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Linear Algebra and its Applications 428 (2008) 2339–2356
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Extremal L_p-norms of linear operators and self-similar functions ☆

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Received 22 December 2006; accepted 23 September 2007
Available online 14 November 2007
Submitted by M. Karow

Abstract

We prove that for any $p \in [1, +\infty]$ a finite irreducible family of linear operators possesses an extremal norm corresponding to the p-radius of these operators. As a corollary, we derive a criterion for the L_p-contractibility property of linear operators and estimate the asymptotic growth of orbits for any point. These results are applied to the study of functional difference equations with linear contractions of the argument (self-similarity equations). We obtain a sharp criterion for the existence and uniqueness of solutions in various functional spaces, compute the exponents of regularity, and estimate moduli of continuity. This, in particular, gives a geometric interpretation of the p-radius in terms of spectral radii of certain operators in the space $L_p[0, 1]$.

AMS classification: 52A21; 47A05; 39B22; 26A16

Keywords: Linear operators; Spectral radius; Extremal norms; Contractibility; Functional equations; Regularity

1. Introduction

The concept of fractal object and self-similarity is intensively studied in the literature and applied in various fields of mathematics. According to the general definition of Hutchinson [1], a fractal is a compact set which coincides with the union of its images under the action of several

☆ The research is supported by the RFBR grant No. 05-01-00066 and by the grant for Leading Scientific Schools No. 5813.2006.1.
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0024-3795/$ - see front matter
doi:10.1016/j.laa.2007.09.023

contraction operators. In particular, if those operators are affine, then it is referred to as affine fractal. Under certain assumptions on the operators, an affine fractal is a continuous curve, i.e., it can be parametrized by a continuous function $v : [0, 1] \to \mathbb{R}^d$ that satisfies a certain functional equation. This point of view on affine fractals originated in [2,3] and has been developed in great detail. In particular, the classical Koch and De Rham curves [4] possess this property, continuous refinable functions in wavelets theory [5,6] and in the study of subdivision algorithms [7,8] can also be interpreted as fractal curves. In some special cases the notion of parametrized fractal curves was generalized to L_p-functions [9,10,11,12] and to self-similar measures [13,14].

In this paper, we introduce the concept of general L_p-fractal curves which are solutions of self-similarity equations in the space $L_p[0, 1]$. As we shall see, this generalization is quite reasonable. First, there is a sharp criterion of L_p-solvability (given by Theorem 2). Moreover, an L_p-solution of a self-similarity equation is unique, whenever it exists, is stable and can be computed by iterative approximations (Remark 7). Any finite family of affine operators and any partition of the segment [0, 1] produce an equation of self-similarity, whose summable solution (if it exists) is a fractal curve. This definition covers many special cases of known self-similar functions, including solutions of refinement equations, and enables us to analyze them in a general framework. In Section 4, we establish a criterion of continuity of the fractal curves, compute their regularity, and estimate the moduli of continuity.

The results on general fractal curves are formulated in terms of the p-radii ρ_p of families of linear operators. In Sections 2 and 3, we derive several auxiliary results concerning the p-radii. First we define an L_p-extremal norm for linear operators and prove its existence in Theorem 1. This theorem extends a well-known result of Barabanov [15] from the case $p = \infty$ to all $p \in [1, +\infty]$. Then we establish the L_p-contractibility property for any family of linear operators (Proposition 1) and apply this result in Section 3 to estimate the growth of the L_p-averaged norm of the orbit of an arbitrary point under the action of that family of operators. This will allow us to estimate the rate of convergence of the iterative approximation method (Theorem 3) and to make a very detailed analysis of the regularity of fractal curves in Section 5. This approach is also applicable to analyze the local regularity of fractal curves (Remark 12).

2. Extremal norms and L_p-contractibility

In this section, we introduce some notation and prove the main theorems concerning extremal norms and the L_p-contractibility. Throughout the paper we deal with finite families $\mathscr{B} = \{B_1, \ldots, B_m\}$, $m \geqslant 1$ of linear operators acting in the d-dimensional Euclidean space $\mathbb{R}^d$. We consider an arbitrary norm $|\cdot|$ in $\mathbb{R}^d$ and the corresponding operator norm $\|B\| = \sup_{|x|=1} |Bx|$. For a given $k \in \mathbb{N}$ and for any sequence $\sigma \in \{1, \ldots, m\}^k$ we write Π_σ for the product $B_{\sigma(1)} \cdots B_{\sigma(k)}$. Also for any $p \in [1, +\infty)$ denote by $\mathscr{F}_k(p) = \mathscr{F}_k(p, \mathscr{B})$ the value $\left[m^{-k} \sum_\sigma \|\Pi_\sigma\|^p\right]^{1/p}$, which is the L_p-averaged norm of all products of the operators B_j of length k; $\mathscr{F}_k(\infty) = \max_{\sigma \in \{1,\ldots,m\}^k} \|\Pi_\sigma\|$.

Definition 1. For given $p \in [1, +\infty]$ the *p-radius of linear operators* of the family $\mathscr{B}$ is the value $\rho_p = \rho_p(\mathscr{B}) = \lim_{k\to\infty}[\mathscr{F}_k(p, \mathscr{B})]^{1/k}$.

This limit exists for any operators and does not depend on the norm in $\mathbb{R}^d$ [16]. If $m = 1$ or all the operators B_j are equal to some operator B, then by well-known Gelfand's formula $\rho_p(\mathscr{B})$ equals to the (usual) spectral radius $\rho(B)$, which is the largest modulus of eigenvalues. The p-

radius of given operators is a non-decreasing function in p. Moreover, the function $f(x) = \rho_{1/x}$ is concave in x on the segment [0, 1] [17].

Remark 1. The value ρ_∞ is often called the *joint spectral radius*. The joint spectral radius (JSR) first appeared in a short paper of Rota and Strang [18] in 1960. For almost 30 years this object was nearly forgotten. It was not before late 1980s that JSR emerged simultaneously and independently in three different topics: in the theory of linear switched systems [15,19], in subdivision algorithms for approximation and curve design [3,8] and in the study of wavelets and of refinement equations [5]. The paper of Daubechies and Lagarias had a broad impact and aroused a real interest to JSR. Now this notion has a lot of applications. The computational issue of JSR has also been studied in many works. It was shown by Blondel and Tsitsiklis [20] that there is no algorithm for approximate calculation of JSR with a prescribed relative accuracy $\varepsilon > 0$ that is polynomial with respect to both ε^{-1} and the dimension d. Nevertheless, there are algorithms that are polynomial either in ε^{-1} or in d. These algorithms originated with Protasov in [21,22]. The algorithm presented in [21] is polynomial in ε^{-1}, it is based on approximation of the invariant bodies by polytopes; the algorithm from [22] is polynomial in the dimension d, it is based on the Kronecker lifting. The second algorithm was obtained later and independently by Zhou [17] in 1998 and by Blondel and Nesterov [23] in 2005. Other approaches for the computation and estimation of JSR were derived by Gripenberg [24], Maesumi [25], Blondel et al. [26], etc.

The notion of JSR was extended to the p-radius, first for $p = 1$ by Y. Wang [9], and then for all $p \in [1, +\infty]$ by Jia [10] and independently by Lau and J. Wang [11]. A polynomial in the dimension d algorithm for approximate computation of ρ_p with a given accuracy was elaborated by Protasov [22]. For even integers p this problem can be solved easier (see Remark 8 for more detail).

One of the main tools in the study of JSR is the concept of extremal norm developed by Barabanov [15], who proved the existence of such a norm for the joint spectral radius ρ_∞. In this section we generalize this notion for the p-radii ρ_p for all $p < \infty$. In Theorem 1, we show that for any $p \in [1, +\infty]$ a finite irreducible family $\mathscr{B}$ of linear operators possesses an extremal norm corresponding to the p-radius $\rho_p(\mathscr{B})$. Then we derive a criterion of L_p-contractibility in terms of the p-radius (Proposition 1). In Section 3, we apply the p-radius and the extremal norms to estimate the asymptotic growth of orbits for any point $u \in \mathbb{R}^d$ under the action of the family $\mathscr{B}$. In Sections 4 and 5, these results are used to analyze the solutions of a wide class of functional equations. This gives a geometric interpretation of the p-radius in terms of spectral radii of certain linear operators in $L_p[0, 1]$.

For any point $u \in \mathbb{R}^d$ and any number $k \geqslant 1$ the set $\{\Pi_\sigma u \mid \sigma \in \{1, \ldots, m\}^k\}$ containing m^k points is the *orbit* of u. For any $p \in [1, +\infty]$ we denote

$$\mathscr{F}_k(p, u) = \mathscr{F}_k(p, \mathscr{B}, u) = \left[m^{-k} \sum_\sigma |\Pi_\sigma u|^p \right]^{1/p}$$

with the standard modifications for $p = \infty$. Thus, $\mathscr{F}_k(p, u)$ is the averaged (in L_p-norm) length of all the m^k elements of the orbit. In particular, $\mathscr{F}_k(1, u)$ is the arithmetic mean of lengths of the vectors $\Pi_\sigma u$, $\mathscr{F}_k(2, u)$ is the quadratic mean, and $\mathscr{F}_k(\infty, u) = \max_{\sigma \in \{1,\ldots,m\}^k} |\Pi_\sigma u|$. For $k = 1$ we use the short notation $\mathscr{F}_1 = \mathscr{F}$. Thus

$$\mathscr{F}(|\cdot|, p, \mathscr{B}, u) = \left(\frac{1}{m} \sum_{j=1}^m |B_j u|^p \right)^{1/p}, \quad \mathscr{F}(|\cdot|, \infty, \mathscr{B}, u) = \max_j |B_j u|.$$

In this section, we assume the family $\mathscr{B}$ and the parameter $p \in [1, +\infty]$ to be fixed and use the short notation $\mathscr{F}(|\cdot|, u) = \mathscr{F}(|\cdot|, p, \mathscr{B}, u)$.

Definition 2. A norm $|\cdot|$ in $\mathbb{R}^d$ is called extremal for a given family $\mathscr{B}$ and for a given $p \in [1, +\infty]$ if there is $\lambda \geqslant 0$ such that $\mathscr{F}(|\cdot|, u) = \lambda |u|$ for any $u \in \mathbb{R}^d$.

We shall see in Theorem 1 that any irreducible family of operators possesses an extremal norm. A family is called *irreducible* if its operators do not have a common invariant nontrivial (different from $\{0\}$ and $\mathbb{R}^d$) real linear subspace.

In the proof of Theorem 1 below we use some elementary facts from convex analysis. Let $\mathscr{S}$ be the set of convex bodies (compact sets with a nonempty interior) in $\mathbb{R}^d$ centrally symmetric with respect to the origin. For $M \in \mathscr{S}$, $u \in \mathbb{R}^d$ we denote $(u, M) = \sup_{x \in M}(u, x)$. The function of support of the body M is $\varphi_M(u) = (u, M)$; $M^* = \{u \in \mathbb{R}^d, (u, M) \leqslant 1\}$ is the polar of M. This is well known that for any $M \in \mathscr{S}$ one has $M^* \in \mathscr{S}$ and $(M^*)^* = M$.

We write $|x|_M$ for the Minkowski norm in $\mathbb{R}^d$ with a given unit ball $M \in \mathscr{S}$. This norm is defined as $|x|_M = \inf\{\alpha > 0 | \alpha^{-1}x \in M\}$. For any $M \in \mathscr{S}$, $u \in \mathbb{R}^d$ we have $|u|_{M^*} = (u, M)$ (see, for instance, [27]).

Theorem 1. *An irreducible family of operators $\mathscr{B}$ for every $p \in [1, +\infty]$ possesses an extremal norm. For any such a norm the factor λ is positive and equals to $\rho_p(\mathscr{B})$.*

Proof. Suppose that $|\cdot|$ is an extremal norm. Since for any $u \in \mathbb{R}^d$, $|u| = 1$ and for any $\sigma \in \{1, \ldots, m\}^k$ one has $\|\Pi_\sigma\| \geqslant |\Pi_\sigma u|$, it follows that $\mathscr{F}_k(p) \geqslant \mathscr{F}_k(p, u) = \lambda^k$ for any k. Therefore, $\rho_p \geqslant \lambda$. Further, take an arbitrary basis $\{e_i\}_{i=1}^d$ of $\mathbb{R}^d$ such that $|e_j| = 1$ for all j, for each $\sigma \in \{1, \ldots, m\}^k$ take a vector $x_\sigma \in \mathbb{R}^d$ such that $|x_\sigma| = 1$, $|\Pi_\sigma x_\sigma| = \|\Pi_\sigma\|$ and write the expansion of x_σ in that basis: $x_\sigma = \sum_{i=1}^d \alpha_{\sigma,i} e_i$.

Denote also $C = \max\{|\alpha_i| \,|\, \exists x = \sum_{j=1}^d \alpha_j e_j, |x| = 1, i = 1, \ldots, d\}$. Using the triangle inequality for the l_p-norm, $p < \infty$, we get

$$\mathscr{F}_k(p) = \left[m^{-k} \sum_\sigma \left|\Pi_\sigma \left(\sum_i \alpha_{\sigma,i} e_i\right)\right|^p\right]^{1/p}$$
$$\leqslant \sum_i \left[m^{-k} \sum_\sigma \left|\Pi_\sigma(\alpha_{\sigma,i} e_i)\right|^p\right]^{1/p} \leqslant \sum_i C \left[m^{-k} \sum_\sigma |\Pi_\sigma e_i|^p\right]^{1/p}.$$

Therefore, $\mathscr{F}_k(p) \leqslant C \sum_i \mathscr{F}_k(p, e_i)$. The limit passage extends this for $p = \infty$. Since $\mathscr{F}_k(p, e_i) = \lambda^k$, we see that $\mathscr{F}_k(p) \leqslant dC\lambda^k$ for any k, hence $\rho_p \leqslant \lambda$. Thus, $\lambda = \rho_p$ for any extremal norm.

To prove the existence we use Theorem 2 of [22] which states that for any irreducible family of operators $A_1, \ldots, A_m$ and for any p there exists an *invariant body* $M \in \mathscr{S}$ such that

$$\overset{p}{\bigoplus} \sum_{j=1}^m A_j M = m^{1/p} \lambda M \tag{1}$$

for some $\lambda \geqslant 0$. The symbol $\overset{p}{\oplus}$ denotes the Firey summation: for $M_1, \ldots, M_m \in \mathscr{S}$ the sum $M = \overset{p}{\oplus} \sum_j M_j$ is a convex body defined by its function of support as follows: $\varphi_M(u) =$

$\left[\sum_j \varphi_{M_j}^p(u)\right]^{1/p}$, $u \in \mathbb{R}^d$ (with the standard modification for $p = \infty$). In particular, $\overset{1}{\oplus}\sum_j M_j$ is the Minkowski sum and $\overset{\infty}{\oplus}\sum_j M_j$ is the convex hull.

The family of adjoint operators $B_1^*, \ldots, B_m^*$ is irreducible as well, therefore it possesses an invariant body $M \in \mathscr{S}$. This means that for any $u \in \mathbb{R}^d$ we have

$$(u, m^{1/p}\lambda M) = \left[\sum_j (u, B_j^* M)^p\right]^{1/p} = \left[\sum_j (B_j u, M)^p\right]^{1/p}. \tag{2}$$

Dividing (2) by $m^{1/p}$ and taking into account that $(B_j u, M) = |B_j u|_{M^*}$, we obtain $\lambda|u|_{M^*} = \mathscr{F}(|\cdot|_{M^*}, u)$. Thus, the norm with the unit ball M^* is an extremal one. □

Remark 2. For $p = \infty$ the existence of extremal norms originated with Barabanov [15]. Theorem 1 extends this result to all p. From relation (2) we see that the extremal norms are, in a sense, dual to invariant bodies, whose existence were proved by Protasov in [21] (in case $p = \infty$) and in [22] (for all p). The polar to an invariant body is the unit ball of the extremal norm corresponding to the adjoint operators. For the case $p = \infty$ this duality phenomenon was observed and analyzed in [28]. The results of [21,22] can be implemented for computing and approximating the extremal norms.

Remark 3. Any invariant norm can be interpreted as a fixed point of a certain homogeneous continuous operator on the set of all norms in $\mathbb{R}^d$. First, let us note that if a family of linear operators $\mathscr{B}$ is irreducible, then for any norm $|\cdot|$ the functional $u \mapsto \mathscr{F}(|\cdot|, u)$ is also a norm in $\mathbb{R}^d$. We write $\mathscr{N}$ for the set of all norms in $\mathbb{R}^d$. Clearly, $\mathscr{N}$ is an open pointed convex cone. Thus, we have a map $P\colon |\cdot| \to \mathscr{F}(|\cdot|, u)$ of the cone $\mathscr{N}$ into itself. For any norm $|\cdot|$ from $\mathscr{N}$ we have $P(|\cdot|) \in \mathscr{N}$, where $P(|\cdot|)[u] = \mathscr{F}(|\cdot|, u)$ for all $u \in \mathbb{R}^d$.

By Theorem 1, the nonlinear operator P always has an "eigenvector" $|\cdot|$, for which $P(|\cdot|) = \mathscr{F}(|\cdot|, u) = \lambda|\cdot|$. Such an eigenvector may not be unique. For instance, a rotation of the plane by the angle $\pi/2$ has many extremal norms (for example, the L_p-norms in $\mathbb{R}^2$ for all $p \in [1, +\infty]$). Nevertheless, Theorem 1 guarantees that all possible eigenvectors have the same eigenvalue $\lambda = \rho_p(\mathscr{B})$. We see that like the usual spectral radius the p-radius can also be viewed as an eigenvalue, although of an infinite-dimensional nonlinear operator.

Everything said above concerns irreducible families. Operators with common invariant subspaces may fail to have extremal norms. The corresponding examples are well-known and elementary. The next proposition, however, guarantees the existence of "almost extremal" norms in this case. This leads to a weaker geometric characteristic of the p-radius: the averaged L_p-contractibility property which holds for any families of operators.

Proposition 1. *For any p the following properties of a family of operators $\mathscr{B}$ are equivalent*:

(a) $\rho_p(\mathscr{B}) < 1$;
(b) *there is a norm $|\cdot|$ in $\mathbb{R}^d$ and $\gamma < 1$ such that* $\mathscr{F}(|\cdot|, u) \leqslant \gamma|u|, \quad u \in \mathbb{R}^d$.

For $p = \infty$ this fact is well-known. The inequality $\rho_\infty < 1$ is equivalent to the simultaneous contractibility of the operators in a certain norm [16]. To prove Proposition 1 we need to make some observations. First of all, from (b) it follows that $\mathscr{F}_k(p, \mathscr{B}, u) \leqslant \gamma^k |u|$ for any k, which implies $\rho_p \leqslant \gamma < 1$. It remains to derive the implication (a) $\Rightarrow$ (b). For irreducible families this follows from Theorem 1. Suppose now that the family $\mathscr{B}$ is reducible; then the matrices B_i in a suitable basis have a block lower triangular form:

$$B_i = \begin{pmatrix} B_i^1 & 0 & 0 & \cdots & 0 \\ * & B_i^2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & \cdots & \cdots & * & B_i^l \end{pmatrix}, \quad i = 1, \ldots, m, \tag{3}$$

where each family of matrices $\mathscr{B}^j = \{B_1^j, \ldots, B_m^j\}$ is irreducible. We denote by $V_1, \ldots, V_l$ the subspaces of $\mathbb{R}^d$ corresponding to factorization (3). Any vector $x \in \mathbb{R}^d$ is represented in a unique way as a sum $x = \sum_{j=1}^l x_j$, where $x_j \in V_j$ for all j. We denote $\rho_{p,j} = \rho_p(\mathscr{B}^j)$.

Lemma 1. *For any family $\mathscr{B}$ and for any $p \in [1, +\infty]$ one has $\rho_p = \max_{j=1,\ldots,l} \rho_{p,j}$.*

The proof for the case $p = \infty$ can be found in [6], and for all other p in [22].

Proof of Proposition 1. Consider factorization (3) and the expansion $u = \sum_{j=1}^l u_j$ of any vector $u \in \mathbb{R}^d$ along the system of subspaces $\{V_j\}_{j=1}^l$. For each j the family of operators $\mathscr{B}^j$ acing in V_j is irreducible, hence Theorem 1 provides an extremal norm $|\cdot|$ on V_j, for which $\mathscr{F}(p, \mathscr{B}^j, u_j) = \rho_{p,j}|u_j|, u_j \in V_j$. For any $\alpha > 0$ we introduce a norm $|u|_\alpha = \sum_{j=1}^l \alpha^{j-1}|u_j|$ in $\mathbb{R}^d$ and the corresponding operator norm $\|\cdot\|_\alpha$. For any k we have

$$B_k u = \sum_{j=1}^l B_k^j u_j + A_k u,$$

where A_k is a linear operator in $\mathbb{R}^d$ and $\|A_k\|_\alpha \to 0$ as $\alpha \to 0$. Therefore

$$\mathscr{F}(|\cdot|_\alpha, p, \mathscr{B}, u) \leqslant \sum_{j=1}^l \mathscr{F}(|\cdot|_\alpha, p, \mathscr{B}^j, u_j) + C(\alpha)|u|_\alpha$$
$$= \sum_{j=1}^l \rho_{p,j}\alpha^{j-1}|u_j| + C(\alpha)|u|_\alpha,$$

where $C(\alpha) \to 0$ as $\alpha \to 0$. Since $\rho_{p,j} \leqslant \rho_p$ we see that the last expression does not exceed

$$\rho_p \sum_{j=1}^l \alpha^{j-1}|u_j| + C(\alpha)|u|_\alpha = (\rho_p + C(\alpha))|u|_\alpha.$$

If $\rho_p < 1$, then choosing α so small that $\rho_p + C(\alpha) < 1$, we conclude the proof. $\square$

Remark 4. As it was pointed out by the anonymous Referee, Proposition 1 is not actually new, it was proved in 2004 in the work [29, Proposition 2.17]. We are grateful to the Referee for this remark and recognize the priority of the work [29]. This is interesting to compare the two proofs, which appear to be totally different. The authors of [29] constructed an "almost extremal" norm as a limit of certain sequence of norms. We obtain the same norm by factorizing the operators $B_1, \ldots, B_m$ to the form (3) and by taking a weighted sum of the extremal norms of the factors. One can apply an algorithm of approximation of invariant bodies elaborated in [22] and get approximations for the extremal norms of the all factors in (3). This yields an approximation of the "almost extremal" norm for the initial family $\mathscr{B}$. In this sense our proof has one advantage: it gives a way to approximate the desired norm numerically.

3. Asymptotics of orbits of linear operators

In Sections 4 and 5, we study solutions of self-similarity equations. Before this we need to do some preliminary job and to analyze the asymptotic growth of the value $\mathscr{F}_k(p, \mathscr{B}, u)$ as $k \to \infty$. In this section we present several results on this direction, that are, in our opinion, of some independent interest. Assume we have an arbitrary family of linear operators $\mathscr{B} = \{B_1, \ldots, B_m\}$ in $\mathbb{R}^d$. How to estimate the L_p-averaged norm of the images $\{B_{\sigma(1)} \cdots B_{\sigma(k)}u, \sigma \in \{1, \ldots, m\}^k\}$ for an arbitrary $u \in \mathbb{R}^d$? In other words, we deal with the averaged norms of the orbits $\mathscr{F}_k(p, \mathscr{B}, u)$. Theorem 1 provides an immediate solution in case of irreducible families of operators. Since all norms in $\mathbb{R}^d$ are equivalent, we have

Corollary 1. *For an irreducible family $\mathscr{B}$ and for every $p \in [1, +\infty]$ there are constants $c_1, c_2 > 0$ such that for any $u \in \mathbb{R}^d$ we have $c_1(\rho_p)^k|u| \leqslant \mathscr{F}_k(p, u) \leqslant c_2(\rho_p)^k|u|, k \in \mathbb{N}$.*

Thus, for an irreducible family the value $\mathscr{F}_k(p, u)$ is asymptotically equivalent to $(\rho_p)^k$. This means, in particular, that for all vectors $u \neq 0$ this value has the same rate of growth as $k \to \infty$. For general operators this is, a priory, not true: $\mathscr{F}_k(p, u)$ may grow faster than $(\rho_p)^k$, moreover, it may have different rates of growth for different vectors u. Corresponding examples are elementary already in case $m = 1$, $\mathscr{B} = \{B\}$. If u is an eigenvector of the operator B associated to an eigenvalue λ, then $\mathscr{F}_k(p, u) = |B^k u| \asymp k^{r-1}\lambda^k$, where r is the multiplicity of λ. So, the growth may be different for different u and it may exceed λ^k.

To attack the problem of asymptotics of $\mathscr{F}_k(p, u)$ for general families of operators $\mathscr{B}$ we use factorization (3). According to Lemma 1, $\rho_p = \max_{j=1,\ldots,l} \rho_{p,j}$. The total number of subscripts $j \in \{1, \ldots, l\}$, for which this maximum is attained, will be denoted by s and called *valency* of the family $\mathscr{B}$. The valency depends on p. However, neither the values $\rho_{p,j}$ nor s depend on the basis chosen for factorization (3).

For an arbitrary $u \in \mathbb{R}^d$ we write V_u for the smallest by inclusion common linear invariant subspace of the family $\mathscr{B}$ containing u.

Proposition 2. *For any family $\mathscr{B}$ of linear operators, for every $p \in [1, +\infty]$ and $u \in \mathbb{R}^d$ there are positive constants c_1, c_2 such that*

$$c_1|u| \leqslant \mathscr{F}_k(p, u) \leqslant c_2 k^{s-1}(\rho_p)^k|u|, \quad k \geqslant d, \tag{4}$$

where ρ_p is the p-radius of the family $\mathscr{B}$ restricted to V_u, s is its valency on V_u. Moreover, there is a uniform constant c_2 for all vectors of the subspace V_u.

Thus, the p-radius ρ_p is the exponent of growth of the L_p-averaged norm of the images $B_{\sigma(1)} \cdots B_{\sigma(k)}u$. The value $\mathscr{F}_k(p, u)$ for all k is asymptotically between $(\rho_p)^k$ and $k^{s-1}(\rho_p)^k$. The parameters ρ_p and s depends only on the subspace V_u.

Corollary 2. *For any family $\mathscr{B}$ and for any $p \in [1, +\infty]$ there are $c_1, c_2 > 0$ such that*

$$c_1(\rho_p)^k \leqslant \mathscr{F}_k(p) \leqslant c_2 k^{s-1}(\rho_p)^k, \quad k \geqslant d, \tag{5}$$

where $\rho_p = \rho_p(\mathscr{B})$ and s is the valency of $\mathscr{B}$.

Proof of Proposition 2 and of Corollary 2. It can be assumed that $V_u = \mathbb{R}^d$, otherwise we consider restrictions of the operators to V_u. We start with the lower bound in (4). Since u does not belong to common invariant subspaces of $\mathscr{B}$, it follows that there are products $\Pi_q, q = 1, \ldots, d$ of the operators of the family $\mathscr{B}$, each product Π_q is of length $n_q \leqslant d$, such that the vectors $\{\Pi_q u\}_{q=1}^d$ form a basis of $\mathbb{R}^d$. Now consider factorization (3) and the corresponding subspaces $\{V_j\}_{j=1}^l$. Since the family $\mathscr{B}^j$ is irreducible on V_j we see that it possesses an extremal norm $|\cdot|$ on V_j. We introduce the following norm in $\mathbb{R}^d$: $|x| = \sum_{j=1}^l |x_j|, x_j \in V_j$. For any $x \in \mathbb{R}^d$ and for any i, j it follows that $|B_i x| \geqslant |B_i^j x_j| = \rho_{p,j}|x_j|$. Choose j so that $\rho_{p,j} = \rho_p$; then for any $y \in V_j$ we have $\mathscr{F}_k(p, \mathscr{B}, y) \geqslant \mathscr{F}_k(p, \mathscr{B}^j, y) = |y|(\rho_p)^k, k \in \mathbb{N}$. Now we write the vector y in the basis $\{\Pi_q u\}$: $y = \sum_{q=1}^d \lambda_q \Pi_q u$. Using the inequality $\mathscr{F}_{n+k}(p, \mathscr{B}, x) \leqslant \mathscr{F}_n(p, \mathscr{B})\mathscr{F}_k(p, \mathscr{B}, x)$, we obtain

$$\begin{aligned} |y|(\rho_p)^k \leqslant \mathscr{F}_k(p, \mathscr{B}, y) &= \mathscr{F}_k\left(p, \mathscr{B}, \sum_q \lambda_q \Pi_q u\right) \leqslant \sum_q |\lambda_q| \mathscr{F}_k\left(p, \mathscr{B}, \Pi_q u\right) \\ &\leqslant \sum_q m^{n_q/p}|\lambda_q|\mathscr{F}_{k+n_q}(p, \mathscr{B}, u) \\ &\leqslant \left[\sum_q m^{n_q/p}|\lambda_q|\mathscr{F}_{n_q}\left(p, \mathscr{B}\right)\right]\mathscr{F}_k(p, \mathscr{B}, u). \end{aligned}$$

It remains to set $c_1 = |y| \cdot \left[\sum_q m^{n_q/p}|\lambda_q|\mathscr{F}_{n_q}(p, \mathscr{B})\right]^{-1}$, which proves the lower bound in (4), and hence in (5) as well. Now consider the upper bound. Each operator B_k can be written as

$$(B_k x)_i = B_k^i x_i + \sum_{j \leqslant i-1} D_{k,j}^i x_j,$$

where $D_{k,j}^i : V_j \to V_i$ are linear operators. Whence $|(B_k x)_i| \leqslant |B_k^i x_i| + C \sum_{j \leqslant i-1} |x_j|$, where we set $C = \max_{i,j,k} \|D_{k,j}^i\|$. Applying the triangle inequality in the l_p-norm, we obtain

$$\mathscr{F}(p, \mathscr{B}, x)_i \leqslant \rho_{p,i}|x_i| + C \sum_{j \leqslant i-1} |x_j|, \quad i = 1, \ldots, m, \tag{6}$$

where $\mathscr{F}(p, \mathscr{B}, x)_i = \left[m^{-1} \sum_{k=1}^m |(B_k x)_i|^p\right]^{1/p}$. Obviously, $\mathscr{F}(p, \mathscr{B}, x) \leqslant \sum_{i=1}^l \mathscr{F}(p, \mathscr{B}, x)_i$. Let us now denote $z_k = (\mathscr{F}_k(p, \mathscr{B}, u)_1, \ldots, \mathscr{F}_k(p, \mathscr{B}, u)_l)^{\mathrm{T}} \in \mathbb{R}^l$, $z_0 = (|u_1|, \ldots, |u_l|)^{\mathrm{T}}$. Let also P be a lower triangular $l \times l$-matrix with the diagonal entries $\rho_{p,1}, \ldots, \rho_{p,l}$ and with all entries under the diagonal equal to C. Substituting z_{k-1} for x in (6) we obtain $z_k \leqslant P z_{k-1}$ and hence $z_k \leqslant P^k z_0$, where $a \leqslant b$ means that $a_i \leqslant b_i$ for all i. Since the largest eigenvalue of P equals to $\max_i \rho_{p,i} = \rho_p$ and its multiplicity is s it follows that for $k \geqslant d$ the norm of the vector $P^k z_0$ does not exceed $C_0 k^{s-1}(\rho_p)^k|z_0|$, where C_0 is a constant. Hence the sum of entries of the vector z_k is at most $c_2 k^{s-1}(\rho_p)^k|u|$ for some $c_2 > 0$. Thus

$$\mathscr{F}(p, \mathscr{B}, u) \leqslant \sum_{i=1}^{m} \mathscr{F}(p, \mathscr{B}, u)_i \leqslant c_2 k^{s-1} (\rho_p)^k |u|,$$

which proves the upper bound of (4). The upper bound of (5) is now derived in the same way as in the proof of Theorem 1. □

For any $x \in \mathbb{R}^d$ we set $C(x)$ to be the largest number such that $\mathscr{F}_k(p, x) \geqslant C(x)(\rho_p(\mathscr{B}))^k$ for all $k \geqslant d$. Proposition 2 implies the following:

Corollary 3. *The function $C(x)$ is upper semicontinuous and positive, whenever x does not belong to common invariant subspaces of the family $\mathscr{B}$.*

The condition $k \geqslant d$ imposed in (4) and in (5) is necessary only for the upper bound in the case $\rho_p = 0$, when these inequalities may fail for $k < d$. This exceptional case, however, can be easily recognized due to the following:

Corollary 4. *The p-radius of a family $\mathscr{B}$ is zero if and only if all products of these operators of length d vanish. In this case all the blocks in (3) are one-dimensional and all B_k^j are zeros.*

Proof. Assume first that the family $\mathscr{B}$ is irreducible. Corollary 1 for $k = 1$ yields that $\rho_p = 0$ precisely when $\mathscr{F}_1(p, u) = 0$ for all $u \in \mathbb{R}^d$, i.e., when all the operators B_j are zeros. If $\mathscr{B}$ is reducible, then we consider factorization (3). Applying Lemma 1 we obtain: $\rho_p = 0$ if and only if $\rho_{p,j} = 0$ for each j. Therefore, all the blocks B_j^r are zeros, consequently all products of the operators $B_1, \ldots, B_m$ of length l (and hence, of length d) vanish. □

4. Self-similar functions in L_p

In this section, we apply the p-radius and extremal norms to the problem of solvability of a wide class of functional equations.

For an arbitrary $p \in [1, +\infty]$ we consider the space $L_p[0, 1]$ of vector-functions from the segment $[0, 1]$ to $\mathbb{R}^d$ with the norm $\|v\|_p = \left(\int_0^1 |v(t)|^p \mathrm{d}t\right)^{1/p}$ and $\|v\|_\infty = \text{ess sup}_{[0,1]} |v(t)|$, where $|\cdot|$ is a given norm in $\mathbb{R}^d$. We denote the Lebesgue measure of a set $H \subset \mathbb{R}$ by $|H|$. Let us have a partition of the segment $[0, 1]$ with nodes $0 = b_0 < \cdots < b_m = 1$. We use the notation $\Delta_j = [b_{j-1}, b_j]$ and $r_j = |\Delta_j| = b_j - b_{j-1}$. This partition will be referred to as $\{\Delta_j\}_{j=1}^m$. Let $g_k \colon [0, 1] \to \Delta_k$ be the affine function that maps the segment $[0, 1]$ to Δ_j. So

$$g_k(t) = tb_k + (1 - t)b_{k-1}, \quad k = 1, \ldots, m. \tag{7}$$

Suppose we are given a family of affine operators $\widetilde{\mathscr{A}} = \{\widetilde{A}_1, \ldots, \widetilde{A}_m\}$ acting in $\mathbb{R}^d$. Let $\mathscr{A} = \{A_1, \ldots, A_m\}$ be the family of the associated linear operators in $\mathbb{R}^d$. In the sequel we denote by A the linear part of an affine operator $\widetilde{A}$.

Definition 3. A fractal curve (a self-similar function) of a family of affine operators $\widetilde{\mathscr{A}}$ corresponding to a given partition $\{\Delta_j\}_{j=1}^m$ is a summable function $v \colon [0, 1] \to \mathbb{R}^d$ satisfying the equation

$$v(t) = \widetilde{A}_k v(g_k^{-1}(t)), \qquad t \in \Delta_k, \quad k = 1, \ldots, m \tag{8}$$

almost everywhere on $[0, 1]$.

A fractal curve is a fixed point of the affine *operator of self-similarity* $\widetilde{\mathbf{A}}$ acting in $L_p[0, 1]$ by the formula $[\widetilde{\mathbf{A}} f](t) = \widetilde{A}_k f(g_k^{-1}(t))$, $t \in \Delta_k$. By $\mathbf{A}$ we denote the linear part of this operator. In Theorem 3, we show that the spectral radius $\rho(\mathbf{A})$ of this operator equals to the p-radius ρ_p of the family $\mathscr{A}$.

Functional self-similarity equations of the type (8) have been intensively studied in the literature, they found many applications in the image processing, wavelets, approximation theory, mathematical physics, ergodic theory, probability, number theory etc. A detailed review of applications of fractal curves would lead us too far from the subject of this paper. See, for example, the works [2]–[14], [30]–[34] and references therein. We mention only that Definition 3 covers most of classical fractal curves, such as the Cantor singular function (three operators in $\mathbb{R}^1$), the Koch and the de Rham curves (two operators in $\mathbb{R}^2$), etc.

Most of fractal curves analyzed in the literature concern two special cases of Definition 3:

(1) arbitrary partition $\{\Delta_j\}$ of the segment [0, 1] and arbitrary operators in $\mathbb{R}^1$, or some special operators in $\mathbb{R}^2$;
(2) uniform partition, when $r_1 = \cdots = r_m = \frac{1}{m}$, and special operators in $\mathbb{R}^d$ (for instance, the block Toeplitz operators in the study of refinement equations and wavelets, etc.) The results obtained for this case concern mostly continuous solutions. Refinement equations will be discussed in more detail in Remark 6.

In Definition 3, we attempt to cover all these cases and to put them in a general framework. As we shall see, this approach is quite reasonable. First of all, there is an existence and uniqueness theorem for the solutions in the spaces $L_p[0, 1]$ and $C[0, 1]$ (Theorems 2 and 4). The solutions continuously depend on the initial data (Remark 7). Their exponents of regularity are explicitly computed in terms of the corresponding p-radii, their moduli of continuity can be estimated and their differentiability and smoothness can be analyzed by a general scheme. Finally, any affine fractal can be parametrized by a solution of Eq. (8) (Remark 9).

We formulate the main theorem under the assumption that the family $\widetilde{\mathscr{A}}$ is irreducible, i.e., these operators do not have common affine invariant real subspaces, different from the whole $\mathbb{R}^d$. As we shall see in Remark 5, this assumption is made without loss of generality. Note that the irreducibility of affine operators does not imply the irreducibility of their linear parts. To prove the theorem we need one simple auxiliary result and some further notation.

Lemma 2. *For any* $f \in L_p[0, 1]$, $p < \infty$ *and any partition* $\Delta = \{\Delta_j\}_{j=1}^N$ *of the segment* $[0, 1]$ *let* $S(f, \Delta)$ *be a step (piecewise constant) function that on each segment* Δ_j *equals to* $\frac{1}{|\Delta_j|}\int_{\Delta_j} f(t)\mathrm{d}t$. *Then* $\|f - S(f, \Delta)\|_p \to 0$ *as* $\mathrm{diam}(\Delta) \to 0$, *where* $\mathrm{diam}(\Delta) = \max_j |\Delta_j|$.

The proof is quite elementary and we leave it to the reader. For any $\sigma \in \{1, \dots, m\}^k$ we denote the corresponding segment $\Delta_\sigma = g_{\sigma(1)} \circ \cdots \circ g_{\sigma(k)}[0, 1]$, where the affine functions $g_j(t)$ are defined in (7). Thus, $|\Delta_\sigma| = r_{\sigma(1)} \cdots r_{\sigma(k)}$, where $r_j = |\Delta_j|$ is the length of the segment Δ_j. We write Δ^k for the partition of kth degree $\{\Delta_\sigma, \sigma \in \{1, \dots, m\}^k\}$. This partition splits the segment [0, 1] into m^k pieces. For any positive vector $z = (z_1, \dots, z_m)$ and for any $\alpha \in \mathbb{R}$ we set $z^\alpha \mathscr{A} = \{z_1^\alpha A_1, \dots, z_m^\alpha A_m\}$. Denote also $r = (r_1, \dots, r_m)$. For example, $mr\mathscr{A} = \{mr_1 A_1, \dots, mr_m A_m\}$.

Theorem 2. *For an irreducible family of affine operators Eq.* (8) *possesses a summable solution* $v(t)$ *if and only if* $\rho_1(mr\mathscr{A}) < 1$. *This solution is unique.*

If for some $p \in [1, +\infty]$ *one has* $\rho_p((mr)^{1/p}\mathscr{A}) < 1$, *then* $v \in L_p$. *For* $p < \infty$ *the converse is also true*: *if* $v \in L_p$, *then* $\rho_p < 1$. *If* $v \in L_\infty$, *then* $\rho_\infty \leqslant 1$.

Proof. Take an arbitrary $p < \infty$ and denote $B_j = (mr_j)^{1/p} A_j$, $\mathscr{B} = (mr)^{1/p}\mathscr{A}$, $\rho_p = \rho_p(\mathscr{B})$. For any $f_1, f_2 \in L_p[0, 1]$ we have

$$\|\mathbf{A}(f_1 - f_2)\|_p = \left[\int_0^1 \sum_j r_j |A_j(f_1(t) - f_2(t))|^p \mathrm{d}t\right]^{1/p} = \left[\int_0^1 \mathscr{F}^p(p, \mathscr{B}, f_1(t) - f_2(t))\mathrm{d}t\right]^{1/p}.$$

Thus

$$\|\mathbf{A}(f_1 - f_2)\|_p = \|\mathscr{F}(p, \mathscr{B}, f_1(\cdot) - f_2(\cdot))\|_p. \tag{9}$$

Taking a limit, we extend this equality for $p = \infty$. If $\rho_p < 1$, then Proposition 1 provides us with a norm $|\cdot|$ in $\mathbb{R}^d$ such that $\mathscr{F}(p, \mathscr{B}, f_1(t) - f_2(t)) \leqslant \gamma |f_1(t) - f_2(t)|$ for all $t \in [0, 1]$ and hence $\|\mathbf{A}(f_1 - f_2)\|_p \leqslant \gamma \|f_1 - f_2\|_p$, where $\gamma < 1$. Therefore, $\widetilde{\mathbf{A}}$ is a contraction operator in $L_p[0, 1]$ which possesses a unique fixed point in this space. This proves the sufficiency.

To establish the necessity we involve Proposition 2. Assume Eq. (8) possesses a solution $v \in L_p$ and set $a = \int_0^1 v(t)\mathrm{d}t$. By the same symbol we denote the function $a(t) \equiv a$, $t \in [0, 1]$. For any $k \geqslant 1$ and any segment Δ_σ of the partition Δ^k we have

$$|\Delta_\sigma|^{-1} \int_{\Delta_\sigma} v(t)\mathrm{d}t = \int_0^1 \widetilde{A}_{\sigma(1)} \cdots \widetilde{A}_{\sigma(k)} v(t)\mathrm{d}t = \widetilde{A}_{\sigma(1)} \cdots \widetilde{A}_{\sigma(k)} \int_0^1 v(t)\mathrm{d}t = \widetilde{A}_{\sigma(1)} \cdots \widetilde{A}_{\sigma(k)} a.$$

Therefore, the step function $f_k = \widetilde{\mathbf{A}}^k a$ equals to the average $|\Delta_\sigma|^{-1} \int_{\Delta_\sigma} v(t)\mathrm{d}t$ on each segment Δ_σ. The diameter of the partition Δ^k tends to zero, hence by Lemma 2 in case $p < \infty$ one has $\|f_k - v\|_p \to 0$ as $k \to \infty$. In case $p = \infty$ the sequence $\|f_k - v\|_\infty$ may not converge to zero, but it is bounded.

Assume there is a common linear subspace L of the operators A_j that contains the vectors $v(t) - a$ for almost all $t \in [0, 1]$. This subspace can be defined by several equations $(l_q, u) = 0$, $q = 1, \ldots, n$, where l_q are some linear functionals in $\mathbb{R}^d$. For any q we have $(l_q, v(t)) = (l_q, a)$ for almost all t. Therefore, for any $j \in \{1, \ldots, m\}$ Eq. (8) implies

$$(l_q, \widetilde{A}_j a) = \left(l_q, \int_0^1 \widetilde{A}_j v(t)\mathrm{d}t\right) = \left(l_q, |\Delta_j|^{-1} \int_{\Delta_j} v(t)\mathrm{d}t\right) = (l_q, a).$$

This yields that the affine plane $\widetilde{L} = a + L$ contains all the points $\widetilde{A}_j a$, $j = 1, \ldots, m$. Hence, for any $x \in L$ we have $\widetilde{A}_j(a + x) = \widetilde{A}_j a + A_j x \in \widetilde{L}$, because $\widetilde{A}_j a \in \widetilde{L}$ and $A_j x \in L$. Thus, $\widetilde{L}$ is a common invariant affine plane for the family $\widetilde{A}$, which contradicts the assumptions. Whence there is a set μ, $|\mu| > 0$ such that $v(t) - a$ does not belong to common invariant subspaces of the family $\mathscr{A}$ (and therefore to ones of the family $\mathscr{B}$), whenever $t \in \mu$. Now apply Corollary 3 and denote $C(t) = C(v(t) - a)$. We see that the function $C(t)$ is positive on μ, consequently, there is an $\varepsilon > 0$ and a set of positive measure $\mu_\varepsilon \subset \mu$ such that $C(t) \geqslant \varepsilon$ for all $t \in \mu_\varepsilon$. Thus, $\mathscr{F}_k(p, \mathscr{B}, v(t) - a) \geqslant \varepsilon(\rho_p)^k$ for all $t \in \mu_\varepsilon$. Iterating equality (9), we obtain

$$\|\mathbf{A}^k(f_1 - f_2)\|_p = \|\mathscr{F}_k(p, \mathscr{B}, f_1(\cdot) - f_2(\cdot))\|_p, \quad k \in \mathbb{N}, \tag{10}$$

which for $f_1 = v$, $f_2 = a$ yields $\|\widetilde{\mathbf{A}}^k v - \widetilde{\mathbf{A}}^k a\|_p = \|\mathscr{F}_k(p, \mathscr{B}, v(\cdot) - a)\|_p$. Thus

$$\|v - \widetilde{\mathbf{A}}^k a\|_p \geqslant |\mu_\varepsilon|^{1/p}\varepsilon(\rho_p)^k, \quad k \in \mathbb{N}. \tag{11}$$

If $p < \infty$, then the left-hand side tends to zero as $k \to \infty$, and so $|\mu_\varepsilon|^{1/p}\varepsilon\rho_p^k \to 0$ as $k \to \infty$, which implies $\rho_p < 1$. In case $p = \infty$ the left-hand side is bounded uniformly, hence $\rho_\infty \leqslant 1$. $\square$

Remark 5. Actually instead of the irreducibility of the operators we used a weaker assumption: the family $\widetilde{\mathscr{A}}$ does not possess common invariant affine subspaces (different from $\mathbb{R}^d$) containing the point $a = \int_0^1 v(x)\mathrm{d}x$. Nevertheless, in the study of arbitrary fractal curves it suffices to consider only irreducible families. Let $\widetilde{L}$ be the smallest by inclusion affine plane containing a and invariant with respect to all operators of $\widetilde{\mathscr{A}}$. Since for any k the image of the function $f_k = \widetilde{\mathbf{A}}^k a$ lies in $\widetilde{L}$ and f_k converge to v in $L_1[0, 1]$ as $k \to \infty$, it follows that $v(t) \in \widetilde{L}$ a.e. Hence one could consider the restrictions of the operators $\widetilde{A}_j$ to the subspace $\widetilde{L}$. On this subspace our family is irreducible.

Indeed, if the operators have a common invariant plane $\widetilde{L}_0 \subset \widetilde{L}$ (we take this plane to be smallest by inclusion), then $a \notin \widetilde{L}_0$. Theorem 2 implies that $\rho_1(\mathscr{B}) < 1$ on the space L, and hence on L_0 as well. Then Eq. (8) possesses a solution on the plane $\widetilde{L}_0$ which does not coincide with v since $a \notin \widetilde{L}_0$. This contradicts the uniqueness.

Thus, restricting, if necessary, our operators to the subspace $\widetilde{L}$ we obtain a fractal curve generated by an irreducible family. This justifies the irreducibility assumption in Theorem 2. In the sequel, unless the opposite is stated, we assume that the family $\widetilde{\mathscr{A}}$ is irreducible.

The criterion of L_p-solvability of Eq. (8) looks especially simple for uniform partitions of the segment [0,1], when $r_1 = \cdots = r_m = \frac{1}{m}$.

Corollary 5. *For the uniform partition equation* (8) *possesses an* L_p*-solution* ($p < \infty$) *if and only if* $\rho_p(\mathscr{A}) < 1$.

Remark 6. The best studied case of fractal curves are refinable functions which are compactly supported solutions $\varphi: \mathbb{R} \to \mathbb{R}$ of the *refinement equations*:

$$\varphi(t) = \sum_{k=0}^{N} c_k \varphi(mt - k), \quad t \in \mathbb{R} \tag{12}$$

(for the sake of simplicity we consider here one-dimensional univariate refinement equations with finitely many terms). Let n be the smallest integer that is not less than $\frac{N}{m-1}$. The vector-function $v(t) = (\varphi(t), \varphi(t+1), \ldots, \varphi(t+n-1))^{\mathrm{T}}$, $t \in [0, 1]$ is a fractal curve. It satisfies Eq. (8) for the uniform partition of the segment [0, 1] and for affine operators $\widetilde{A}_s = T_s|_V$, $s = 1, \ldots, n$, where V is a certain affine subspace of $\mathbb{R}^n$ (defined individually for each equation) end T_s are linear operators defined by their matrices (*block Toeplitz matrices*) as follows: $(T_s)_{ij} = c_{mi-m-j+s+1}$, $i, j \in \{1, \ldots, n\}$. Refinement equations have been studied in great detail in connection with wavelets theory, subdivision algorithms, etc. (see [11,29,30] for many references). These equations belong to the special case of the self-similarity equation, when $r_1 = \cdots = r_m = \frac{1}{m}$ and the operators $\widetilde{A}_s$ are defined by the block Toeplitz matrices T_s. For refinement equations most of our results are known. The criterion of L_p-solvability was obtained in [11], the criterion of continuity of the solution was derived in [6], the formula for the Hölder exponent was in [5,6,30]. So, the results of Sections 4 and 5 can be considered as natural generalizations of known facts on refinement equations from the uniform partitions to arbitrary partitions and from the operators defined by the Toeplitz matrices to arbitrary affine operators. Here it should be mentioned that the proofs of the corresponding results for refinement equations used essentially the special properties of the uniform partition and of the Toeplitz matrices, therefore, we have to introduce a different technique. Also the estimation for the moduli of continuity (Theorem 5) and the conditions of Lipschitz continuity (Corollary 7) are new for refinement equations.

Corollary 5 gives one more geometric interpretation of the p-radius. A family of affine operators has an L_p-fractal curve for the uniform partition of $[0, 1]$ if and only if the p-radius of the corresponding linear operators is smaller than 1. The following theorem establishes a stronger fact. The p-radius of a family $\mathscr{A}$ equals to the usual spectral radius of the corresponding linear self-similarity operator $\mathbf{A}$ in the space $L_p[0, 1]$.

Theorem 3. *Let an irreducible family of affine operators $\widetilde{\mathscr{A}}$ in $\mathbb{R}^d$ and a partition of the segment* $[0, 1]$ *be given*, $\mathbf{A}$ *be the corresponding self-similarity operator in* $L_p[0, 1]$, *where* $p \in [1, +\infty]$. *Then the p-radius* $\rho_p((mr)^{1/p}\mathscr{A})$ *is equal to the spectral radius* $\rho(\mathbf{A})$. *In particular*, *for the uniform partition we have* $\rho(\mathbf{A}) = \rho_p(\mathscr{A})$. *Moreover*

$$C_1(\rho_p)^k \leqslant \|\mathbf{A}^k\|_p \leqslant C_2(\rho_p)^k k^{s-1}, \qquad k \in \mathbb{N}, \quad k \geqslant d, \tag{13}$$

where $\rho_p = \rho_p((mr)^{1/p}\mathscr{A})$, *s is the valency of the family* $(mr)^{1/p}\mathscr{A}$, *and* $C_1, C_2 > 0$.

Proof. Combining equality (10) with Proposition 2, we conclude that

$$\|\mathbf{A}^k(f_1 - f_2)\|_p \leqslant c_2\|f_1 - f_2\|_p(\rho_p)^k k^{s-1}, \qquad k \in \mathbb{N}, \quad k \geqslant d \tag{14}$$

for any $f_1, f_2 \in L_p[0, 1]$, which gives the upper bound in (13). Applying now the inverse inequality (11) we arrive at the lower bound of (13). Taking the power $1/k$ in (13) and invoking Gelfand's formula $\rho(\mathbf{A}) = \lim_{k\to\infty}[\|\mathbf{A}^k\|_p]^{1/k}$, we complete the proof. □

Remark 7. The solution $v(t)$ of self-similarity equation (8) in the space $L_p[0, 1]$, whenever it exists, can be computed by iterative approximations starting with an arbitrary initial function $f \in L_p[0, 1]$. Indeed, inequality (14) applied for $f_1 = f$, $f_2 = v$ gives $\|\widetilde{\mathbf{A}}^k f - v\|_p \leqslant C_2(\rho_p)^k k^{s-1}$. This provides a sharp upper bound for the rate of convergence of this method. Since $\rho_p < 1$, it follows that the iterative approximation method converges exponentially for any initial function f, provided the fractal curve belongs to L_p. In particular, it converges faster than $(\rho_p + \varepsilon)^k$ for any $\varepsilon > 0$. Since the operator $\mathbf{A}$ continuously depends on the operators of the family $\widetilde{\mathscr{A}}$ and on the partition $\{\Delta_j\}$, it follows that the solution $v \in L_p$, whenever it exists, continuously depends on those parameters in the metric $L_p[0, 1]$.

Remark 8. It was shown by Protasov [22] and later independently and in some weaker form by Zhou [17] that for even integers p, starting with $p = 2$, the p-radius can be found as an eigenvalue of a suitable finite-dimensional linear operator. This means that for even integers p the p-radius can be effectively computed. Therefore, for instance, the L_2-solvability of Eq. (8) can be effectively decided by Theorem 2 for any affine operators and for any partition. Under the additional assumption that all the operators of the family $\mathscr{A}$ have a common invariant cone (for example, if all their matrices are nonnegative in some basis), Blondel and Nesterov [23] proved that the p-radius can be computed for odd integers p as well. Hence, in this case the L_1-solvability of Eq. (8) can be effectively decided.

Remark 9. Fractal curves can also be considered as natural generalizations of the notion of affine fractals. Recall that an affine fractal of a family of affine operators $\widetilde{\mathscr{A}}$ is a compact set $K \subset \mathbb{R}^d$ such that $K = \sum_{i=1}^{m} \widetilde{A}_i K$. According to the classical result of Hutchinson [1], a family $\widetilde{\mathscr{A}}$ possesses a unique fractal, provided all the operators of $\widetilde{\mathscr{A}}$ are contractions. Proposition 1 relaxes this condition to $\rho_\infty(\mathscr{A}) < 1$. On the other hand, Theorem 2 guarantees the solvability of Eq. (8) in the space $L_\infty[0, 1]$, whenever $\rho_\infty(\mathscr{A}) < 1$. This implies that any affine fractal K can be parametrized by the solution $v(t)$ of the corresponding equation of the type (8). More precisely, if $\rho_\infty < 1$, then there is a bounded Borel function v satisfying Eq. (8) almost everywhere on [0, 1] and such that $\{v(t), t \in [0, 1]\} = K$. See the recent work [35, Section 6] for details.

5. Smooth fractal curves

The technique proposed in the previous section enables us to go further in the study of fractal curves and to derive criteria of their continuity, differentiability, etc. In some cases this leads to a very sharp information on the regularity of fractal curves. First of all, we have to introduce one more condition on affine operators. According to Theorem 2, the fractal curve belongs to L_∞, provided $\rho_\infty(\mathscr{A}) < 1$. In this case, by Proposition 1 there is a norm in $\mathbb{R}^d$, for which the affine operators of the family $\widetilde{\mathscr{A}}$ are all contractions. In particular, each operator $\widetilde{A}_j$ has a unique fixed point $v_j \in \mathbb{R}^d$.

Definition 4. A family of affine operators $\widetilde{\mathscr{A}}$ such that $\rho_\infty(\mathscr{A}) < 1$ is said to satisfy the cross-condition if

$$\widetilde{A}_j v_m = \widetilde{A}_{j+1} v_1 \quad \text{for any } j = 1, \ldots, m-1. \tag{15}$$

This condition for contraction affine operators was put to good use in many works and is sometimes called Barnsley condition. We are going to see that this condition, together with the inequality $\rho_\infty < 1$, is responsible for continuity of fractal curves.

Theorem 4. *The solution of* (8) *is continuous if and only if* $\rho_\infty(\mathscr{A}) < 1$ *and* (15) *holds.*

Proof. Combining relation (9) for $p = \infty$ with Proposition 1, we conclude that if $\rho_\infty < 1$, then $\widetilde{\mathbf{A}}$ is a contraction operator in the space $L_\infty[0, 1]$. Furthermore, if condition (15) is fulfilled, then $\widetilde{\mathbf{A}}$ respects the space $C[0, 1]$. Therefore, it has a unique fixed point in this space. Conversely, if $v \in C[0, 1]$, then the step functions $f_k = \widetilde{\mathbf{A}}^k a$ converge to v, i.e., $\|v - \widetilde{\mathbf{A}}^k a\|_\infty \to 0$ as $k \to \infty$. Using (11) for $p = \infty$ we get $\varepsilon\rho_\infty^k \to 0$, which yields $\rho_\infty < 1$. Finally, Eq. (8) applied at the nodes of the partition yields $v(b_j) = \widetilde{A}_j v(1) = \widetilde{A}_j v_m$ and $v(b_j) = \widetilde{A}_{j+1} v(0) = \widetilde{A}_{j+1} v_1$, which proves (15). $\square$

Remark 10. Theorems 2 and 4 admit also the case $d = 0$, when the affine space degenerates to one point. In this case all the operators A_j are identical zeros, and the function $v(t)$ is an identical constant. In the sequel it will be more convenient to exclude this trivial case and assume that $d \geqslant 1$.

Having ensured the continuity one can compute the exponents of regularity of fractal curves in the space $C[0, 1]$. We use the modulus of continuity of a function $f \in C[0, 1]$ which is the value $\omega(f, h) = \sup\{|f(t_1) - f(t_2)|, t_1, t_2 \in [0, 1], |t_1 - t_2| \leqslant h\}, h > 0$, and the Hölder exponent: $\alpha_f = \{\sup \alpha | \omega(f, h) \leqslant Ch^{\alpha}\}$.

Theorem 5. *If a fractal curve v is continuous, then its Hölder exponent α_v is a unique solution of the equation $\rho_{\infty}(r^{-\alpha}\mathscr{A}) = 1$. The modulus of continuity satisfies the inequality*

$$C_1 h^{\alpha} \leqslant \omega(v, h) \leqslant C_2 |\ln h|^{s-1} h^{\alpha}, \quad h \in \left(0, \frac{1}{2}\right), \tag{16}$$

where s is the valency of the family $r^{-\alpha}\mathscr{A}$ for $p = \infty$, $\alpha = \alpha_v$, $C_1, C_2 > 0$ are some constants.

Proof. Let $v \in C[0, 1]$ and let α be the solution of the equation $\rho_{\infty}(r^{-\alpha}\mathscr{A}) = 1$. For a given segment $\Delta \subset [0, 1]$ we denote by $\mathrm{dev}(v, \Delta) = \sup_{t_1, t_2 \in \Delta} |v(t_1) - v(t_2)|$ the deviation of the function v on Δ. This is the diameter of the set $\{v(t),\ t \in \Delta\}$. Also we set $\mathrm{dev}_0(v, \Delta) = \sup_{t \in \Delta} \left| v(t) - \frac{1}{\Delta}\int_{\Delta} v(\tau)\mathrm{d}\tau \right|$. From the triangle inequality it follows that $\mathrm{dev}(v, \Delta) \leqslant 2\,\mathrm{dev}_0(v, \Delta)$. On the other hand, since the point $\frac{1}{\Delta}\int_{\Delta} v(\tau)\mathrm{d}\tau$ belongs to the convex hull of the set $\{v(t), t \in \Delta\}$, we have $\mathrm{dev}(v, \Delta) \geqslant \mathrm{dev}_0(v, \Delta)$. In the proof of Theorem 2 we showed that there is a subset $\mu \subset [0, 1]$ of positive measure such that for any $t \in \mu$ the vector $v(t) - a$ does not belong to common invariant subspaces of the operators $\mathscr{A}$. For any $t \in \mu$ Proposition 2 implies $\mathscr{F}_k(\infty, r^{-\alpha}\mathscr{A}, v(t) - a) \geqslant C$, where $C > 0$ does not depend on k. Therefore, for any k there is $\sigma \in \{1, \ldots, m\}^k$ such that

$$\left| \left[\Pi_{j=1}^{k} r_{\sigma(j)}^{-\alpha} A_{\sigma(j)} \right] (v(t) - a) \right| \geqslant C.$$

Whence $\left| \left[\Pi_{j=1}^{k} A_{\sigma(j)} \right] (v(t) - a) \right| \geqslant C|\Delta_{\sigma}|^{\alpha}$, and so

$$\mathrm{dev}_0(v, \Delta_{\sigma}) = \sup_{t \in \Delta_{\sigma}} |v(t) - \widetilde{\mathbf{A}}^k a| = \sup_{t \in \Delta_{\sigma}} |\mathbf{A}^k[v - a](t)| \geqslant C|\Delta_{\sigma}|^{\alpha}.$$

Thus, for any k there is a segment Δ_{σ} of the partition Δ^k such that $\mathrm{dev}(v, \Delta_{\sigma}) \geqslant C|\Delta_{\sigma}|^{\alpha}$, which proves the lower bound in (16). To establish the upper bound we take arbitrary points $t_1, t_2 \in (0, 1)$ such that $0 < t_2 - t_1 < r_0$, where $r_0 = \min_{j=1,\ldots,m} r_j$, and denote by n the smallest natural number, for which the partition Δ^n has a node on the interval (t_1, t_2). Denote this node by b. Let n_1 be the smallest natural number such that the partition Δ^{n_1} has a node on the interval (t_1, b), similarly we define n_2. Since $t_2 - t_1 < r_0$, we see that $n_1, n_2 \geqslant 2$. Thus, the segment $[t_1, t_2]$ is covered by two segments $\Delta_{\sigma_1} \in \Delta^{n_1 - 1}$ and $\Delta_{\sigma_2} \in \Delta^{n_2 - 1}$. Therefore

$$|v(t_1) - v(t_2)| \leqslant 2 \max\{\mathrm{dev}(v, \Delta_{\sigma_1}), \mathrm{dev}(v, \Delta_{\sigma_2})\} \tag{17}$$

For any $i \in \{1, 2\}$ we have

$$\mathrm{dev}(v, \Delta_{\sigma_i}) \leqslant 2\,\mathrm{dev}_0(v, \Delta_{\sigma_i}) = 2 \sup_{t \in [0,1]} \left| \Pi_{\sigma_i}(v(t) - a) \right|,$$

where $\Pi_{\sigma_i} = A_{\sigma_i(1)} \cdots A_{\sigma_i(n_i - 1)}$. Proposition 2 applied to the operators $r^{-\alpha}\mathscr{A}$ for $p = \infty$ yields $\left| \Pi_{\sigma_i}(v(t) - a) \right| \leqslant |v(t) - a|(\Delta_{\sigma_i})^{\alpha} c_2 (n_i - 1)^{s-1} \leqslant C(\Delta_{\sigma_i})^{\alpha} \ln^{s-1} \Delta_{\sigma_i}$, where C is independent of t and n_i. Thus, $\mathrm{dev}(v, \Delta_{\sigma_i}) \leqslant 2C(\Delta_{\sigma_i})^{\alpha} \ln^{s-1} \Delta_{\sigma_i}$. Substitute this in (17):

$$|v(t_1) - v(t_2)| \leqslant 4C(\Delta_{\sigma})^{\alpha} \ln^{s-1} \Delta_{\sigma}, \quad \sigma \in \{\sigma_1, \sigma_2\}, \qquad |\Delta_{\sigma}| = \max\{|\Delta_{\sigma_1}|, |\Delta_{\sigma_2}|\}.$$

Since $|\Delta_{\sigma}| \leqslant {r_0}^{-1}|t_1 - t_2|$ (because $|\Delta_{\sigma_i}| \leqslant {r_0}^{-1}|t_i - b|$ for $i = 1, 2$), we arrive at the upper bound in (16). □

Corollary 6. *If a fractal curve v is continuous, then $\alpha_v > 0$.*

Corollary 7. *If a fractal curve v is Lipschitz continuous, then $\rho_\infty(r^{-1}\mathscr{A}) = 1$ and (15) holds. Conversely, if (15) holds, $\rho_\infty(r^{-1}\mathscr{A}) = 1$, and $s = 1$, then v is Lipschitz continuous.*

Remark 11. One aspect may seem strange in Theorem 5. In case $\rho_\infty(r^{-1}\mathscr{A}) < 1$ we have $\alpha_v > 1$, which is, of course, impossible. In fact, this case cannot appear because of cross-condition (15). This condition implies that $\sum_{j=1}^{m} A_j(v_m - v_1) = v_m - v_1$ and therefore, $\rho_1(m\mathscr{A}) \geqslant 1$. This, in turn, yields for any $p \geqslant 1$

$$\rho_p((mr)^{1/p}r^{-1}\mathscr{A}) \geqslant \rho_1((mr)^1 r^{-1}\mathscr{A}) = \rho_1(m\mathscr{A}) \geqslant 1. \tag{18}$$

For $p = \infty$ this gives $\rho_\infty(r^{-1}\mathscr{A}) \geqslant 1$.

Remark 12. The proof of Theorem 5 can easily be modified to analyze the local regularity of continuous self-similar functions. The local regularity of fractal objects is used in problems of real data modeling, mathematical physics, signal processing, etc. (see, for instance, [31]–[34] and references therein). In this paper, we do not obtain any precise statements on the local regularity, and refer instead to the recent work [35], where we treated the case of uniform partition of the segment ($r_1 = \cdots = r_m$). In this case we can express the local Hölder exponent of a self-similar function $v(t)$ at any point $t \in [0, 1]$ by means of certain joint spectral characteristics of the operators $\{A_j\}$. It is also possible to describe the distribution of points $t \in [0, 1]$ with a given value of the local regularity. In particular, the Hölder exponent is the same at almost all points of the segment $[0, 1]$ and is equal to $-\log_m \bar{\rho}$, where $\bar{\rho}$ is the Lyapunov exponent of those operators [35, Section 8].

Definition 3 makes it possible to test easily any fractal curve on differentiability. The crucial observation is that the derivative is also a fractal curve in a smaller space.

We use the notation $C^l[0, 1]$ for the space of l times continuously differentiable functions from $[0, 1]$ to $\mathbb{R}^d$, and $W_p^l[0, 1]$, $p \in [1, +\infty]$ for the Sobolev space of functions possessing absolutely continuous $(l - 1)$st derivative and lth derivative in L_p, $W_p^0 = L_p$.

If a fractal curve is differentiable, then it is continuous, and Theorem 4 implies that each of the operators $\widetilde{A}_j$ has a unique fixed point v_i and condition (15) is satisfied. Let $\widetilde{V}$ be the smallest *affine* subspace of the linear operators $r^{-1}\mathscr{A}$ containing the point $v_m - v_1$.

Proposition 3. *The solution v of (8) belongs to W_p^1 if and only if the family $\widetilde{\mathscr{A}}$ satisfies (8) and the p-radius ρ_p of the operators $r^{\frac{1}{p}-1}m^{\frac{1}{p}}\mathscr{A}$ on the subspace V is smaller than 1. In this case* $\dim \widetilde{V} = d - 1$ *and the derivative $v'(t)$ is a fractal curve for the family $r^{-1}\mathscr{A}$ on $\widetilde{V}$.*

Proof. If $v \in W_p^1$, then differentiating (8) we get

$$v(t) = r_k^{-1}\widetilde{A}_k v(g_k^{-1}(t)), \qquad t \in \Delta_k, \quad k = 1, \ldots, m \tag{19}$$

almost everywhere on $[0, 1]$. The smallest by inclusion affine plane containing $v'(t)$ for almost all t also contains the point $\int_0^1 v'(x)\mathrm{d}x = v(1) - v(0) = v_m - v_1$ and hence coincides with $\widetilde{V}$ (Remark 5). Theorem 2 now yields $\rho_p(r^{\frac{1}{p}-1}m^{\frac{1}{p}}\mathscr{A}|_V) < 1$. Combining this with (18) we conclude that $V \neq \mathbb{R}^d$ and so $\dim V \leqslant d - 1$. On the other hand $v(t) = v_1 + \int_0^t v'(\tau)\mathrm{d}\tau$, therefore, the affine hull of the set $\widetilde{V} \cup \{v_1\}$ is the entire $\mathbb{R}^d$. Hence dim $V = d - 1$. Conversely, if $\rho_p \left(r^{\frac{1}{p}-1} m^{\frac{1}{p}} \mathscr{A}\big|_V \right) < 1$, then by Theorem 2 Eq. (19) has an L_p-solution on the plane $\widetilde{V}$. Its primitive is a solution of (8), which by the uniqueness coincides with v. □

Note that the space $\widetilde{V}$ is uniquely defined, whenever $v(t)$ is continuous. Hence, to check whether $v \in W_p^1$ one needs to compute the corresponding p-radius. Furthermore, Theorem 4 and Proposition 3 applied to the operators $r^{-1}\mathscr{A}$ restricted to the affine subspace $\widetilde{V}$ yield a criterion for the function v to belong to $C^1[0, 1]$ and to $W_p^2[0, 1]$. Iterating we obtain a criterion of solvability of Eq. (8) in the spaces W_p^l and C^l, $l \geqslant 1$. The derivative $v^{(l)}$ lies in an affine subspace $\widetilde{V}_l \subset \mathbb{R}^d$ of dimension $d - l$. In particular, dim $\widetilde{V}_d = 0$, which means that the function $v^{(d)}$ is concentrated at one point, i.e., is an identical constant. Therefore, $v(t)$ is a dth primitive of a constant, i.e., belongs to $\mathscr{P}_d$, which is the space of *polynomial curves* of degree at most d (all the coordinate functions are algebraic polynomials in t, their largest degree is d). Thus, we have established.

Proposition 4. *If a fractal curve v belongs to $W_1^d[0, 1]$, then $v \in \mathscr{P}_d$.*

We see that the regularity of a fractal curve in $\mathbb{R}^d$ is either less than d or infinite. If a fractal curve possesses a summable dth derivative, then it is polynomial, and so it is infinitely smooth. In this case there is a basis in $\mathbb{R}^d$ in which the matrices of the operators A_j are lower triangular with the diagonal elements $r_j, \ldots, r_j^d$; the kth entry of the function v in this basis is a polynomial of degree k. Let us finally remark that any polynomial curve is a fractal one. For any polynomial curve v of degree d and any partition $\{\Delta_j\}_{j=1}^d$ of the segment [0, 1] there are affine operators $\widetilde{A}_j$, for which v satisfies (8). Indeed, from an identical constant by d passages to primitive one can obtain any polynomial curve of degree d.

Acknowledgements

The author wishes to express his thanks to the three anonymous Referees for their helpful comments and advices.

References

[1] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (5) (1981) 713–747.
[2] M. Barnsley, Fractals Everywhere, Academic Press, Boston, 1988.
[3] Ch.A. Micchelli, H. Prautzsch, Uniform refinement of curves, Linear Algebra Appl. 114–115 (1989) 841–870.
[4] G. De Rham, Sur une courbe plane, J. Math. Pures Appl. 35 (9) (1956) 25–42.
[5] I. Daubechies, J. Lagarias, Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals, SIAM J. Math. Anal. 23 (4) (1992) 1031–1079.
[6] D. Collela, C. Heil, Characterization of scaling functions: continuous solutions, SIAM J. Matrix Anal. Appl. 15 (2) (1994) 496–518.
[7] G. Deslauriers, S. Dubuc, Symmetric iterative interpolation processes, Constr. Approx. 5 (1) (1989) 49–68.
[8] N. Dyn, D. Levin, Interpolatory subdivision schemes for the generation of curves and surfaces, Multivariate Approximation and Interpolation (Duisburg, 1989), Birkhäuser, Basel (1990), pp. 91–106.
[9] Y. Wang, Two-scale dilation equations and the mean spectral radius, Random Comput. Dynam. 4 (1) (1996) 49–72.
[10] R.Q. Jia, Subdivision schemes in L_p spaces, Adv. Comput. Math. 3 (1995) 309–341.
[11] K.-S. Lau, J. Wang, Characterization of L_p-solutions for two-scale dilation equations, SIAM J. Math. Anal. 26 (4) (1995) 1018–1046.

[12] I. Sheipak, On the construction and some properties of self-similar functions in the spaces $L_p[0, 1]$, Math. Notes J. 81 (6) (2007) 924–938.
[13] M. Solomyak, E. Verbitsky, On a spectral problem related to self-similar measures, Bull. London Math. Soc. 27 (1995) 242–248.
[14] G.A. Derfel, N. Dyn, D. Levin, Generalized refinement equations and subdivision processes, J. Approx. Theory 80 (1995) 272–297.
[15] N.E. Barabanov, Lyapunov indicator for discrete inclusions, I–III, Autom. Remote Control 49 (2) (1988) 152–157.
[16] M.A. Berger, Y. Wang, Bounded semigroups of matrices, Linear Algebra Appl. 166 (1992) 21–27.
[17] D.-X. Zhou, The p-norm joint spectral radius for even integers, Methods Appl. Anal. 5 (1) (1998) 39–54.
[18] G.C. Rota, G. Strang, A note on the joint spectral radius, Indag. Math. (N.S.) 63 (1960) 379–381.
[19] V.S. Kozyakin, Algebraic unsolvability of problem of absolute stability of desynchronized systems, Autom. Remote Control 51 (6) (1990) 754–759.
[20] V. Blondel, J. Tsitsiklis, Approximating the spectral radius of sets of matrices in the max-algebra is NP-hard, IEEE Trans. Autom. Control 45 (9) (2000) 1762–1765.
[21] V.Yu. Protasov, The joint spectral radius and invariant sets of linear operators, Fundam. Prikl. Mat. 2 (1) (1996) 205–231.
[22] V.Yu. Protasov, The generalized spectral radius. A geometric approach, Izv. Math. 61 (5) (1997) 995–1030.
[23] V.D. Blondel, Yu. Nesterov, Computationally efficient approximations of the joint spectral radius, SIAM J. Matrix Anal. 27 (1) (2005) 256–272.
[24] G. Gripenberg, Computing the joint spectral radius, Linear Algebra Appl. 234 (1996) 43–60.
[25] M. Maesumi, An efficient lower bound for the generalized spectral radius, Linear Algebra Appl. 240 (1996) 1–7.
[26] V.D. Blondel, Y. Nesterov, J. Theys, On the accuracy of the ellipsoid norm approximation of the joint spectral radius, Linear Algebra Appl. 394 (2005) 91–107.
[27] G.G. Magaril-Il'yaev, V.M. Tikhomirov, Convex analysis: theory and applications, Transl. from the Russian by Dmitry Chibisov, Translations of Mathematical Monographs 222, Providence, RI, USA, 2001.
[28] E. Plischke, F. Wirth, N. Barabanov, Duality results for the joint spectral radius and transient behaviour, in: Proceedings of the 44 IEEE CDC Conference, Sevilla, Spain, December 12–15, 2005, pp. 2344–2349.
[29] C.A. Cabrelli, C. Heil, U.M. Molter, Self-similarity and multiwavelets in higher dimensions, Mem. Amer. Math. Soc. 170 (2004) 807.
[30] L. Villemoes, Wavelet analysis of refinement equations, SIAM J. Math. Anal. 25 (5) (1994) 1433–1460.
[31] L.I. Levkovich-Maslyuk, Determination of the scaling parameters of affine fractal interpolation functions with the aid of wavelet analysis, SPIE Proc. 2825 (1996), paper 2825-91.
[32] J. Kigami, Analysis on fractals, Cambridge Tracts in Math., vol. 143, Cambridge University Press, 2001.
[33] B.M. Hambly, J. Kigami, T. Kumagai, Multifractal formalisms for the local spectral and walk dimensions, Math. Proc. Cambridge Philos. Soc. 132 (3) (2002) 555–571.
[34] V.Yu. Protasov, On the regularity of de Rham curves, Izv. Math. 68 (3) (2004) 567–606.
[35] V.Yu. Protasov, Fractal curves and wavelets, Izv. Math. 70 (5) (2006) 975–1013.