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BMN operators and string field theory

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Abstract

We extract from gauge theoretical calculations the matrix elements of the SYM dilatation operator. By the BMN correspondence this should coincide with the 3-string vertex of light cone string field theory in the pp-wave background. We find a mild but important discrepancy with the SFT results. If the modified $O(g_2)$ matrix elements are used, the $O(g_2^2)$ anomalous dimensions are exactly reproduced without the need for a contact interaction in the single-string sector.

1. Introduction

In [1] Berenstein, Maldacena and Nastase studied a pp-wave limit of string theory in the $AdS_5 \times S^5$ background. Type IIB strings on the pp-wave geometry were found to correspond to operators of a $\mathcal{N} = 4$ $SU(N)$ super-Yang–Mills theory with large R charge J in the limit where J^2/N is fixed. They obtained definite predictions for the scaling dimensions of the relevant operators in the free string limit which were subsequently verified on the gauge theory side [1–3].

Subsequent work was made in extending the correspondence on both sides to lowest orders in the effective gauge coupling $\lambda' = g_{\mathrm{YM}}^2 N/J^2$ and genus $g_2 = J^2/N$ parameter [4–8]. On the string theory side the tool used to study interactions was light cone IIB string field theory (SFT) constructed for the pp-wave background in [9] (and also inherently discrete string-bit formulations [10–12]). There exist explicit expressions for the gauge theory parameters g_2, λ' in terms of string theoretical quantities (but see also [13]

$$\lambda' = \frac{1}{(\mu p^+ \alpha')^2}, \qquad g_2 = 4\pi g_s \big(\mu p^+ \alpha'\big)^2. \tag{1}$$

The link was made through a proposal made in [5] of a relation between matrix elements of the SFT Hamiltonian and certain gauge theoretical 3-point functions. This was verified in various cases [14–19] (see also [20–23] for further developments). However the explicit proposal was not derived from ‘first principles’. In fact subsequent work [24] uncovered an error in the construction of SFT matrix elements of [18] which makes the proposal more

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problematic. A direct calculation of the $O(g_2^2)$ anomalous dimensions from the $O(g_s)$ SFT matrix elements failed to give an agreement with the gauge theoretical result. This was not a direct contradiction, however, due to the theoretical possibility of $O(g_2^2)$ contact terms in the SFT Hamiltonian. In this Letter we want to look for a more direct test of the SFT—gauge theory correspondence.

The main aim of this Letter is to extract directly from the gauge theoretical calculations done so far the order $O(g_2)$ matrix elements of the gauge theory dilatation operator. This should be identified with the $O(g_s)$ vertex of light cone string field theory thus allowing for a direct comparison with the formulation of [9]. In addition it might give some insight into the failure of SFT (modulo contact terms) to describe the $O(g_2^2)$ gauge theoretical anomalous dimensions.

The outline of this Letter is as follows. In Section 2 we will recall some features of the BMN operator-string correspondence, in Section 3 we will extract the $O(g_2)$ matrix elements and show that they are sufficient to reconstruct full $O(g_2^2)$ anomalous dimensions without the need for explicit $O(g_2^2)$ 'contact interactions' in the single-string sector. We conclude the Letter with a discussion.

2. BMN operator-string correspondence

The dictionary established in [1] between string theory and gauge theoretical operators associates to each physical state of the string an explicit (single-trace) operator of the gauge theory. The operators which we will consider here are

$$O^J = \frac{1}{\sqrt{JN^J}} \operatorname{tr} Z^J, \tag{2}$$

$$O_i^J = \frac{1}{\sqrt{N^{J+1}}} \operatorname{tr}\bigl(\phi_i Z^J\bigr), \tag{3}$$

$$O_{ij,n}^J = \frac{1}{\sqrt{JN^{J+2}}} \left(\sum_{p=0}^{J} e^{2\pi i p n/J} \operatorname{tr}\bigl(\phi_i Z^p \phi_j Z^{J-p}\bigr) \right), \tag{4}$$

here $Z = (\phi_5 + \phi_6)/\sqrt{2}$ and ϕ_i are other transverse coordinates. These operators correspond respectively to the states $|0, p^+\rangle$, $a_0^{i\,\dagger}|0, p^+\rangle$ and $a_n^{i\dagger} a_{-n}^{j\dagger}|0, p^+\rangle$.

Double trace operators correspond to two-string states and at zero genus ($g_2 = 0$) can be identified unambigously. The operators that we will use here are

$$\mathcal{T}_{12}^{J,r} = \mathcal{O}_{12,n}^{rJ} \mathcal{O}^{(1-r)J}, \tag{5}$$

$$\mathcal{T}_{12,m}^{J,r} = \mathcal{O}_1^{rJ} \mathcal{O}_2^{(1-r)J}. \tag{6}$$

Here $r \in (0, 1)$ denotes the fraction of light cone momentum carried by the first string. Presumably (bosonic) multi-string states have to be symmetrized (this will not be important here).

The light cone string Hamiltonian is

$$H_{\text{string}}^{\text{l.c.}} = \frac{\mu}{2}(\Delta - J). \tag{7}$$

Therefore we should identify it (up to the factor $2/\mu$ and the constant shift) as equivalent to the gauge theory dilatation operator D.

At zero-genus all the single- and double-string states are eigenstates of $H_{\text{string}}^{\text{l.c.}}$ as the respective gauge theory operators are eigenstates of D. Once we turn on the interaction, the dilatation operator will start to mix the operators and $H_{\text{string}}^{\text{l.c.}}$ will start to mix the corresponding single- and multi-string states. We expect the action of the full interacting operators D and $H^{\text{l.c.}}_{\text{string}}$ on the gauge theory operators, and (multi-)string states respectively to coincide[1]

$$DO_\alpha = D_{\alpha\beta} O_\beta, \tag{8}$$

$$\left(\frac{2}{\mu} H^{\text{l.c.}}_{\text{string}} + J\right)|\alpha\rangle = h_{\alpha\beta}|\beta\rangle, \tag{9}$$

i.e., we should have $D_{\alpha\beta} = h_{\alpha\beta}$.

In [9] the terms linear in g_s in $H^{\text{l.c.}}_{\text{string}}$ were constructed

$$H^{\text{l.c.}}_{\text{string}} = H_2^{\text{l.c.}} + g_s H_3^{\text{l.c.}}, \tag{10}$$

where $H_2^{\text{l.c.}}$ is the free Hamiltonian and $H_3^{\text{l.c.}}$ represents the 3-string vertex. The following matrix elements were computed in [18] and will be relevant later[2]

$$\langle \mathcal{O}^J_{12,n} | H_3^{\text{l.c.}} | \mathcal{T}^{J,r}_{12,m} \rangle = \frac{4\mu}{\pi}(1-r)\sin^2(\pi n r), \tag{11}$$

$$\langle \mathcal{O}^J_{12,n} | H_3^{\text{l.c.}} | \mathcal{T}^{J,r}_{12} \rangle = \frac{4\mu}{\pi}\sqrt{r(1-r)}\,\sin^2(\pi n r). \tag{12}$$

Up till now most comparisons between string field theory and gauge theory were performed either on the level of 3-point correlation functions or by computing scaling dimensions.

The former method was based on a proposal which linked the structure constants C_{ijk} and appropriate SFT Hamiltonian matrix elements [5]

$$\langle i | H_3^{\text{l.c.}} | j, k \rangle = \mu g_2 (\Delta_i - \Delta_j - \Delta_k) C_{ijk}. \tag{13}$$

Although plausible and supported by various calculations it has not been strictly proven from first principles nor shown how it could be systematically extended beyond leading order.

The latter method of comparison based on determining anomalous dimensions is difficult because the first nontrivial corrections to the scaling dimensions are of order $O(g_2^2)$ while the SFT Hamiltonian in the pp-wave background has been only determined to $O(g_2)$ order. Indeed $H_3^{\text{l.c.}}$ with the matrix elements (11), (12) could not reproduce [5,7] the g_2^2 correction to the anomalous dimension of the $O^J_{ij,n}$ operator obtained in a SYM calculation [6]

$$\frac{g_2^2 \lambda'}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32\pi^2 n^2}\right). \tag{14}$$

In fact the disagreement between the scaling dimensions calculated in gauge theory and ones obtained from the cubic interaction Hamiltonian has been attributed to the possible appearance of nontrivial contact terms of order $O(g_s^2)$. Indeed additional $O(g_s^2)$ terms appear also in flat space light cone SFT [25,26]. However there they only involve four-string fields while here it seems that the disagreement can be cured only by terms which involve only *two-string* fields.

Therefore it is interesting to directly extract the $O(g_2)$ matrix elements of the gauge theory dilatation operator as these, according to the BMN operator-string correspondence, should be identified with the $O(g_s)$ SFT Hamiltonian matrix elements.

[1] Up to possible rescalings of the individual states.

[2] These are the corrected matrix elements from v3 of [18].

3. Gauge theory results

We will now extract the matrix of the gauge theory dilatation operator up to order $O(g_2)$. Let O_α be the set of all operators (single- and multi-trace) with R charge J which are eigenstates of the free (planar) dilatation operator, $\overline{O}_\alpha$ the corresponding complex conjugates and let us denote by O'_A the operators with definite scaling dimension

$$DO'_A = \Delta_A O'_A, \tag{15}$$

where D is the dilatation operator. These O'_A's may be rewritten as linear combination of the original operators and vice versa

$$O_\alpha = V_{\alpha A} O'_A \quad (O = VO'), \qquad O' = V^{-1} O. \tag{16}$$

Similar formulas hold for the barred operators (with a different matrix[3] $V^*_{\alpha A}$). Thus the matrix elements of the gauge theory dilatation operator in the original basis O_α are

$$DO_\alpha \equiv D_{\alpha\beta} O_\beta = \left(V \Delta V^{-1}\right)_{\alpha\beta} O_\beta. \tag{17}$$

This should be identified with $\frac{2}{\mu}\langle\beta|H^{\text{l.c.}}_{\text{string}}|\alpha\rangle + J\delta_{\alpha\beta}$.

We will now show how to extract the matrix $V\Delta V^{-1}$ from 2-point correlation functions. Using the expansions (16) we get

$$\left\langle O_\alpha(0)\overline{O}_\beta(x)\right\rangle = V_{\alpha A} V^*_{\beta B} \frac{\delta_{AB} C_A}{|x|^{2(J+2+\Delta_A)}}. \tag{18}$$

Here the C_A's are some undetermined normalization constants. Expanding to linear order in the logarithm gives

$$\left\langle O_\alpha(0)\overline{O}_\beta(x)\right\rangle = \frac{1}{|x|^{2(J+2)}}\left(M'_{\alpha\beta} + M''_{\alpha\beta}\log(x\Lambda)^{-2}\right), \tag{19}$$

where the matrices M' and M'' are given by

$$M' = VCV^\dagger, \qquad M'' = VC\Delta V^\dagger \tag{20}$$

and $V^\dagger$ denotes here the transpose of V^*. The dilatation operator matrix is then given by

$$D_{\alpha\beta} = \left(M'' M'^{-1}\right)_{\alpha\beta}. \tag{21}$$

The matrices M' and M'' have been calculated in [6]. For our purposes it is enough to find their elements to order $O(g_2)$. To this order there are only nonzero elements in the $\mathcal{O}^J_{12,n} - \mathcal{T}^{J,r}_{12,m}$ sector and the $\mathcal{O}^J_{12,n} - \mathcal{T}^{J,r}_{12}$ sector. It is easy to see that to order $O(g_2)$ we may treat them independently.

3.1. The $\mathcal{O}^J_{12,n} - \mathcal{T}^{J,r}_{12,m}$ sector

The calculations of [6,7] yield (see, e.g., (3.15) in [6])

$$M' = \begin{pmatrix} 1 & g_2 x \\ g_2 x & 1 \end{pmatrix},$$
$$M'' = \lambda' \begin{pmatrix} n^2 & g_2 x\left(\frac{m^2}{r^2} + n^2 - \frac{mn}{r}\right) \\ g_2 x\left(\frac{m^2}{r^2} + n^2 - \frac{mn}{r}\right) & \frac{m^2}{r^2} \end{pmatrix} \tag{22}$$

[3] We do not need to assume anything about the relation of V^* to V.

with

$$x = \frac{r^{3/2}\sqrt{1-r}\,\sin^2(\pi n r)}{\sqrt{J}\pi^2(m-nr)^2}. \tag{23}$$

The dilatation matrix to order $O(g_2)$ is thus

$$\begin{pmatrix} \lambda' n^2 & 0 \\ 0 & \lambda' \frac{m^2}{r^2} \end{pmatrix} + g_2 \begin{pmatrix} 0 & \lambda' x \frac{m}{r^2}(m-nr) \\ -\lambda' x \frac{nr}{r^2}(m-nr) & 0 \end{pmatrix}. \tag{24}$$

Several comments are in order here. The off-diagonal elements should be identified with the SFT Hamiltonian acting on string states corresponding to the single- and double-trace operators in gauge theory (as in (8), (9)). These quantities do not agree even with the corrected matrix elements of [18]. There is some relation, however. We note that the sum of the off-diagonal elements is equal to

$$\frac{4g_s}{\pi\sqrt{J}}\sqrt{\frac{1-r}{r}}\,\sin^2(\pi n r) \tag{25}$$

which exactly coincides with (11) up to the normalization factor of $\sqrt{Jr(1-r)}$. In fact we see that the rhs of the proposal (13) is antisymmetric w.r.t. exchange of initial and final states. A minor generalization which would still hold even for the modified matrix elements (24) would be

$$\frac{1}{2}\left(\langle i|H_3^{\text{l.c.}}|j,k\rangle - \langle j,k|H_3^{\text{l.c.}}|i\rangle\right) = \mu g_2(\Delta_i - \Delta_j - \Delta_k)C_{ijk}. \tag{26}$$

Secondly the matrix (24) does not have a definite symmetry. From the SFT point of view this would signify that the amplitude of splitting strings is different from joining. This does not necessarily mean that the gauge theory dilatation operator is non-Hermitian since the natural scalar product is nonzero only between the barred and non-barred sectors. We will return to this point in the discussion.

3.2. The $\mathcal{O}^J_{12,n} - \mathcal{T}^{J,r}_{12}$ sector

In this case the relevant formulas (see, e.g., (3.15) in [6]) are

$$M' = \begin{pmatrix} 1 & g_2 y \\ g_2 y & 1 \end{pmatrix}, \qquad M'' = \lambda' \begin{pmatrix} n^2 & g_2 y n^2 \\ g_2 y n^2 & 0 \end{pmatrix} \tag{27}$$

with

$$y = \frac{1}{\sqrt{J}}\left(\delta_{n,0} r - \frac{\sin^2(\pi n r)}{\pi^2 n^2}\right). \tag{28}$$

The dilatation matrix to order $O(g_2)$ is thus

$$\begin{pmatrix} \lambda' n^2 & 0 \\ 0 & 0 \end{pmatrix} + g_2 \begin{pmatrix} 0 & 0 \\ \lambda' y n^2 & 0 \end{pmatrix}. \tag{29}$$

Again we see that it is nonsymmetric and that here the sum of off-diagonal elements gives (minus) the SV matrix element (12).

Let us now assume that the cubic $O(g_s)$ SFT vertex is given by the above formulas (24) and (29). We will show that this is enough to reproduce the exact gauge-theoretic scaling dimension to order $O(g_2^2)$.

3.3. Scaling dimensions to order $O(g_2^2)$

The formulas for scaling dimension follow easily (as in [5,7]) from first order perturbation theory in the off-diagonal elements of the Hamiltonian (dilatation matrix), but keeping in mind the fact that the Hamiltonian is nonsymmetric. Indeed assuming that $D_{\alpha\beta} = \Delta_\alpha \delta_{\alpha\beta} + g_2 H^{(1)}_{\alpha\beta} + g_2^2 H^{(2)}_{\alpha\beta}$ we obtain

$$\Delta = \Delta_\alpha + g_2^2 \sum_\beta \frac{H^{(1)}_{\alpha\beta} H^{(1)}_{\beta\alpha}}{\Delta_\alpha - \Delta_\beta} + g_2^2 H^{(2)}_{\alpha\alpha}. \tag{30}$$

We assume that $H^{(2)}_{\alpha\alpha} = 0$ (no contact interactions in the single-string sector). We will now show that the full $O(g_2^2)$ result is obtained. It is interesting to compare with Section 5.2 in [5]. Now $\mathcal{T}^{J,r}_{12}$ does *not* contribute as the product of the off-diagonal elements in (29) vanishes. Only the operators $\mathcal{T}^{J,r}_{12,m}$ give a contribution. Since $\Delta_n - \Delta^r_m = \lambda'(n^2 - m^2/r^2)$ we have to calculate

$$g_2^2 \sum_{m,r} \frac{-\lambda' x^2 \frac{nm}{r^3}(m-nr)^2}{n^2 - \frac{m^2}{r^2}} = -\frac{g_2^2 \lambda'}{J\pi^4} \sum_{m,r} r^2(1-r)\sin^4(\pi n r)\frac{nm}{(m-nr)^2(n^2r^2-m^2)}. \tag{31}$$

We now use the formula

$$\sum_{m=-\infty}^{\infty} \frac{nm}{(m-nr)^2(n^2r^2-m^2)} = \frac{\pi}{4nr^2}\left(-n\pi r \csc^2(n\pi r) + \cot(n\pi r)\left(2n^2\pi^2 r^2 \csc^2(n\pi r) - 1\right)\right) \tag{32}$$

and replace $(1/J)\sum_r$ by an integral. The result is

$$\frac{g_2^2 \lambda'}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32\pi^2 n^2}\right) \tag{33}$$

in agreement with (14). We see that the full $O(g_2^2)$ result was obtained just from the cubic $O(g_2)$ interaction. The positive sign of the correction for $n = 1$ could only appear due to the fact that the matrix (24) is nonsymmetric. In comparison to the work of [6] the above result (33) was derived here only from a small subset of data. This is a strong argument in favour of a SFT interpretation—$O(g_2^2)$ elementary interactions (contact terms) in the single-string sector, which seem unlikely by comparison to the flat space SFT indeed do not appear here (by the above calculation we demonstrated that $H^{(2)}_{nn} = 0$). On the gauge theory side, if it were not for the SFT interpretation we would not have any reason to expect a vanishing $O(g_2^2)$ term in the single trace (single-string) sector.

However the main problem which remains is how to reconcile the asymmetric SFT vertex reconstructed here from the gauge theory calculations of [6] with the construction of light cone SFT in the pp-wave background.

4. Discussion

In this Letter we have reconstructed the order $O(g_2)$ matrix elements of the dilatation operator directly from gauge theory calculations. By the BMN operator-string correspondence this should give the 3-string $O(g_s)$ vertex of light cone SFT in the pp-wave background. We find a disagreement with the continuum SFT matrix elements of [18] even at order $O(g_s)$.

From this point of view we may return to the problem of the failure of SFT to reproduce the correct gauge theory scaling dimensions. Previously this was attributed to the possible existence of $O(g_s^2)$ contact terms. However from the flat space perspective such contact terms in the *single-string* sector are unlikely.

Here we show that there is a disagreement even at order $O(g_s)$, although a mild one. With the 'new' $O(g_s)$ matrix elements the full $O(g_2^2)$ anomalous dimensions can be reconstructed without any additional $O(g_2^2)$ contact terms. As was mentioned earlier we believe that this is an argument in favour of a SFT interpretation.

The deviation from the matrix elements of the SFT vertex constructed in [9] is not very large. The symmetric component coincides with the corrected SFT matrix elements of [18] (up to a sign). So perhaps there is room for reconciling these results with SFT.

A curious feature of the gauge theoretical dilatation matrix which we obtained is that it does not have any simple symmetry properties. Matrix elements which would correspond on the string theory side to 'splitting' and 'joining' of strings are different. From the point of view of string theory this asymmetry may not be unacceptable as, in contrast to flat space, the pp-wave background is not symmetric w.r.t. light cone time reversal ($x^+ \to -x^+$) since then the RR field strength changes sign. On the gauge theory side there is no obvious contradiction with Hermiticity because the natural scalar product is off-diagonal and is nonvanishing only for operators with *opposite* R charge. It would be interesting to see how it is possible to understand explicitly that lack of symmetry within the SFT framework.

A remaining open problem is to reproduce the dilatation matrix elements derived here from 'continuum' SFT. As this Letter was being written [12] appeared which gave a refined discrete string bit approach to the BMN-string correspondence. It would also be interesting to examine the interrelation with the framework of [23].

Note added

After this Letter was submitted the result (33) was reproduced in the discrete string bit formalism [27]. However there the mechanism of reproducing (33) was different and the agreement was reached using a specific form of $O(g_s^2)$ contact term in the discrete string bit formalism. Here (33) follows just from the $O(g_s)$ dilatation operator matrix elements. This might suggest that one could perhaps perform some further field redefinition in [27] and reproduce directly the gauge theory dilatation matrix elements (24) and (29). In that basis the $O(g_s^2)$ term should vanish in the single trace (string) sector.

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