite groups in the class $\mathfrak{A}^{\infty}$.

Lemma 3.1. *Let G be an infinite soluble-by-finite $\mathfrak{A}^{\infty}$-group which is not abelian, and let T be the largest periodic normal subgroup of G. Then T is finite abelian and either G is nilpotent-by-finite or $G = \langle x \rangle \ltimes (A \times T)$, where A is a torsion-free abelian subgroup and x is an element of infinite order such that $[T, x] = \{1\}$, $C_{\langle x \rangle}(A) = \{1\}$ and each non-trivial subgroup of $\langle x \rangle$ acts rationally irreducibly on AT/T.*

Proof. Let R be the Hirsch–Plotkin radical (i.e. the largest locally nilpotent normal subgroup) of G. Suppose first that G/R is finite, so that R is finitely generated and hence it is nilpotent. Thus G is a (finitely generated) nilpotent-by-finite group, so that in this case T is finite and hence also abelian.

Assume now that G/R is infinite, so that in particular R is abelian. Since G/R contains abelian non-trivial normal subgroups, $Z(G)$ is properly contained in R and so by Lemma 2.3 we have that G/R is finite-by-cyclic. Moreover, as each normal subgroup of G properly containing R is not abelian, it follows that all non-trivial normal subgroups of G/R have finite index and hence G/R is infinite cyclic and G is soluble. Thus T lies in R and so it is abelian. Let x be an element of infinite order of G such that $G = \langle x \rangle \ltimes R$. Clearly, $C_G(R) = R$ and so $C_{\langle x \rangle}(R) = \{1\}$. Assume for a contradiction that

$$C_R(x) \cap [R, x] \neq \{1\},$$

so that there exists an element a of R such that $[a, x] \neq 1$ and $[a, x, x] = 1$. Then $\langle a, x \rangle$ is a nilpotent non-abelian subgroup and hence the index $|G : \langle a, x \rangle|$ is finite, a contradiction because G is not nilpotent-by-finite. Therefore

$$C_R(x) \cap [R, x] = \{1\},$$

and since $R \cap \langle x \rangle = \{1\}$, it follows also that

$$C_R(x) \cap \big([R, x]\langle x \rangle\big) = \{1\}.$$

On the other hand, the subgroup $[R, x]\langle x \rangle$ is not abelian, so that it has finite index in G and hence $Z(G) = C_R(x)$ is finite. Assume that $C_R(x)$ is properly contained in T, and let y be an element of $T \setminus C_R(x)$; then the subgroup $\langle x, y \rangle = \langle x \rangle \langle y \rangle^{\langle x \rangle}$ has finite index in G and hence the index $|R : \langle y \rangle^{\langle x \rangle}|$ is also finite. On the other hand, $\langle y \rangle^{\langle x \rangle}$ has obviously finite exponent and so R itself has finite exponent, a contradiction by Corollary 2.7. Therefore $T = C_R(x)$ is finite. In particular, $R = A \times T$ for a suitable torsion-free subgroup A.

Assume now that $C_R(x) \neq C_R(x^n)$ for some positive integer n. Clearly, the subgroup $C_R(x^n)$ is normal in G and the factor group $G/C_R(x^n)$ is finite-by-cyclic by Lemma 2.3; then $R/C_R(x^n)$ is finite and so the abelian subgroup $\langle x^n, C_R(x^n) \rangle$ has finite index in G, contradicting again the fact that G is not nilpotent-by-finite. Therefore $C_R(x^n) = C_R(x)$ for each positive integer n. Let n be a positive integer, and let B be any non-trivial subgroup of A such that BT is $\langle x^n \rangle$-invariant. If b is any non-trivial element of B, the subgroup $\langle b, x^n \rangle$ is not abelian and so has finite index in G and hence

$$\langle b \rangle^{\langle x^n \rangle} = R \cap \big\langle b, x^n \big\rangle$$

has finite index in R; thus also the index $|R : BT|$ is finite and $\langle x^n \rangle$ acts rationally irreducibly on R/T. □

Observe that the torsion-free abelian subgroup A in the statement of Lemma 3.1 in general cannot be chosen to be normal in G. To see this, consider the group

$$G = \langle x \rangle \ltimes \big(\langle t \rangle \times \langle a_1 \rangle \times \langle a_2 \rangle\big),$$

where t is an element of order 2 and a_1, a_2, x are elements of infinite order such that $t^x = t$, $a_1^x = a_1 a_2 t$ and $a_2^x = a_1^2 a_2$. Put $U = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle t \rangle$. Then every non-trivial power of x acts rationally irreducibly on the free abelian group $U/\langle t \rangle$, and hence G satisfies the conclusion of the statement of Lemma 3.1. Suppose now for a contradiction that G contains a normal subgroup A such that $U = A \times \langle t \rangle$ and $A^x = A$. Then

$$\big\langle a_1^2 \big\rangle \times \big\langle a_2^2 \big\rangle = U^2 = A^2$$

is likewise a normal subgroup of G and

$$\bar{U} = U/U^2 = \langle \bar{t} \rangle \times \langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle$$

is an elementary abelian group of order 8; the action of x on $\bar{U}$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

over the field $\mathbb{F}_2$ with two elements. On the other hand, $\bar{U} = \langle \bar{t} \rangle \times \bar{A}$, with $\bar{A}^x = \bar{A}$, and hence the action of x on $\bar{U}$ is also given by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & \alpha & 1 \end{pmatrix},$$

where $\alpha\beta = 0$ in $\mathbb{F}_2$. This is a contradiction as the two matrices considered above are not conjugate in the general linear group $GL(3, \mathbb{F}_2)$.

Lemma 3.2. *Let G be a non-abelian $\mathfrak{A}^{\infty}$-group such that $Z(G) = Z_2(G)$. If $G = \langle x, A\rangle$, where A is a torsion-free abelian normal subgroup, then the centralizer $C_A(x)$ is cyclic. Moreover, if $A \cap \langle x\rangle = \{1\}$, then $Z(G) = C_{\langle x\rangle}(A)$ and $\langle x^n\rangle$ acts rationally irreducibly on A for each positive integer n such that $x^n \notin Z(G)$.*

Proof. Let a be any element of A such that $[a, x, x] = 1$. Then $[a, x]$ belongs to $Z(G)$, so that $a \in Z_2(G) = Z(G)$ and $[a, x] = 1$. As

$$[A, x] = \big\{[a, x] \bigm| a \in A\big\},$$

it follows that

$$[A, x] \cap C_A(x) = \{1\}$$

and hence

$$[A, x]\langle x\rangle \cap C_A(x) = A \cap \langle x\rangle.$$

On the other hand, the subgroup $[A, x]\langle x\rangle$ is not abelian, and so it has finite index in G. Therefore also the index $|C_A(x) : A \cap \langle x\rangle|$ is finite and hence the centralizer $C_A(x)$ is cyclic.

Suppose now that $A \cap \langle x\rangle = \{1\}$, so that $C_A(x) = \{1\}$ and $Z(G) = C_{\langle x\rangle}(A)$. Assume that A contains a non-trivial subgroup B such that $B^{x^n} = B$ for some positive integer n and the index $|A : B|$ is infinite. Then the subgroup $\langle B, x^n\rangle$ has infinite index in G and hence it is abelian. Therefore $B \leqslant C_G(x^n)$ and so $n > 1$; in particular, x acts rationally irreducibly on A. Moreover, the centralizer $C_A(x^n)$ is a non-trivial G-invariant subgroup of A, and hence it has finite index in A. On the other hand, $A/C_A(x^n)$ is isomorphic to $[A, x^n]$, so that it is torsion-free. Thus $A = C_A(x^n)$ and x^n belongs to $Z(G)$. The lemma is proved. □

We can now prove that all infinite abelian-by-finite $\mathfrak{A}^{\infty}$-groups are soluble.

Lemma 3.3. *Let G be an infinite abelian-by-finite $\mathfrak{A}^{\infty}$-group. Then G is soluble.*

Proof. Assume for a contradiction that G is not soluble, and let T be the largest periodic normal subgroup of G. Then T is abelian and finite by Lemma 3.1, so that the factor group G/T is also a counterexample and it can be assumed without loss of generality that G has no periodic non-trivial normal subgroups. Let A be any maximal abelian normal subgroup of finite index of G. Then A is torsion-free and $C_G(A)/A$ is finite, so that it follows from Schur's theorem that $C_G(A)$ is abelian and hence $C_G(A) = A$ (see [4, Part 1, Theorem 4.12]). If G contains a metacyclic subgroup of finite index, A must be free abelian of rank 2; in this case, G/A is isomorphic to a finite subgroup of $GL(2, \mathbb{Z})$ and hence it is soluble. This contradiction shows that G is not metacyclic-by-finite, and so Lemma 2.9 yields that in any subgroup of G the centre and the hypercentre coincide.

As G is not soluble, the order of the finite group G/A is even, and so there exists an element $x \in G \setminus A$ such that $x^2 \in A$. Put $H = \langle x, A\rangle$. Then the centralizer $C_A(x)$ is cyclic by Lemma 3.2; moreover, $A/C_A(x)$ is torsion-free and

$$A \cap \langle x\rangle \leqslant C_A(x) = Z(H) = Z_2(H).$$

Therefore Lemma 3.2 can also be applied to the group $H/C_A(x)$, obtaining that x induces on $A/C_A(x)$ a fixed-point-free rationally irreducible automorphism of order 2. It follows that x inverts all elements of $A/C_A(x)$, so that $A/C_A(x)$ is infinite cyclic and A is metacyclic. This last contradiction completes the proof. □

Lemma 3.4. *Let G be a torsion-free abelian-by-finite $\mathfrak{A}^{\infty}$-group. Then the centre of G is not trivial.*

Proof. Assume that the statement is false, and among all counterexamples choose a group G containing a maximal abelian normal subgroup A such that G/A is finite of minimal size. As G is torsion-free, it follows from Schur's theorem that $C_G(A) = A$. If G/A is cyclic, there exists an element x such that $G = \langle A, x\rangle$ and $x^n \in A$ for some positive integer n; then x^n belongs to $Z(G)$, a contradiction. Thus G/A is not cyclic. Since G is soluble by Lemma 3.3, A is contained in a normal subgroup M of G such that the index $|G : M|$ is a prime number. Clearly, $A < M$ and so $Z(M) \neq \{1\}$ by the minimal choice of G. Let y be an element of the set $M \setminus A$. If $Z(\langle A, y\rangle) \neq Z_2(\langle A, y\rangle)$, the group G is metacyclic-by-finite by Lemma 2.9, so that the index

$$\bigl|\langle A, y\rangle : Z_2\bigl(\langle A, y\rangle\bigr)\bigr|$$

is finite and hence $\langle A, y\rangle$ must be abelian; this contradiction shows that $Z(\langle A, y\rangle) = Z_2(\langle A, y\rangle)$, so that $C_A(y)$ is cyclic by Lemma 3.2. It follows that $Z(M) = \langle a\rangle$ is an infinite cyclic normal subgroup of G; as $Z(G) = \{1\}$, the factor group $G/\langle a\rangle$ is finite-by-cyclic by Lemma 2.3. Thus $\langle a\rangle$ is contained in a cyclic normal subgroup $\langle b\rangle$ of G such that $G/\langle b\rangle$ is infinite cyclic, and hence $G = \langle b, g\rangle$, where $b^g = b^{-1}$. Therefore $g^2 \in Z(G)$, and this last contradiction completes the proof of the lemma. □

Lemma 3.5. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group. If G is torsion-free, then either G is nilpotent or $G = \langle x\rangle \ltimes A$, where A is an abelian normal subgroup and x is an element such that $Z(G) = \langle x^m\rangle$ for some positive integer m and x^n acts rationally irreducibly on A for each proper divisor n of m.*

Proof. Suppose that the group G is not nilpotent, and let $\bar{Z}(G)$ be its hypercentre. Of course, $\bar{Z}(G) = Z_c(G)$ for some positive integer c, so that the index $|G : \bar{Z}(G)|$ must be infinite (see [4, Part 1, Theorem 4.25]) and hence $Z(G) = \bar{Z}(G)$ by Corollary 2.4. Let H be an abelian normal subgroup of finite index of G, and consider the transfer homomorphism τ of G into H. By Lemma 3.4 there exists in G a central element $z \neq 1$, and we have

$$z^\tau = z^k \neq 1,$$

where $|G : H| = k$. In particular, $G^\tau \neq \{1\}$ and so G has an infinite cyclic factor group G/A. Clearly, A is abelian and $G = \langle x\rangle \ltimes A$ for some element x. Application of Lemma 3.2 yields that

$$Z(G) = C_{\langle x\rangle}(A) = \bigl\langle x^m\bigr\rangle$$

for some positive integer m, and x^n acts rationally irreducibly on A for each proper divisor n of m. □

Corollary 3.6. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group. If G is torsion-free and non-trivial, then it has an infinite cyclic homomorphic image.*

Lemma 3.7. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group with no periodic non-trivial normal subgroups. If all torsion-free subgroups of G are abelian, then G has a unique maximal torsion-free subgroup.*

Proof. Since G does not contain periodic non-trivial normal subgroups, it is clear that G has a unique maximal torsion-free normal subgroup A and the index $|G : A|$ is finite. Moreover, A is abelian and $C_G(A) = A$ by Schur's theorem. Assume for a contradiction that there exists an element $g \in G \setminus A$ of infinite order. The subgroup $\langle g, A\rangle$ is not abelian, and so it contains an element $x \neq 1$ of finite order. As the index $|G : \langle x, A\rangle|$ is finite, the set of all elements of finite order of $\langle x, A\rangle$ cannot be a subgroup; thus the hypercentre $\bar{Z}(\langle x, A\rangle)$ must have infinite index in $\langle x, A\rangle$, and hence $Z(\langle x, A\rangle = \bar{Z}(\langle x, A\rangle)$ by Corollary 2.4. Application of Lemma 3.2 yields that $Z(\langle x, A\rangle) = C_{\langle x\rangle}(A)$ is finite. On the other hand, g^n belongs to A for some positive integer n, and hence $g^n \in Z(\langle x, A\rangle)$. This contradiction proves that all elements of infinite order of G belong to A. □

Lemma 3.8. *Let G be a group and let A be an abelian normal subgroup of G. If x is an element of G such that $(xa)^m = 1$ for some positive integer m and for each element a of A, then the group $A/[A,x]$ has finite exponent dividing m. Moreover, $[A,x^k] = [A,x]$ for every positive integer k coprime to m.*

Proof. Let a be any element of A. Since A is an abelian normal subgroup of G, for each positive integer n, we have

$$\left[a,x^n\right] = \left[a,x^{n-1}\right]\left[a^{x^{n-1}},x\right] = \cdots = \left[aa^x\ldots a^{x^{n-1}},x\right],$$

and so $[a,x^n]$ belongs to $[A,x]$. Moreover,

$$1 = (xa)^m = x^m a^{x^{m-1}}\ldots a^x a = a^{x^{m-1}}\ldots a^x a = \left[a,x^{m-1}\right]\left[a,x^{m-2}\right]\ldots[a,x]a^m$$

and hence

$$a^{-m} = \left[a,x^{m-1}\right]\left[a,x^{m-2}\right]\ldots[a,x]$$

lies in $[A,x]$. Therefore A^m is contained in $[A,x]$ and $A/[A,x]$ has finite exponent dividing m.

Let k be any positive integer coprime to m. Then $x = x^{kh}$ for some positive integer h. Since $[A,x]$ is $\langle x\rangle$-invariant, the subgroup $[A,x^k]$ is contained in $[A,x]$, and so

$$[A,x] = \left[A,\left(x^k\right)^h\right] \leqslant \left[A,x^k\right] \leqslant [A,x].$$

Therefore $[A,x^k] = [A,x]$. □

Lemma 3.9. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group with no periodic non-trivial normal subgroups. If all torsion-free subgroups of G are abelian, then either G is abelian or $Z(G) = \{1\}$ and $G = H \ltimes A$, where A is a torsion-free abelian normal subgroup and H is a finite cyclic subgroup whose non-trivial elements act rationally irreducibly on A.*

Proof. By Lemma 3.7 the group G has a unique maximal torsion-free subgroup A, and of course it can be assumed that A is properly contained in G. Let g be any element of $G \setminus A$. Then g has finite order and $ag \neq ga$ for each $a \in A$; in particular, $C_G(A) = A$, the centre of G is trivial and g acts fixed-point-freely on A. Therefore a classical result of Zassenhaus [10] yields that all abelian subgroups of G/A are cyclic. Moreover, the elements of finite order of $\langle g,A\rangle$ cannot form a subgroup, so that the hypercentre of $\langle g,A\rangle$ has infinite index in $\langle g,A\rangle$, and hence it follows from Corollary 2.4 that

$$Z\left(\langle g,A\rangle\right) = Z_2\left(\langle g,A\rangle\right).$$

Thus g acts rationally irreducibly on A by Lemma 3.2.

Assume first that the finite group G/A has even order. Then there exists an element x of $G \setminus A$ of order 2, so that $a^x = a^{-1}$ for each element a of A. Thus A is infinite cyclic and G is infinite dihedral.

Suppose now that G/A has odd order, so that all its Sylow subgroups are cyclic and hence G/A is the semidirect product of two cyclic subgroups with coprime orders (see for instance [5, 5.3.6 and 10.1.10]). Then

$$G = \langle y\rangle \ltimes \left(\langle x\rangle \ltimes A\right),$$

where x and y have coprime orders m and n, respectively. Clearly, the commutator subgroup $[A,x] = \langle x,A\rangle'$ is normal in G, and so we can consider the factor group

$$\bar{G} = G/[A,x] = \langle\bar{y}\rangle \ltimes \left(\langle\bar{x}\rangle \times \bar{A}\right).$$

Since $A/[A,x]$ has finite exponent dividing m by Lemma 3.8, it is finite and it follows from Maschke's theorem that $\bar{A}$ has a $\bar{G}$-invariant complement in $\langle\bar{x}\rangle \times \bar{A}$. Therefore without loss of generality it can be assumed that $\langle\bar{x}\rangle$ is a normal subgroup of $\bar{G}$, so that $x^y = x^k u$, where u is an element of $[A,x]$ and k is a positive integer prime to m. On the other hand, $[A,x^k] = [A,x]$ by Lemma 3.8 and hence $u = [x^k, a]$ for some $a \in A$. Thus

$$x^y = x^k\big[x^k, a\big] = a^{-1}x^k a,$$

so that $x^{ya^{-1}} = x^k$ and $(ya^{-1})^n$ is an element of A normalizing $\langle x\rangle$. Thus $(ya^{-1})^n = 1$, and replacing y by ya^{-1} we may suppose that $x^y = x^k$. It follows that $\langle x, y\rangle$ is a finite subgroup, so that it is abelian and hence even cyclic. Moreover, $G = \langle x, y\rangle \ltimes A$, and the lemma is proved. □

Lemma 3.10. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group, and let T be the largest periodic normal subgroup of G. If all torsion-free subgroups of G are abelian, then also the torsion-free subgroups of G/T are abelian.*

Proof. Let H/T be any torsion-free (non-trivial) subgroup of G/T. Then the subgroup H has an infinite cyclic homomorphic image by Corollary 3.6, and so $H = \langle h\rangle \ltimes A$, where A is an abelian normal subgroup containing T and h is an element of infinite order. As A is finitely generated, there exists a positive integer n such that A^n is torsion-free, so that $\langle h, A^n\rangle$ is likewise torsion-free and hence it is abelian. Thus A^n is contained in $Z(H)$ and so $H/Z(H)$ is finite-by-cyclic. It follows that H is finite-by-nilpotent, so that the torsion-free group H/T is nilpotent and hence even abelian, as it is abelian-by-finite. The lemma is proved. □

Lemma 3.11. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group with no periodic non-trivial normal subgroups. If G contains torsion-free non-abelian subgroups and $Z(G) = Z_2(G)$, then G is torsion-free.*

Proof. Assume for a contradiction that the statement is false, and choose a counterexample G containing a maximal abelian normal subgroup A such that the factor group G/A is finite with smallest possible order. Suppose that G contains a non-abelian subgroup H whose hypercentre has finite index, so that the elements of finite order of H form a finite subgroup K (see [4, Part 1, p. 113]). Since H has finite index in G, it follows from Dietzmann's Lemma that K^G is finite, so that $K = \{1\}$ and H is a torsion-free nilpotent group. But G is abelian-by-finite and so H must be abelian. Therefore every non-abelian subgroup of G is infinite over its hypercentre, and hence Corollary 2.4 shows that $Z(H) = Z_2(H)$ for each subgroup H of G.

The centralizer $C_G(A)$ has finite commutator subgroup by Schur's theorem, so that it is abelian, and hence $C_G(A) = A$. Moreover, G is soluble by Lemma 3.3, and hence it contains a normal subgroup M such that $A \leqslant M$ and G/M has prime order p. Suppose first that $M = A$. Then $G = \langle z\rangle \ltimes A$, where z is an element of order p, and Lemma 3.2 yields that $Z(G)$ is trivial; on the other hand, G contains a torsion-free non-abelian subgroup U, and if u is an element of $U \setminus A$, the power u^p obviously belongs to the centre of G, a contradiction. Therefore A is properly contained in M.

Assume now that all torsion-free subgroups of M are abelian. Then by Lemma 3.7 the elements of infinite order of M (together with the identity) form a characteristic abelian subgroup, which of course must coincide with A; it follows from Lemma 3.9 that $M = \langle g\rangle \ltimes A$, where g is an element of finite order m and each non-trivial power of g acts rationally irreducibly on A. Since G contains torsion-free non-abelian subgroups, there exists an element x of infinite order such that $G = \langle M, x\rangle$; clearly, $x^p \in M$ and so x^p even belongs to A. The centralizer $C_A(x)$ is cyclic by Lemma 3.2. As all elements of infinite order of M lie in A, we have $(ga)^m = 1$ for each element a of A, and so it follows from Lemma 3.8 that the factor group $A/[A,g]$ has finite exponent dividing m. Note also that $[A,g] = M'$ is a normal subgroup of G.

Suppose that p divides m, so that $\langle g\rangle$ contains an element h of order p. Then both cosets xA and hA have order p, and $\langle hA\rangle$ is a normal subgroup of G/A, so that $[x,h]$ belongs to A. Thus

$$C_A(x)^h = C_A\big(x[x,h]\big) = C_A(x),$$

and so A is cyclic since $C_A(x)$ is cyclic and h acts rationally irreducibly on A. It follows that G is infinite dihedral, which is impossible as G contains torsion-free non-abelian subgroups. Therefore p does not divide m and x induces on $M/[A,g]$ an automorphism whose order is coprime with the exponent of the group. As in the last part of the proof of Lemma 3.9, it can be assumed that $g^x = g^k$ for some positive integer k prime to m and hence there exists a positive integer s such that

$$x^{-s} g x^s = g^{ks} = g.$$

Thus x^{sp} belongs to $C_A(g)$ and $\langle x^{sp}\rangle$ is a normal subgroup of M. On the other hand, g acts rationally irreducibly on A, so that A is cyclic and G is dihedral. This contradiction proves that M contains torsion-free non-abelian subgroups, and hence it is torsion-free by the minimal choice of the order of G/A.

Application of Lemma 3.5 yields that $M = \langle y\rangle \ltimes B$, where B is an abelian normal subgroup and $Z(M) = \langle y^r\rangle$ for some positive integer r. Clearly, y^r belongs to A since $C_G(A) = A$. Let w be any non-trivial element of finite order of G. Then $C_A(w) = \{1\}$ by Lemma 3.2 and so $wy^r \neq y^r w$. It follows that $C_G(y^r) = M$, so that $|G:M| = 2$ and hence $w^2 = 1$. Therefore $\langle y^r, w\rangle$ is an infinite dihedral group and the index $|G:\langle y^r, w\rangle|$ is finite, which is impossible since M is torsion-free and non-abelian. This last contradiction completes the proof of the lemma. □

Lemma 3.12. *Let G be an abelian-by-finite $\mathfrak{A}^\infty$-group such that $Z(G) = Z_2(G)$. Then either G is abelian or cyclic-by-finite or it satisfies one of the following conditions*:

(a) *$G = (\langle x\rangle \ltimes (T \times \langle y\rangle))\langle g\rangle$, where T is a finite abelian subgroup and x and y are elements of infinite order such that $[T,x] = [T,g] = \{1\}$, $g^3 \in T$, $1 \neq [x,y] \in T$, $x^g = y$ and $y^g = zy^{-1}x^{-1}$ for some $z \in T$*;

(b) *G is isomorphic to a subgroup of a direct product $T \times K$, where T is a finite abelian group and $K = \langle g\rangle \ltimes A$ is a semidirect product of a torsion-free abelian normal subgroup A and a finite cyclic subgroup $\langle g\rangle$ whose non-trivial elements act rationally irreducibly on A*;

(c) *G is isomorphic to a subgroup of a direct product $T \times K$, where T is a finite abelian group and $K = \langle g\rangle \ltimes A$ is a semidirect product of a torsion-free abelian normal subgroup A and an infinite cyclic subgroup $\langle g\rangle$ such that $C_{\langle g\rangle}(A) = \langle g^m\rangle$ for some integer $m > 1$ and g^n acts rationally irreducibly on A for each proper divisor n of m.*

Proof. Suppose that neither G is abelian nor cyclic-by-finite; in particular, G is not nilpotent since $Z(G) = Z_2(G)$. Assume first that G has no periodic non-trivial normal subgroups. If all torsion-free subgroups of G are abelian, application of Lemma 3.9 yields that G satisfies condition (b) with $T = \{1\}$ and $K = G$. On the other hand, if G contains a torsion-free non-abelian subgroup, then G is torsion-free by Lemma 3.11 and in this case it follows from Lemma 3.5 that G satisfies condition (c) with $T = \{1\}$ and $K = G$.

Suppose now that the largest periodic normal subgroup T of G is not trivial. The subgroup T is finite abelian by Lemma 3.1 and for each element a of G the subgroup $\langle a, T\rangle$ has infinite index in G, so that T is contained in the centre of G and G/T cannot be nilpotent.

Assume that all torsion-free subgroups of G are abelian, so that the same property holds for G/T by Lemma 3.10. Application of Lemma 3.9 yields that $Z(G/T) = \{1\}$ and

$$G/T = H/T \ltimes A/T,$$

where A/T is a torsion-free abelian normal subgroup and H/T is a finite cyclic subgroup whose non-trivial elements act rationally irreducibly on A/T. In particular, A is nilpotent with class at most 2, $Z(G) = T$ and $G = \langle g, A\rangle$ for some element g of H. If A is not abelian, the group G is metacyclic-by-finite by Lemma 2.9 and so A/T is a free abelian group of rank 2, and hence

$$A = \langle x\rangle \ltimes \big(\langle y\rangle \times T\big),$$

where $[x, y]$ is a non-trivial element of T. Moreover, g acts on A/T as an irreducible square matrix of order 2, so that g^3 belongs to T and the elements x and y can be chosen in such a way that $x^g = y$ and $y^g = zy^{-1}x^{-1}$ for some $z \in T$. Therefore in this case G satisfies condition (a) of the statement. On the other hand, if A is abelian, the factor group $A/C_A(g)$ is isomorphic to $[A, g]$ and $C_A(g) = Z(G) = T$, so that $[A, g]$ is a torsion-free normal subgroup of G and $G/[A, g]$ is a finite abelian group. Since G can be embedded in the direct product $G/[A, g] \times G/T$, it follows that condition (b) is satisfied.

Assume finally that G contains torsion-free non-abelian subgroups. As $Z(G) = Z_2(G)$ and $T \leqslant Z(G)$, we have also that $Z(G/T) = Z(G)/T$, and so $Z(G/T) = Z_2(G/T)$. Moreover, G/T obviously contains torsion-free non-abelian subgroups and hence it is torsion-free by Lemma 3.11. Thus T is the set of all elements of finite order of G. It follows that G/T has an infinite cyclic homomorphic image by Corollary 3.6. Therefore $G = \langle g\rangle \ltimes A$, where g is an element of infinite order and A is an abelian normal subgroup of G. Since G is abelian-by-finite, there exists an integer $m > 1$ such that $C_{\langle g\rangle}(A) = \langle g^m\rangle$. Applications of Lemma 3.5 yields that $Z(G) = T \times \langle g^m\rangle$, and so $C_A(g) = T$. Then the subgroup $[A, g]$ is isomorphic to A/T, and hence it is torsion-free abelian. It follows that $[A, g]\langle g\rangle$ is a torsion-free normal subgroup of G and $G/[A, g]\langle g\rangle$ is a finite abelian group. Clearly, G can be embedded in the direct product $G/[A, g]\langle g\rangle \times G/T$, and in this case the group G satisfies condition (c) of the statement. The lemma is proved. □

The above lemma completes the description of abelian-by-finite $\mathfrak{A}^\infty$-groups, and our final result deals with the case of nilpotent-by-finite $\mathfrak{A}^\infty$-groups which are not finite extensions of abelian groups.

Lemma 3.13. *Let G be a nilpotent-by-finite $\mathfrak{A}^\infty$-group which is not abelian-by-finite, and let T be the largest periodic normal subgroup of G. Then either G is nilpotent or*

$$G = \big(\langle x\rangle \ltimes \big(T \times \langle z\rangle \times \langle y\rangle\big)\big)\langle g\rangle,$$

where $[T, x] = [T, g] = \{1\}$, $[z, x] = [z, g] = 1$, $[x, y] = tz^n$ for some element t of T and some positive integer n, $g^3 \in T\langle z\rangle$, $x^g = y$ and $y^g = uz^m y^{-1}x^{-1}$ for some element u of T and some non-negative integer m.

Proof. Assume that G is not nilpotent. The subgroup T is finite abelian by Lemma 3.1, and hence it is contained in the largest nilpotent normal subgroup F of G. Clearly, F has no abelian subgroups of finite index and T is the set of all elements of finite order of F. Application of Lemma 2.8 yields that

$$F = \langle x\rangle \ltimes \big(T \times \langle y\rangle \times \langle z\rangle\big),$$

where x, y, z are elements of infinite order of G such that $[T, x] = \{1\}$, $[x, z] = 1$ and $[x, y] = tz^n$ for some $t \in T$ and some positive integer n. As

$$Z(F) = T \times \langle z\rangle,$$

for each element g of G the subgroup $\langle Z(F), g\rangle$ has infinite index in G, and hence it is abelian. It follows that $Z(F) = Z(G)$. The factor group $\bar{G} = G/Z(G)$ is abelian-by-finite, so that it is soluble by Lemma 3.3 and hence the Fitting subgroup $\bar{F} = \langle\bar{x}\rangle \times \langle\bar{y}\rangle$ of $\bar{G}$ coincides with its centralizer. If $Z(\bar{G}) \neq \{1\}$, the centre $Z(G)$ is properly contained in $Z_2(G)$ and so the hypercentre of G has finite index by Lemma 2.4; then G is finite-by-nilpotent, and so even nilpotent because T is contained in $Z(G)$. This contradiction proves that $Z(\bar{G}) = \{1\}$, and hence it follows from Lemma 3.4 that $\bar{G}$ is not torsion-free. Moreover, $\bar{G}$ has no periodic non-trivial normal subgroups since $\bar{F}$ is torsion-free and $C_{\bar{G}}(\bar{F}) = \bar{F}$, and so all torsion-free subgroups of $\bar{G}$ are abelian by Lemma 3.11. Thus Lemma 3.9 yields that $\bar{G} = \langle\bar{g}\rangle \ltimes \bar{F}$, where $\bar{g}$ is an element of finite order whose non-trivial powers act rationally irreducibly on $\bar{F}$. As $\bar{F}$ is free abelian of rank 2, we have $\bar{g}^3 = 1$ (so that g^3 belongs to $T\langle z\rangle$) and the elements x and y can be chosen in such a way that $x^g = y$ and $y^g = uz^m y^{-1}x^{-1}$ for some element u of T and some non-negative integer m. The statement is proved. □

We are now in a position to prove the main result of the paper.

Proof of the theorem. The group G is obviously finitely generated. If G is not soluble-by-finite, then it satisfies condition (a) of the statement by Lemma 2.10. Suppose that G is soluble-by-finite, and assume first that it is not nilpotent-by-finite. It follows from Lemma 3.1 that

$$G = \langle x \rangle \ltimes (A \times T),$$

where A is torsion-free abelian and x is an element of infinite order such that $[T, x] = \{1\}$, $C_{\langle x \rangle}(A) = \{1\}$ and each non-trivial subgroup of $\langle x \rangle$ acts rationally irreducibly on AT/T. Then the torsion-free group G/T has the same properties as the group K in statement (b5). Moreover, $C_{AT}(x) = T$ so that the subgroup $[A, x] = [AT, x]$ is isomorphic to AT/T and hence it is torsion-free. It follows that $[A, x]\langle x \rangle$ is a torsion-free normal subgroup of G, and $G/[A, x]\langle x \rangle$ is a finite abelian group. As

$$[A, x]\langle x \rangle \cap T = \{1\},$$

the group G embeds into the direct product of $G/[A, x]\langle x \rangle$ and G/T and so it satisfies condition (b5).

Assume now that the group G is nilpotent-by-finite. If the hypercentre of G has finite index, it follows from Lemma 2.8 that G satisfies one of the conditions (b1) and (b2) of the statement. On the other hand, if the index $|G : \bar{Z}(G)|$ is infinite, we have $Z(G) = Z_2(G)$ by Corollary 2.4 and Lemma 3.13 yields that either G satisfies condition (b4) or it contains an abelian subgroup of finite index. Therefore we may finally suppose that G is abelian-by-finite. In this latter case it follows from Lemma 3.12 that G satisfies one of the conditions (b3), (b5), (b6). The proof of the theorem is complete. □

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