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# On Commutative Nonassociative Algebras

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

## INTRODUCTION

Let $A$ be a commutative but nonassociative algebra. Consider the subset $M(A)$ of $A$ consisting of all $a \in A$ satisfying the "weakened Jordan identity"

$$\begin{aligned} xy \cdot a^2 + 2xa \cdot ya &= x(y \cdot a^2) + 2(y \cdot xa)a \\ &= x(ya \cdot a) + y(xa \cdot a) + (xy \cdot a)a \end{aligned}$$

for $x, y \in A$. The set $M(A)$ is not artificially constructed: it plays for a finite-dimensional real algebra $A$ a certain role in the analytic theory of homogeneous convex cones.

In the following $M(A)$ is investigated for an arbitrary algebra without 2- and 3-torsion. The proofs are based on a new identity of degree 5 that holds for arbitrary commutative algebras. Among other results about the structure of $M(A)$ it is shown that $M(A)$ is closed under squaring.

As an application a class of power-associative algebras is considered and the connection with convex cones is indicated.

## 1. TWO IDENTITIES FOR COMMUTATIVE ALGEBRAS

1.1. Let $k$ be a unital commutative and associative ring and let $A$ be a commutative (but nonassociative) algebra over $k$ with respect to the product $(u, v) \mapsto uv = A_u v$. In particular the left multiplication in $A$ is denoted by $A_u$. Define the *Jordanator*

$$\begin{aligned} [a, b, c; d] := {} & ab \cdot cd + ac \cdot bd + bc \cdot ad \\ & - a(bc \cdot d) - b(ac \cdot d) - c(ab \cdot d) \end{aligned}$$



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for $a$, $b$, $c$, $d$ in $A$. Note that the Jordanator is linear in each argument and symmetric in the first three arguments. Just as the commutator measures commutativity and the associator associativity, the Jordanator measures how far the algebra is from being Jordan: $A$ is Jordan if and only if all Jordanators are zero.

A verification shows that the identity

$$\begin{aligned} x[u, v, w; y] + y[u, v, w; x] - [u, v, w; xy] \\ = [x, y, uv; w] + [x, y, vw; u] + [x, y, uw; v] \\ - [x, y, u; vw] - [x, y, v; uw] - [x, y, w; uv] \end{aligned} \tag{1.1}$$

holds for all $x, y, u, v, w$ in $A$. Note that (1.1) is symmetric in $u$, $v$, $w$ and in $x$, $y$.

*Remark.* In the absence of 2- and 3-torsion the identity (1.1) is equivalent to $2x[u, u, u; x] - [u, u, u; x^2] = 3[x, x, u^2; u] - 3[x, x, u; u^2]$.

1.2. As a first application consider the 4-linear map $j$ defined via

$$j(u, v, w; x) := 2[u, v, w; x] - [u, v, x; w] - [u, w, x; v] - [v, w, x; u].$$

Note that $j$ is symmetric in the first three arguments. In order to prove the identity

$$j(u, u, u; x^2) = \\ 3u \cdot j(u, u, x; x) - x \cdot j(u, u, u; x) + 3j(u, u, ux; x) - 3j(u, x, u^2; x) \tag{1.2}$$

use

$$\begin{aligned} 3u \cdot j(u, u, x; x) - x \cdot j(u, u, u; x) \\ = 3u[u, u, x; x] + 3x[u, u, x; u] - 6u[u, x, x; u] - 2x[u, u, u; x] \end{aligned}$$

and apply (1.1).

Defining

$$J(A) := \{x; x \in A, j(u, u, u; x) = 0 \text{ for all } u \in A\}$$

then $x \in J(A)$ is equivalent to $2[u, u, u; x] = 3[u, u, x; u]$ hence to

$$2u(u \cdot ux) + u^3x = 2u(u^2x) + u^2 \cdot ux$$

for $u$ in $A$. As a trivial consequence one concludes from (1.2) the

THEOREM. *In the absence of* 2- *and* 3-*torsion* $J(A)$ *is a Jordan subalgebra of* $A$.

*Remark.* In the case of a finite-dimensional algebra over a field the theorem was proved in [5, Satz 2.3] by different methods. Later K. McCrimmon [8] gave a proof in the general situation by a calculation. For power-associative algebras one gets $[u, u, u; x] + 3[u, u, x; u] = 0$ by linearization of $[u, u, u; u] = 0$ and consequently $j(u, u, u; x) = 3[u, u, u; x]$.

## 2. Some Relations

2.1. In the Introduction it was mentioned that a certain subset $M(A)$ of $A$ is of special interest for some applications to real analysis. In order to discuss $M(A)$ and related questions three relations $R$, $S$, and $RS$ are defined on $A$ via

$$\begin{aligned} aRb &:\Leftrightarrow [u, v, a; b] = 0 \qquad \text{for all} \quad u, v \in A, \\ aSb &:\Leftrightarrow [a, b, u; v] = 0 \qquad \text{for all} \quad u, v \in A, \\ aRSb &:= a(RS)b :\Leftrightarrow aRb \text{ and } aSb. \end{aligned}$$

Note that the relation $S$ is symmetric. Now the subset $M(A)$ of $A$ is defined by

$$M(A) := \{x;\ x \in A,\ xRSx\}.$$

Hence an element $x$ of $A$ belongs to $M(A)$ if and only if

$$[u, v, x; x] = [x, x, u; v] = 0 \qquad \text{for all} \quad u, v \in A.$$

Obviously $x \in M(A)$ implies $\alpha x \in M(A)$ for all $\alpha \in k$. The following example shows that $M(A)$ in general is not closed under addition or multiplication. However, the center $C(A)$ is contained in $M(A)$ and $C(A) + M(A) \subset M(A)$.

Example (K. McCrimmon). Consider an algebra $A$ over a field of characteristic $\neq 2$ with basis $x, y, z, w$ and multiplication $x^2 = y^2 = z^2 = w^2 = 0$, $xy = z$, $zw = w$, $xz = xw = yz = yw = 0$. A verification yields $M(A) = kx \cup ky \cup kw$.

2.2. In the following $u, v, w$ are arbitrary elements of $A$. From (1.1) it follows that

$$[u, v, w; xy] = x[u, v, w; y] + y[u, v, w; x] \qquad \text{whenever} \quad xSy. \tag{2.1}$$

Lemma 2.1. *Let $x, y, z \in A$ such that $xSy$, $zRSx$, and $zRSy$ hold. Then $zRS(xy)$.*

*Proof.* Put $w = z$; then (2.1) yields $zR(xy)$. Next replace $u, v, w, x, y$ in (1.1) by $x, y, v, u, z$. Hence $xSy$ and the symmetry of the Jordanotor in the first three arguments leads to

$$\begin{aligned} &[z, xy, u; v] + [u, xv, z; y] + [u, yv, z; x] \\ &\qquad = [z, x, u; yv] + [z, y, u; xv] + [u, v, z; xy]. \end{aligned}$$

Hence $zS(xy)$ because of $zRy$, $zRx$, $zSx$, $zSy$, and $zR(xy)$.

The simple Lemma 2.1 turns out to be the key for the

LEMMA 2.2. *Suppose that $a$ is an element of $M(A)$ satisfying $(a^2)Ra$. Then $(a^m)\,RS(a^n)$ holds for all $m, n \geqslant 1$.*

For the *proof* it suffices to show by induction on $r$ the statement

$$(a^m)\,RS(a^n) \qquad \text{for} \quad 1 \leqslant m, n \leqslant r. \tag{2.2}$$

Here the case $r = 1$ comes from the hypothesis. The substitution $u = a^r$, $v = w = a$, $x = u$, $y = v$ in (1.1) leads to

$$2[u, v, a^{r+1}; a] + [u, v, a^2; a^r] = 2[u, v, a; a^{r+1}] + [u, v, a^r; a^2] \tag{2.3}$$

because of $aSa$.

First consider the case $r = 2$. Lemma 2.1 can be applied to $x = y = z = a$ because of $aRSa$ and yields $aRS(a^2)$. By hypothesis $(a^2)\,Ra$ holds and consequently $(a^2)\,RSa$ holds. Finally Lemma 2.1 gives $(a^2)\,RS(a^2)$ for $x = y = a$ and $z = a^2$. Hence (2.2) is proved for $r = 2$.

Next suppose that (2.2) is proved for some $r \geqslant 2$. In particular $aS(a^r)$ as well as $(a^n)\,RSa$ and $(a^n)\,RS(a^r)$, $n \leqslant r$, hold. For $x = a$, $y = a^r$, $z = a^n$ Lemma 2.1 yields $(a^n)\,RS(a^{r+1})$ for $n \leqslant r$. Hence $(a^n)\,RS(a^m)$ holds for $n \leqslant r$, $m \leqslant r + 1$.

In particular $(a^2)\,R(a^r)$, $aR(a^{r+1})$, and $(a^r)\,R(a^2)$ hold. Therefore (2.3) implies $(a^{r+1})\,Ra$ and hence $(a^{r+1})\,RSa$. The choice $x = a$, $y = a^m$, $m \leqslant r$, $z = a^{r+1}$ in Lemma 2.1 shows that $(a^{r+1})\,RS(a^m)$ implies $(a^{r+1})\,RS(a^{m+1})$ in case $m \leqslant r$. Hence $(a^{r+1})\,RS(a^m)$ for $m \leqslant r + 1$ and (2.2) is proved for $r + 1$ instead of $r$.

COROLLARY. *In the absence of 3-torsion $(a^m)\,RS(a^n)$ holds for $a \in M(A)$ and $m, n \geqslant 1$.*

*Proof.* For $r = 1$ identity (2.3) yields $3[u, v, a^2; a] = 0$ and hence $(a^2)\,Ra$ holds.

2.3. Suppose now that there is no 2-and 3-torsion in $A$. A *submodule* of $A$ is by definition a submodule of the underlying $k$-module of $A$. Say that two elements $x$ and $y$ of $A$ *commute* whenever $x(yu) = y(xu)$ holds for $u \in A$. Denote by $A[a]$ the submodule of $A$ generated by the powers $a, a^2, \ldots$ of $a$.

THEOREM. *Suppose that $a$ is an element of $M(A)$. Then*

(a) *$xRSy$ holds for $x, y \in A[a]$, in particular $A[a] \subset M(A)$.*

(b) *$A[a]$ becomes an associative subalgebra of $A$.*

(c) *Each $A_x$, $x \in A[a]$, is a polynomial in the commuting endomorphisms $A_a$ and $A_{a^2}$.*

(d) *Any two elements of* $A[a]$ *commute.*

(e) (i) $A_{xy}A_u + A_{xu}A_y + A_{yu}A_x = A_uA_{xy} + A_yA_{xu} + A_xA_{yu}$ ,

(ii) $A_{xy}A_u + A_{xu}A_y + A_{yu}A_x = A_{xy\cdot u} + A_xA_uA_y + A_yA_uA_x$ ,

(iii) $A_{xy}A_u + A_{xu}A_y + A_{yu}A_x = A_{x\cdot yu} + A_yA_xA_u + A_uA_xA_y$

*hold for* $x, y \in A[a]$ *and* $u \in A$.

*Proof.* Since the Jordanator is linear in each argument the corollary implies $[x, y, u; v] = [u, v, x; y] = 0$ for $u, v \in A$. Hence parts (a) and (e) are proved. The endomorphisms $A_a$ and $A_{a^2}$ commute because of (e)(i). From (e)(iii) an induction leads to part (c) and hence to part (d). Now part (b) can be proved as in the case of Jordan algebras.

Define the quadratic representation $P = P_A$ of $A$ via

$$P(u)v := 2u(uv) - u^2v$$

for $u, v \in A$.

COROLLARY. *Suppose that* $a$ *is an element of* $M(A)$. *Then* $P(xy) = P(x)\, P(y)$ *for all* $x, y \in A[a]$.

*Proof.* From (e)(ii) and (c) follows

$$A_{xy\cdot xy} = A_{xy}^2 + A_{xx}A_y{}^2 - 4A_x{}^2A_y{}^2 + A_{yy}A_x{}^2 + 2A_xA_yA_{xy}$$

as well as

$$A_{xx\cdot yy} = A_{xx}A_{yy} + 4A_{xy}A_xA_y - 4A_x{}^2A_y{}^2.$$

Here the left-hand sides coincide because of (b), hence

$$2A_xA_yA_{xy} = A_{xy}^2 + A_{xx}A_y{}^2 + A_x{}^2A_{yy} - A_{xx}A_{yy}\,.$$

Together with the first identity the corollary is proved.

*Question.* Are there *interesting* algebras that are not Jordan algebras and nontrivial mappings $f\colon A \times A \to A$ satisfying $f(x, y) \in M(A)$ for all $x, y \in M(A)$? Candidates for $f$ are of course $f(x, y) = xy$ and $f(x, y) = P(x)y$.

2.4. As an application consider the case $k = \mathbb{R}$ and suppose that $A$ is finite dimensional over $\mathbb{R}$. Hence

$$\exp u := \sum_{n=1}^{\infty} \frac{1}{n!}\, u^n, \qquad u \in A,$$

is well defined. Let $a$ be an element of $M(A)$. From part (b) of the theorem one concludes

$$\exp(\alpha+\beta)a = (\exp \alpha a)(\exp \beta a), \qquad \alpha, \beta \in \mathbb{R}.$$

Hence $\alpha \to P(\exp \alpha a)$ defines a one-parameter subgroup of $GLA$ because of the corollary and it follows that

$$P(\exp a) = \exp 2A_a, \qquad a \in M(A).$$

## 3. Submodules of $M(A)$

3.1. *Suppose in this section that there is no* 2- *and* 3-*torsion in* $A$. The subset $Z$ of $A$ is called $R$-*closed* ($S$-closed, $RS$-closed, resp.) whenever $xRy$ ($xSy$, $xRSy$, resp.) holds for all $x, y \in Z$. Because of the symmetry of $S$ it follows that any submodule of $A$ contained in $M(A)$ is $S$-closed.

Denote by Der $A$ the Lie algebra of all derivations of $A$ and put $D_{u,v} := A_uA_v - A_vA_u$ for $u, v \in A$. By a verification it follows

$$D_{x,y} \in \operatorname{Der} A \Leftrightarrow [u, v, x; y] = [u, v, y; x] \qquad \text{for all} \quad u, v \in A. \tag{3.1}$$

Lemma 3.1. *A submodule $Z$ of $A$ is $RS$-closed if and only if $Z \subset M(A)$ and $D_{x,y} \in$ Der $A$ for all $x, y \in Z$.*

*Proof.* Clearly, if $Z$ is $RS$-closed then $xRSx$ for all $x \in Z$ and therefore $Z \subset M(A)$. Moreover, by (3.1) it follows that $D_{x,y} \in$ Der $A$ holds for all $x, y \in Z$.

Suppose now that $Z \subset M(A)$ and $D_{x,y} \in$ Der $A$ holds for all $x, y \in Z$. For $x, y \in Z$ by linearization follows $[u, v, x; y] + [u, v, y; x] = 0$ for $u, v \in A$. Hence $[u, v, x; y] = 0$ in view of (3.1) and therefore $Z$ is $R$-closed. But as a submodule of $M(A)$ the module $Z$ is $S$-closed, too.

From Lemma 2.1 follows immediately the

Corollary. *Suppose that the submodule $Z$ of $A$ is $RS$-closed. Then $xRS(yz)$ holds for all $x, y, z \in Z$.*

*Remark.* Part (a) of the theorem in Section 2.3 shows that $A[a]$ is $RS$-closed whenever $a \in M(A)$.

3.2. For a subset $Z$ of $A$ define

$$\operatorname{Mon} Z := \{A_{z_1}A_{z_2} \cdots A_{z_n}z_{n+1};\ n \geq 0,\ z_1, \ldots, z_{n+1} \in Z\}.$$

In particular $Z \subset \operatorname{Mon} Z$ and $z \in Z$, $x \in \operatorname{Mon} Z$ implies $zx \in \operatorname{Mon} Z$. Note that in general Mon $Z$ does not coincide with the monoid generated by $Z$.

LEMMA 3.2. *Suppose that the submodule $Z$ of $A$ is $RS$-closed. Then $zRSx$ holds for all $z \in Z$ and $x \in \operatorname{Mon} Z$.*

*Proof.* Because of $Z \subset \operatorname{Mon} Z$ it suffices to prove

$$zRSx \text{ for all } z \in Z \Rightarrow zRS(yx) \text{ for all } z, y \in Z \tag{3.2}$$

for $x \in A$. For $z, y \in Z$ it follows that $zRSx$ and $yRSx$ hold, in particular $ySx$. Since $Z$ is supposed to be $RS$-closed $zRSy$ holds. Hence by Lemma 2.1 one concludes $zRS(yx)$ and the conclusion in (3.2) is proved.

3.3. In [6] it was shown that for arbitrary unital algebras $A$ there is a Peirce-decomposition with respect to a complete system of orthogonal idempotents as in the Jordan case provided these idempotents are in a Peirce-set of $A$. Here $U \subset A$ is Peirce-*set of* $A$ (see [6, Sect. 3.1]) if for $x, y, z \in U$ the conditions

$$[x, y, z; u] = 0 \qquad \text{for all} \quad u \in A, \tag{P.1}$$

$$2[x, y, u; v] = [u, v, x; y] + [u, v, y; x] \qquad \text{for all } u, v \in A, \tag{P.2}$$

are fulfilled. Obviously, any $RS$-closed subset of $A$ is a Peirce-set.

Let $e_1, \ldots, e_r$ be a complete system of orthogonal idempotents of $A$, i.e., $e_i \neq 0$ and $e_i e_j = \delta_{ij} e_i$, $e_1 + \cdots + e_r = e$, where $e$ is the unit element of $A$. Then the Peirce-spaces are defined by

$$A_{ii} := \{x;\, x \in A,\, e_i x = x\},$$
$$A_{ij} := \{x;\, x \in A,\, e_i x = \tfrac{1}{2}x = e_j x\}, \qquad i \neq j.$$

From [6, Satz 3.3] follows

LEMMA 3.3. *Suppose that $A$ contains a unit element $e$ and suppose that the submodule $Z$ of $A$ is $RS$-closed. Let $e_1, \ldots, e_r$ be a complete system of orthogonal idempotents of $Z$. Then*

$$A = \sum_{i \leqslant j} A_{ij}$$

*is a direct sum of $k$-modules and*

$$A_{ij} A_{jl} \subset A_{jl} \qquad \textit{for} \quad j \neq l,$$
$$A_{ij} A_{ij} \subset A_{ii} + A_{jj},$$
$$A_{ij} A_{lm} = 0 \qquad \textit{otherwise}.$$

In particular the $A_{ii}$'s are subalgebras annihilating each other.

3.4. As an application of the Peirce-decomposition it will be shown that some submodules of $M(A)$ are already subalgebras of $A$. This result has been proved in a special situation in connection with homogeneous convex cones in finite-dimensional real vector spaces first by Dorfmeister [2, IV, Satz 4.5].

THEOREM. *Suppose that $A$ is an unital algebra and suppose that the submodule $Z$ of $A$ is RS-closed and satisfies*

(i) $$D_{x,y}z \in Z \qquad \text{for all} \quad x, y, z \in Z.$$

*Let $e_1, \ldots, e_r$ be a complete system of orthogonal idempotents of $Z$ such that*

(ii) $$A_{ii} \subset Z \qquad \text{for} \quad i = 1, \ldots, r.$$

*Then $Z$ is a Jordan subalgebra of $A$.*

*Proof.* Since $Z$ is $R$-closed it follows that

$$3(x^2 \cdot xy - x \cdot x^2y) = [x, x, x; y] = 0 \qquad \text{for} \quad x, y \in Z.$$

Hence it suffices to show that $Z$ is closed under multiplication, i.e., $x^2 \in Z$ whenever $x \in Z$.

Let $x$ be an element of $Z$ with Peirce-components $x_{ij}$, $i \leqslant j$, and put $x_i := x_{ii}$. Hence $x_i \in Z$ in view of (ii). From (i) it follows then

$$\tfrac{1}{2}x_i x_{ij} = x_i(xe_j) = D_{x_i,x}\, e_j \in Z \qquad \text{for} \quad i \neq j,$$

hence

(iii) $$x_i x_{ij} \in Z \qquad \text{for} \quad i \neq j.$$

In particular $x_{ij} \in Z$ for $i \neq j$. But from (ii) it follows now that *all* Peirce-components $x_{ij}$ are in $Z$. Again by (ii) Lemma 3.3 leads to

(iv) $$x_{ij} x_{ij} \in Z \qquad \text{for} \quad i \leqslant j.$$

Next for $l \neq i$, $l \neq j$, the condition (i) yields

(v) $$x_{ij} x_{jl} = 2D_{x_{ij},x_{jl}}\, e_l \in Z.$$

Since all products that are not listed under (iii)–(v) are zero the theorem is proved.

*Remark.* Consider a submodule $Z$ of $A$ contained in $M(A)$ satisfying $D_{x,y} \in \operatorname{Der} A$ and $D_{x,y}z \in Z$ for $x, y, z \in Z$. If $Z$ contains a complete system of orthogonal idempotents $e_1, \ldots, e_r$ such that $A_{ii} = ke_i$, then $Z$ is a Jordan subalgebra.

## 4. A Class of Power-Associative Algebras

4.1. Consider first a $k$-algebra $A$ satisfying $a^2a^2 = a^4$ for $a \in A$. In the absence of 2- and 3-torsion this is equivalent to

$$[a, b, c; d] + [b, c, d; a] + [c, d, a; b] + [d, a, b; c] = 0$$

for $a, b, c, d \in A$, in particular $[a, a, b; b] + [b, b, a; a] = 0$. Hence

$$[a, b, x; y] + [a, b, y; x] = 0 \qquad \text{for} \quad a, b \in A \text{ provided } xSy,$$

and *the relation RS is symmetric*. Moreover $xSx$ implies $xRx$ and consequently

$$M(A) = \{x; x \in A, [x, x, a; b] = 0 \text{ for } a, b \in A\}. \tag{4.1}$$

LEMMA 4.1. *Suppose that the submodule $Z$ of $A$ is RS-closed. Then for each $z \in Z$ the submodule $Z + kz^2$ is RS-closed.*

*Proof.* If $x \in Z$ then $xRSz$ and $zSz$ hold. Hence by Lemma 2.1 one gets $xRS(z^2)$ and by the symmetry of $RS$ it follows that $(z^2)\,RSx$. But part (a) of the theorem in Section 2.3 yields $(z^2)\,RS(z^2)$. Hence $(x + \alpha z^2)\,RS(y + \beta z^2)$ for $x, y \in Z$ and $\alpha, \beta \in k$ and $Z + kz^2$ is $RS$-closed.

COROLLARY. *If $Z$ is a maximal RS-closed submodule of $A$ then $Z$ is a Jordan subalgebra of $A$.*

4.2. In order to discuss a class of examples change notations and consider a commutative and associative $k$-algebra $A$ and an unitary $k$-module $X'$ that is supposed to be an $A$-left module via $(a, x') \mapsto ax'$ where $a \in A$ and $x' \in X'$. The direct sum $X := A \oplus X'$ of $k$-modules turns out to be an $A$-left module with respect to componentwise multiplication. The elements of $X$ are written as $x = t(x) \oplus x'$, where $t\colon X \to A$ denotes the projection of $X$ onto the first component and where $x' \in X'$. By definition $t(ax) = a \cdot t(x)$ and $t(a) = a$ for $a \in A$ and $x \in X$.

For the moment no assumption on the absence of torsion is necessary. However in the absence of 2-torsion there are additional results. In that what follows the elements $a, b, c$ are in $A$ and $x, y, z$ are in $X$. Consider a symmetric and $k$-bilinear mapping $q\colon X' \times X' \to A$ satisfying

$$a \cdot q(x', x') = q(ax', x'). \tag{4.2}$$

Define a symmetric and $k$-bilinear mapping $n\colon X \times X \to A$ by

$$n(x, y) := t(x)\, t(y) + q(x', y') \tag{4.3}$$

and put $n(x) := n(x, x)$. In particular $n(x, a) = a \cdot t(x)$, $n(a) = a^2$. Finally put

$$d_a(x, y) := D_{x,y}a := n(ax, y) - a \cdot n(x, y) = q(ax', y') - a \cdot q(x', y') \tag{4.4}$$

Note that $d_a(x, y)$ and $D_{x,y}$ depend only on $x'$ and $y'$.

LEMMA 4.2. (a) *For $a, b, c \in A$ and $x \in X$ it follows that*

(i) $a \cdot n(x, x) = n(ax, x)$,

(ii) $ab \cdot n(x, x) = n(ax, bx)$,

(iii) $d_a(x, x) = d_a(bx, x) = d_a(bx, cx) = 0$.

(b) *In the absence of 2-torsion one has in addition*:

(i) *For a given $a \in A$ the mapping $d_a$: $X \times X \to A$ is skew symmetric and $A$-bilinear.*

(ii) *For given $x, y \in X$ the mapping $D_{x,y}$: $A \to A$ is a derivation of $A$.*

*Proof.* (a) Part (i) follows directly from (4.2) and (4.3). Using (i) and its linearized form one obtains $n(abx, x) + n(ax, bx) = 2b \cdot n(ax, x) = 2ab \cdot n(x, x) = 2n(abx, x)$, hence $n(ax, bx) = n(abx, x) = ab \cdot n(x, x)$. (iii) is a consequence of (i) and (ii).

(b) From (iii) of part (a), $d_a$ is skew. Using the linearized form of (i) and (ii) in part (a) one obtains $2[d_a(bx, y) - b \cdot d_a(x, y)] = 2n(abx, y) - 2a \cdot n(bx \cdot y) - 2b \cdot n(ax, y) + 2ab \cdot n(x, y) = 2ab \cdot n(x, y) - n(ax, by) - n(bx, ay) = 0$. Hence part (i) is proved. Part (ii) follows from a similar calculation.

4.3. Define a product on $X$ via

$$\begin{aligned} xy &:= t(x)y + t(y)x - n(x, y) \\ &= [t(x)\,t(y) - q(x', y')] \oplus [t(x)\,y' + t(y)x'], \end{aligned} \tag{4.5}$$

in particular

$$x^2 = 2t(x)x - n(x). \tag{4.5'}$$

Hence $X$ is a commutative $k$-algebra, the *algebra associated with the triple* $(A, X', q)$, and the multiplication with elements of $A$ coincides with their action as left-modules. From (4.4) one concludes

$$a \cdot xy - ax \cdot y = d_a(x, y) = D_{x,y}a. \tag{4.6}$$

Hence the mapping $d_a$ measures how far the product (4.5) is from being $A$-bilinear. Part (a) of Lemma 4.2 can be translated into

$$a \cdot x^2 = ax \cdot x, \qquad ab \cdot x^2 = ax \cdot bx. \tag{4.7}$$

As in the case of Jordan algebras associated with quadratic forms define

$$\bar{x} := 2t(x) - x = t(x) \oplus (-x').$$

LEMMA 4.3. *The mapping $x \mapsto \bar{x}$ is an $A$-linear involution of $X$ satisfying $\bar{a} = a$ for $a \in A$ and*

(i) $t(\bar{x}) = t(x)$, $\quad n(x) = x\bar{x} = n(\bar{x})$, $\quad n(x, \bar{y}) = n(\bar{x}, y) = t(xy)$,

(ii) $n(xz, y) - n(x, \bar{z}y) = d_{t(x)}(z, y) + d_{t(y)}(z, x) + 2d_{t(z)}(x, y)$.

*Proof.* Clearly $t(\bar{x}) = t(x)$ and the mapping $x \mapsto \bar{x}$ is $A$-linear and has period 2. From (4.5) it follows that $x\bar{x} = n(x)$ as well as $t(x\bar{y}) = n(x, y)$. Hence (i) is proved. Next $x \mapsto \bar{x}$ is a homomorphism because of (4.5) and $n(\bar{x}, \bar{y}) = n(x, y)$. The identity (ii) follows from $\bar{z}y = t(z)y - t(y)z + n(y, z)$, (4.4) and $d_a(x, x) = 0$ by a calculation.

4.4 Consider the submodule $A[x] := A + kx + Ax$ which is invariant under the mapping $x \mapsto \bar{x}$. From part (a)(iii) of Lemma 4.2 it follows that

$$d_a(u, v) = 0, \quad \text{i.e.,} \quad n(au, v) = a \cdot n(u, v) \qquad \text{for} \quad u, v \in A[x]. \tag{4.8}$$

THEOREM 4.4. *Let $X$ be the algebra associated with the triple $(A, X', q)$ and let $x$ be in $X$. Then*

(a) *$A[x]$ is an associative subalgebra of $X$ and the product in $A[x]$ is $A$-bilinear, i.e., $A[x]$ is an associative $A$-algebra.*

(b) *$n(uv, w) = n(u, \bar{v}w)$ and $n(uv) = n(u)\, n(v)$ for $u, v, w \in A[x]$.*

(c) *The subalgebra of $X$ generated by $x$ is contained in $A[x]$.*

COROLLARY 1. *$X$ is power associative.*

COROLLARY 2. *$n(x^2) = (n(x))^2$ for $x \in X$.*

*Proof.* (a) From (4.7) and (4.5′) one concludes that $A[x]$ is a subalgebra of $X$ that is associative. The product in $A[x]$ is $A$-bilinear because of (4.6) and (4.8).

(b) The first identity follows from Lemma 4.3 (ii) and (4.8). Hence $n(uv) = n(uv, uv) = n(u, \bar{v} \cdot uv) = n(u, v\bar{v} \cdot u) = n(u, n(v)u) = n(v)\, n(u)$ because of Lemma 4.3(i), (4.8) and the associativity of $A[x]$.

(c) Clear since $x \in A[x]$.

4.5. Suppose now that there is no 2- and 3-torsion. Hence by part (b)(i) of Lemma 4.2 the mapping $(x, y) \mapsto d_a(x, y)$ is skew and $A$-bilinear. One concludes from (4.6) and part (a)(iii) of Lemma 4.2

$$x \cdot ay - ax \cdot y = 2d_a(x, y), \qquad d_a(x, xy) = t(x)\, d_a(x, y). \tag{4.9}$$

In order to prove

$$x^2 \cdot xy - x \cdot x^2y = d_{q(x', x')}(x, y) \oplus 2d_{t(x)}(x, y)\, x' \tag{4.10}$$

put $a := t(x)$ and use (4.6) and (4.9) in the following calculation:

$$\begin{aligned} x^2 \cdot xy - x \cdot x^2y &= 2ax \cdot xy - 2x(ax \cdot y) + x(n(x)y) - n(x) \cdot xy \\ &= 2ax \cdot xy - 2x(a \cdot xy) + 2d_a(x, y)x - d_{n(x)}(y, x) \\ &= -4d_a(x, xy) + d_{n(x)}(x, y) + 2d_a(x, y)x \\ &= [d_{n(x)}(x, y) - 2a \cdot d_a(x, y)] \oplus 2d_a(x, y)\, x'. \end{aligned}$$

Hence (4.10) follows from part(b) of Lemma 4.2.

Using (4.1) one concludes from (4.10)

LEMMA 4.5. *An element $u$ of $X$ belongs to $M(X)$ if and only if*

$$\text{(i)} \qquad d_a(u, y)\, u' = 0,$$

$$\text{(ii)} \qquad d_{t(u)}(x, y)\, u' + d_{t(u)}(u, y)\, x' = 0,$$

$$\text{(iii)} \qquad 2d_{q(u',x')}(u, y) + d_{q(u',u')}(x, y) = 0,$$

*for $a \in A$ and $x, y \in X$.*

4.6. The construction given above depends on the mapping $q: X' \times X' \to A$ satisfying (4.2). A "non-trivial" (i.e., non-Jordan) situation occurs only if $q$ is not $A$-bilinear and in view of part (b) of Lemma 4.2 derivations of $A$ have to be involved.

In order to construct such a mapping put $X' := A \oplus A$ and write the elements of $X'$ as $x' = x_1 \oplus x_2$ where $x_1, x_2 \in A$. For a derivation $D$ of $A$ define $q = q_D: X' \times X' \to A$ by

$$q(x', y') := x_2 \cdot Dy_1 + y_2 \cdot Dx_1 - x_1 \cdot Dy_2 - y_1 \cdot Dx_2 .$$

Using the abbreviations $s(x', y') := x_1y_2 - x_2y_1$ and $Dx' := Dx_1 \oplus Dx_2$ one has

$$q(x', y') = s(Dx', y') + s(Dy', x'). \tag{4.11}$$

Since $s$ is $A$-bilinear and skew symmetric one gets

$$d_a(x', y') = Da \cdot s(x', y'). \tag{4.12}$$

Hence (4.2) holds. Note that in general $q$ is not $A$-bilinear. Denote the algebra associated with the triple $(A, A \oplus A, q_D)$ by $X = X(A, D)$ and write the elements of $X$ as $x = x_0 \oplus x_1 \oplus x_2$ where $x_0 = t(x)$.

LEMMA 4.6. *Suppose that $A$ is unital and let $D$ be a derivation of $A$. Then an element $u$ of $X = X(A, D)$ belongs to $M(X)$ if and only if for $i, j = 1, 2$*

$$u_iu_j \cdot DA = 0, \qquad u_i \cdot Du_0 = 0, \qquad u_i \cdot D^2u_j = 0.$$

*Proof.* Since $A$ is unital it follows that $s(u', x') = 0$ for all $x' \in X'$ implies $u' = 0$. Hence using (4.12) the conditions of Lemma 4.5 are equivalent to

$$\text{(i}'\text{)} \qquad Da \cdot u_i u' = 0 \qquad \text{for} \quad a \in A, \quad i = 1, 2,$$

$$\text{(ii}'\text{)} \qquad Du_0[u_i x' + x_i u'] = 0 \qquad \text{for} \quad x' \in X', \quad i = 1, 2,$$

$$\text{(iii}'\text{)} \quad 2\,Dq(u', x') \cdot u' + Dq(u', u') \cdot x' = 0, \qquad \text{for} \quad x' \in X'.$$

The first two conditions of the lemma are equivalent to ($i'$) and (ii$'$). In order to discuss (iii$'$) substitute (4.11) and use $Ds(x', y') = s(Dx', y') + s(x', Dy')$ so $Dq(x', y') = s(D^2x', y') + s(D^2y', x')$. Hence (iii$'$) is equivalent to $s(D^2x', u')\, u' + s(D^2u' \cdot x')\, u' + s(D^2u', u')\, x' = 0$ for all $x' \in X'$, where the first term vanishes by ($i'$). But this is the last condition of the lemma.

COROLLARY. *$X(A, D)$ is Jordan if and only if $D = 0$.*

*Remark.* The simple commutative and power-associative algebras studied by Kokoris [7] and in an improved and more general version by Albert [1] are special cases of the construction given in Sections 4.2, 4.3, and 4.4. The Kokoris example is an algebra $X(A, D)$ where $k$ is a field of characteristic $p > 3$, $A := k[T]/(T^p)$ for an indeterminate $T$ and where $D$ is the derivative. Using Lemma 4.6 one can prove that $X(A, D)$ contains a Jordan subalgebra of dimension $(3p + 5)/2$ where $3p$ is the dimension of $X(A, D)$.

## 5. Finite-Dimensional Algebras over a Field

5.1. Suppose now that $A$ is a finite-dimensional algebra over the field $k$ of characteristic different from 2 and 3. Assume further that $\sigma: A \times A \to k$ is a symmetric and nondegenerate bilinear form that is associative for $A$. For an endomorphism $T$ of the underlying vector space of $A$ denote by $T^\sigma$ the adjoint endomorphism with respect to $\sigma$. Then $A_u$ is selfadjoint for all $u \in A$.

Define two endomorphisms $J(a, b, c)$ and $K(a, b; c)$ via

$$[a, b, c; d] = J(a, b, c)d = K(a, b; d)c \qquad \text{for} \quad a, b, c, d \in A.$$

Hence

$$J(a, b, c) = A_{ab}A_c + A_{ac}A_b + A_{bc}A_a - A_aA_{bc} - A_bA_{ac} - A_cA_{ab},$$

$$K(a, b; d) = A_{ab}A_d + A_{bd}A_a + A_{ad}A_b - A_aA_dA_b - A_bA_dA_a - A_{ab\cdot d}.$$

A verification leads to

(i) $[J(a, b, c)]^\sigma = -J(a, b, c)$, $[K(a, b; d)]^\sigma = K(a, b; d) - J(a, b, d)$.

PROPOSITION. *$xSy$ holds whenever $xRy$ holds.*

*Proof.* By the definition in Section 2.1 it follows that $xSy$ is equivalent to $J(x, y, u) = 0$ for all $u \in A$ and $xRy$ is equivalent to $K(u, x; y) = 0$ for all $u \in A$. Hence starting with $xRy$ then $K(u, x; y) = 0$ and consequently $J(x, y, u) = 0$ in view of (i). So $xRy$ implies $xSy$.

COROLLARY. $M(A) = \{a; a \in A, [x, x, a; a] = 0 \text{ for } x \in A\}$.

5.2. Finally an application to analysis shall be described. Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ and let $K$ be a regular convex cone in $V$. Consider an arbitrary differentiable and homogeneous mapping $\eta$: $K \to \mathbb{R}^+$ and suppose that the symmetric bilinear form

$$\sigma_x(u, v) := \Delta_x{}^u \Delta_x{}^v \log \eta(x) \qquad (\Delta_x{}^u = \text{directional derivative})$$

of $V$ is positive definite for each $x \in K$. Choose $e \in K$, put $\sigma := \sigma_e$ and define a product $(u, v) \mapsto uv$ on $V$ by

$$\sigma(uv, w) := -\tfrac{1}{2}\Delta_x{}^u \Delta_x{}^v \Delta_x{}^w \log \eta(x) \,|_{x=e} .$$

The induced algebra on $V$ is denoted by $[K, \eta, e]$. It is known that $[K, \eta, e]$ is a commutative algebra with unit element $e$ and associative form $\sigma$. Consider the Lie group $\mathrm{Aut}(K, \eta)$ of $W \in GLV$ satisfying

(1) $WK = K$.

(2) There exists $\alpha(W) \in \mathbb{R}^+$ such that $\eta(Wx) = \alpha(W)\, \eta(x)$, $x \in K$.

For details see [4].

Vinberg [9] introduced in a special situation the *core* of $K$ as the subspace $C(K, \eta, e)$ of $V$ consisting of the elements a such that the left multiplication $A_a$ of $[K, \eta, e]$ belongs to the Lie algebra of $\mathrm{Aut}(K, \eta)$.

THEOREM. *Suppose that* $\mathrm{Aut}(K, \eta)$ *acts transitively on* $K$. *Then the core* $C(K, \eta, e)$ *is contained in* $M([K, \eta, e])$.

*Proof.* By a result of Dorfmeister *et al.* [3, Satz 3.1] the elements

$$a \in C(K, \eta, e)$$

satisfy, in the notation in Section 5.1.,

$$K(a, x; a) - J(a, a, x) = K(a, a; x) = 0 \qquad \text{for} \quad x \in V.$$

But (i) shows that this is equivalent to $J(a, a, x) = K(a, x; a) = 0$ for $x \in V$.

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