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# Studying Baskakov–Durrmeyer operators and quasi-interpolants via special functions

Elena E. Berdysheva*

*Institut für Angewandte Mathematik und Statistik, Universität Hohenheim, D-70593 Stuttgart, Germany*

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**Abstract**

We prove that the kernels of the Baskakov–Durrmeyer and the Szász–Mirakjan–Durrmeyer operators are completely monotonic functions. We establish a Bernstein type inequality for these operators and apply the results to the quasi-interpolants recently introduced by Abel. For the Baskakov–Durrmeyer quasi-interpolants, we give a representation as linear combinations of the original Baskakov–Durrmeyer operators and prove an estimate of Jackson–Favard type and a direct theorem in terms of an appropriate K-functional.
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## 1. Introduction

Let $c \in \mathbb{R}$. Put $I_c = \left[0, -\frac{1}{c}\right]$ for $c < 0$ and $I_c = [0, \infty)$ for $c \geqslant 0$. For $n > 0$, $k \in \mathbb{N}_0$ and $x \in I_c$ we define

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k} (cx)^k (1+cx)^{-\frac{n}{c}-k}, \quad c \neq 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx}, \quad c = 0.$$

* Fax: +49 711 459 23030.
*E-mail address:* elena.berdysheva@uni-hohenheim.de.

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As usual, the binomial coefficients are defined for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$ by $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ if $k \in \mathbb{N}$, and $\binom{\alpha}{0} := 1$. In particular, $\binom{m}{k} = 0$ if $m \in \mathbb{N}$ and $k > m$. The functions $p_{n,k}^{[c]}$ satisfy the property

$$\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1. \tag{1}$$

We will study the following positive linear operators.

**Definition 1.** Let $c \in \mathbb{R}$ and let $n > c$ if $c \geqslant 0$ or $n = -c\ell$ with some $\ell \in \mathbb{N}$ if $c < 0$. Define

$$(\mathbf{M}_{n,c}(f))(x) := \sum_{k=0}^{\infty} \frac{\int_{I_c} p_{n,k}^{[c]}(y)\, f(y)\, dy}{\int_{I_c} p_{n,k}^{[c]}(y)\, dy}\; p_{n,k}^{[c]}(x) \tag{2}$$

for $f \in L_p(I_c)$, $1 \leqslant p \leqslant \infty$.

The restrictions on the parameter $n$ in this definition follow from the following requirements. First, we require that $p_{n,k}^{[c]} \geqslant 0$ for all $k \in \mathbb{N}_0$, which holds true for all $n > 0$ if $c \geqslant 0$, and only for $n = -c\ell$, $\ell \in \mathbb{N}$, if $c < 0$. In the last case $p_{n,k}^{[c]} \equiv 0$ for $k > -\frac{n}{c}$, $k \in \mathbb{N}_0$, so that the sum in definition (2) is finite and consists of the summands with $k = 0, 1, \ldots, -\frac{n}{c}$. Further, we require that the integral $\int_{I_c} p_{n,k}^{[c]}(y)\, dy$ is finite. If $c < 0$, $n = -c\ell$ with some $\ell \in \mathbb{N}$ and $k = 0, 1, \ldots, -\frac{n}{c}$, then

$$\int_{I_c} p_{n,k}^{[c]}(y)\, dy = -\frac{1}{c}\binom{-n/c}{k} \int_0^1 t^k (1-t)^{-\frac{n}{c}-k}\, dt = \frac{1}{n-c}.$$

If $c = 0$, then

$$\int_{I_0} p_{n,k}^{[0]}(y)\, dy = \frac{1}{n}\frac{1}{k!}\int_0^{\infty} t^k e^{-t}\, dt = \frac{1}{n} \tag{3}$$

for all $n > 0$. Finally, if $c > 0$, then

$$\int_{I_c} p_{n,k}^{[c]}(y)\, dy = (-1)^k \frac{1}{c}\binom{-n/c}{k} \int_0^1 t^k (1-t)^{\frac{n}{c}-2}\, dt = \begin{cases} \frac{1}{n-c}, & n > c, \\ \infty, & \text{otherwise}, \end{cases} \tag{4}$$

and thus we need $n > c$.

Note that

$$\begin{aligned} p_{n,k}^{[c]}(x) &= p_{\frac{n}{c},k}^{[1]}(cx), \quad c > 0, \\ p_{n,k}^{[c]}(x) &= p_{-\frac{n}{c},k}^{[-1]}(-cx), \quad c < 0. \end{aligned} \tag{5}$$

By a change of variables we see that

$$\begin{aligned} \left(\mathbf{M}_{n,c}[f(\cdot)]\right)(x) &= \left(\mathbf{M}_{\frac{n}{c},1}\left[f\left(\frac{\cdot}{c}\right)\right]\right)(cx), \quad c > 0, \\ \left(\mathbf{M}_{n,c}[f(\cdot)]\right)(x) &= \left(\mathbf{M}_{-\frac{n}{c},-1}\left[f\left(-\frac{\cdot}{c}\right)\right]\right)(-cx), \quad c < 0. \end{aligned}$$

This means that we have only three essentially different operators: $\mathbf{M}_{n,-1}$, $\mathbf{M}_{n,0}$ and $\mathbf{M}_{n,1}$.

For $c = -1$, the operator (2) is the well-known Bernstein–Durrmeyer operator. It was introduced by Durrmeyer [7] and independently by Lupaş [12] as a modification of the classical Bernstein operator, and studied in detail by Derriennic [6]. The operator is well-defined on the domain $L_p[0, 1]$, $1 \leqslant p < \infty$, and on $C[0, 1]$. The operator $\mathbf{M}_{n,-1}$ reproduces constants and is contractive, i.e. the inequality

$$\|\mathbf{M}_{n,c}(f)\|_p \leqslant \|f\|_p \tag{6}$$

holds true with $c = -1$ and $1 \leqslant p \leqslant \infty$. For $f \in L_p[0, 1]$, $1 \leqslant p < \infty$, or for $f \in C[0, 1]$ the convergence $\|\mathbf{M}_{n,-1}(f) - f\|_p \to 0$, $n \to \infty$, takes place. The Bernstein–Durrmeyer operator is very well studied. There are well-known extensions of this operator to the multivariate case and to the $L_p$-spaces with Jacobi weights. For details and references see e.g. [4].

For $c = 0$, the operator $\mathbf{M}_{n,0}$ was introduced by Mazhar and Totik [13] and is usually called the Szász–Mirakjan–Durrmeyer operator. The operator $\mathbf{M}_{n,c}$ with $c > 0$ was introduced by Heilmann (for $c = 1$ independently by Sahai and Prasad [14]) and is usually called the Baskakov–Durrmeyer operator. The operators $\mathbf{M}_{n,c}$ with $c \geqslant 0$ were studied e.g. by Heilmann [8,9], Heilmann and Müller [10]. Using properties (1) and (3) resp. (4), it is not difficult to show that the operator $\mathbf{M}_{n,c}$ is well-defined if $f \in L_1(I_c)$ or $f \in L_\infty(I_c)$, and (6) holds for $p = 1$ and $\infty$ (e.g. [8]). Consequently, $\mathbf{M}_{n,c}$ is well-defined for all $f \in L_1(I_c) + L_\infty(I_c)$, in particular, for all $f \in L_p(I_c)$, $1 \leqslant p \leqslant \infty$. By the Riesz–Thorin interpolation theorem, (6) holds for all $1 \leqslant p \leqslant \infty$. The operator $\mathbf{M}_{n,c}$ reproduces constants. It was shown in [8, Theorem 3.2] that

$$\|\mathbf{M}_{n,c}(f) - f\|_p \to 0, \quad n \to \infty, \qquad f \in L_p[0, \infty), \quad 1 \leqslant p < \infty. \tag{7}$$

In fact, the right-hand side of (2) is also well-defined for some functions which do not belong to $L_p$-spaces, for example, for polynomials.

In the recent paper of Jetter, Stöckler and the author [4], the Bernstein–Durrmeyer operators (case $c = -1$) on simplices in $L_p$-spaces with Jacobi weights and their natural quasi-interpolants were studied. Using a new representation of the kernel of the Bernstein–Durrmeyer operator in terms of Jacobi polynomials, we showed that the sequence of the kernels is pointwise completely monotonic. Based on this fact we proved various direct results for the natural Bernstein–Durrmeyer quasi-interpolants. The aim of this paper is to obtain for $c \geqslant 0$ statements similar to those from [4].

The paper is organized as follows. In Section 2 we give representations of the kernels of the Durrmeyer operators in terms of special functions and prove the complete monotonicity property. In Section 3 we study a special differential operator $\mathbf{U}_{r,c}$ which plays an important role in investigations on Durrmeyer operators and their linear combinations. The main result of Section 3 is a Bernstein-type inequality which gives an estimate for $\|\mathbf{U}_{r,c}(\mathbf{M}_{n,c}(f))\|_p$, $f \in L_p(I_c)$. In Section 4 we apply results of Sections 2 and 3 to the natural quasi-interpolants of the Baskakov–Durrmeyer and the Szász–Mirakjan–Durrmeyer operators which were recently introduced by Abel [1]. In the case $c > 0$ we give a representation of Abel's quasi-interpolants as linear combinations of the Baskakov–Durrmeyer operators. Finally, we establish an estimate of Jackson–Favard type and a direct theorem in terms of a new K-functional.

In the paper we basically follow the consideration in [4], where the corresponding results for $c = -1$ can be found. Some of the proofs, like in Theorems 1 and 2 and Lemma 8, are similar to those in [4]. On the other hand, for Lemmas 3–7 and Theorems 4 and 5 we give direct proofs which extensively use properties of underlying special functions.

## 2. Kernels of the Durrmeyer operators

In this section we study the kernels of the operators (2). Interchanging integration and summation in (2), we can rewrite it as

$$(\mathbf{M}_{n,c}(f))(x) = (n-c)\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)\int_{I_c} p_{n,k}^{[c]}(y)f(y)\,dy = (n-c)\int_{I_c} T_{n,c}(x,y)\,f(y)\,dy,$$

with the kernel function

$$T_{n,c}(x,y) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)p_{n,k}^{[c]}(y), \quad x, y \in I_c. \tag{8}$$

For $f \in L_1(I_c)$ or $f \in L_\infty(I_c)$, the interchanging of integration and summation can be easily justified by the Lebesgue dominated convergence theorem, basing on the properties (1) and (3) resp. (4); for $f \in L_p(I_c)$ with $1 < p < \infty$ we exploit again the fact that $f \in L_1(I_c) + L_\infty(I_c)$.

In the following lemma we give representations of the function $T_{n,c}(x, y)$ in terms of special functions.

**Lemma 1.** *Let $c \in \mathbb{R}$ and $n$ be as in Definition* 1. *The series in* (8) *converges for all $x, y \in I_c$ and it holds*

$$T_{n,c}(x,y) = [(1+cx)(1+cy)]^{-\frac{n}{c}}\,{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,\frac{c^2xy}{(1+cx)(1+cy)}\right), \quad c \neq 0, \tag{9}$$

$$T_{n,0}(x,y) = e^{-n(x+y)}I_0(2n\sqrt{xy}), \quad c = 0. \tag{10}$$

Here ${}_2F_1$ is the hypergeometric function and $I_0$ is the modified Bessel function of first kind of order 0.

**Proof.** If $c < 0$, as $n = -c\ell$ with some $\ell \in \mathbb{N}$, the sum in (8) is finite (in particular, $T_{n,c}(x, y)$ is a polynomial in $x$ and $y$). If $c \geqslant 0$, since $\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1$ and $p_{n,k}^{[c]} \geqslant 0$ for all $k \in \mathbb{N}_0$ (moreover, $p_{n,k}^{[c]}(x) = 0$ only if $x = 0$ and $k \neq 0$), it follows

$$p_{n,k}^{[c]}(x) \leqslant 1$$

(moreover, $p_{n,k}^{[c]}(x) = 1$ only if $x = 0$ and $k = 0$). Thus, for all $x, y \in I_c$

$$0 < T_{n,c}(x,y) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)p_{n,k}^{[c]}(y) \leqslant \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1. \tag{11}$$

The representations (9), (10) follow immediately from the definitions of the functions ${}_2F_1$, $I_0$ via the power series [2, (15.1.1)] resp. [2, (9.6.10)]. □

The estimate (11) shows, in particular, that the function $T_{n,c}(x, \cdot)$ belongs to the space $L_\infty(I_c)$ for each fixed $x \in I_c$. In fact, $T_{n,c}(x, \cdot) \in L_p(I_c)$ for each $1 \leqslant p \leqslant \infty$. For $c < 0$ it is obvious and for $c \geqslant 0$ it follows form the following lemma.

**Lemma 2.** *For each $x, y \in [0, \infty)$ it holds*

$$T_{n,c}(x, y) \leqslant \frac{\Gamma\left(2\frac{n}{c} - 1\right)}{\left[\Gamma\left(\frac{n}{c}\right)\right]^2} (1 + cx)^{\frac{n}{c}-1} \frac{1}{(1 + cx + cy)^{\frac{n}{c}}}, \quad c > 0, \tag{12}$$

$$T_{n,0}(x, y) \leqslant e^{-n(\sqrt{x}-\sqrt{y})^2}, \quad c = 0. \tag{13}$$

**Proof.** Let first $c > 0$. Using the formula ${}_2F_1(\alpha, \beta, \gamma, z) = (1 - z)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma, z)$ [2, (15.3.3)], we can rewrite (9) as

$$T_{n,c}(x, y) = \frac{[(1 + cx)(1 + cy)]^{\frac{n}{c}-1}}{(1 + cx + cy)^{2\frac{n}{c}-1}}\ {}_2F_1\left(1 - \frac{n}{c}, 1 - \frac{n}{c}, 1, \frac{c^2xy}{(1 + cx)(1 + cy)}\right).$$

Now, since $0 \leqslant \frac{c^2xy}{(1+cx)(1+cy)} < 1$ for $x, y \in [0, \infty)$,

$$\begin{aligned}
&{}_2F_1\left(1 - \frac{n}{c}, 1 - \frac{n}{c}, 1, \frac{c^2xy}{(1 + cx)(1 + cy)}\right) \\
&= \sum_{k=0}^{\infty} \left(\frac{\left(1 - \frac{n}{c}\right)_k}{k!}\right)^2 \left(\frac{c^2xy}{(1 + cx)(1 + cy)}\right)^k \\
&\leqslant \sum_{k=0}^{\infty} \left(\frac{\left(1 - \frac{n}{c}\right)_k}{k!}\right)^2 = {}_2F_1\left(1 - \frac{n}{c}, 1 - \frac{n}{c}, 1, 1\right) = \frac{\Gamma\left(2\frac{n}{c} - 1\right)}{\left[\Gamma\left(\frac{n}{c}\right)\right]^2},
\end{aligned}$$

the last equality follows from the formula ${}_2F_1(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$ ($\gamma \neq 0, -1, -2, \ldots$, $\Re(\gamma - \alpha - \beta) > 0$) [2, (15.1.20)]. An elementary estimate gives (12).

To prove (13), we use (10) and the inequality

$$I_0(z) \leqslant e^z, \quad z \geqslant 0,$$

which follows immediately from the integral representation [2, (9.6.18)]

$$I_0(z) = \frac{1}{\pi} \int_{-1}^{1} e^{-tz} \frac{1}{\sqrt{1 - t^2}}\, dt. \qquad \square \tag{14}$$

In [4] we proved for the kernel of the Bernstein–Durrmeyer operator that the sequence $\{T_{n,-1}(x, y)\}_{n\in\mathbb{N}}$ is pointwise completely monotonic for any fixed $x, y \in [0, 1]$, i.e. the inequalities

$$(-1)^r \Delta_1^r T_{n,-1}(x, y) = \sum_{l=0}^{r} (-1)^\ell \binom{r}{\ell} T_{n+\ell,-1}(x, y) \geqslant 0$$

hold for all $n \in \mathbb{N}$, $r \in \mathbb{N}_0$. Because of (5) it is clear that a similar statement holds also for $\mathbf{M}_{n,c}$ with an arbitrary $c < 0$, namely

$$(-1)^r \Delta_{-c}^r T_{n,c}(x, y) = \sum_{l=0}^{r} (-1)^\ell \binom{r}{\ell} T_{n-\ell c,c}(x, y) \geqslant 0, \quad c < 0. \tag{15}$$

Here we prove an analogue of this result for $c \geqslant 0$. The following statement takes place.

**Theorem 1.** *Let $c \geqslant 0$, $n > c$. For any fixed $x, y \in [0, \infty)$, the function $T_{n,c}(x, y)$ is completely monotonic with respect to $n$.*

**Proof.** We will use the following two simple facts about completely monotonic functions:

(a) The function $\varphi(u) = a^u$, with $0 < a \leqslant 1$, is completely monotonic.
(b) Suppose that a function $f(u, t)$ is completely monotonic with respect to $u$ for any $t$ and integrable with respect to a non-negative measure $dm(t)$. Then the function $J(u) = \int f(u, t)\, dm(t)$ is completely monotonic.

A good reference on completely monotonic functions and sequences is [15].

First we consider the case $c > 0$. Using the quadratic transformation [2, (15.3.27)]

$$ {}_2F_1(\alpha, \beta, \alpha - \beta + 1, z) = (1 + \sqrt{z})^{-2\alpha}\, {}_2F_1\left(\alpha, \alpha - \beta + \frac{1}{2}, 2\alpha - 2\beta + 1, \frac{4\sqrt{z}}{(1 + \sqrt{z})^2}\right), $$

we rewrite (9) as

$$ T_{n,c}(x, y) = \left(\sqrt{c^2xy} + \sqrt{(1 + cx)(1 + cy)}\right)^{-\frac{2n}{c}} \times {}_2F_1\left(\frac{n}{c}, \frac{1}{2}, 1, \frac{4\sqrt{c^2xy(1 + cx)(1 + cy)}}{\left(\sqrt{c^2xy} + \sqrt{(1 + cx)(1 + cy)}\right)^2}\right). $$

Since

$$ 0 \leqslant \frac{4\sqrt{c^2xy(1 + cx)(1 + cy)}}{\left(\sqrt{c^2xy} + \sqrt{(1 + cx)(1 + cy)}\right)^2} < 1 $$

for $x, y \in [0, \infty)$, we can use Euler's representation formula [2, (15.3.1)]

$$ {}_2F_1(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma - \beta)} \int_0^1 t^{\beta - 1}\,(1 - t)^{\gamma - \beta - 1}\,(1 - zt)^{-\alpha}\, dt $$

for $\Re(\gamma) > \Re(\beta) > 0$, in the $z$ plane cut along the real axis from 1 to $\infty$. We get

$$ T_{n,c}(x, y) = \frac{1}{\pi}\left(\sqrt{c^2xy} + \sqrt{(1 + cx)(1 + cy)}\right)^{-\frac{2n}{c}} \times \int_0^1 \left(1 - \frac{4t\sqrt{c^2xy(1 + cx)(1 + cy)}}{\left(\sqrt{c^2xy} + \sqrt{(1 + cx)(1 + cy)}\right)^2}\right)^{-\frac{n}{c}} \frac{dt}{\sqrt{t(1 - t)}}. $$

Thus,

$$ T_{n,c}(x, y) = \frac{1}{\pi}\int_0^1 [\Phi(x, y, t)]^n\, \frac{dt}{\sqrt{t(1 - t)}}, $$

where

$$ \Phi(x, y, t) = \left[\left(\sqrt{c^2xy} + \sqrt{(1 + cx)(1 + cy)}\right)^2 - 4t\sqrt{c^2xy(1 + cx)(1 + cy)}\right]^{-\frac{1}{c}}. $$

According to (a) and (b), the statement of the theorem will follow if we prove that $0 < \Phi(x, y, t) \leqslant 1$ for all $t \in [0, 1]$, $x, y \in [0, \infty)$. For, we see that

$$\left(\sqrt{c^2xy} + \sqrt{(1+cx)(1+cy)}\right)^2 - 4t\sqrt{c^2xy(1+cx)(1+cy)}$$
$$\geqslant \left(\sqrt{c^2xy} - \sqrt{(1+cx)(1+cy)}\right)^2 \geqslant 1,$$

the last inequality follows from

$$\sqrt{(1+cx)(1+cy)} - \sqrt{c^2xy} \geqslant 1$$

which is equivalent to the obvious $cx + cy \geqslant 2c\sqrt{xy}$.

Now let us consider the case $c = 0$. Using (10) and (14), we obtain

$$T_{n,0}(x, y) = \frac{1}{\pi}\int_{-1}^{1} [\Phi(x, y, t)]^n \, \frac{1}{\sqrt{1-t^2}}\, dt,$$

with

$$\Phi(x, y, t) = e^{-(x+y+2t\sqrt{xy})}.$$

Again, to prove the complete monotonicity it is enough to prove that $0 < \Phi(x, y, t) \leqslant 1$ for all $t \in [-1, 1]$, $x, y \in [0, \infty)$. The first inequality is obvious while the second inequality $e^{-(x+y+2t\sqrt{xy})} \leqslant 1$ follows from $x + y + 2t\sqrt{xy} \geqslant (\sqrt{x} - \sqrt{y})^2 \geqslant 0$. The proof of Theorem 1 is now completed. □

## 3. The differential operator $\mathbf{U}_{r,c}$ and the Bernstein inequality

Let $c \in \mathbb{R}$ and $r \in \mathbb{N}_0$. We define the following differential operators of order $2r$:

$$\mathbf{U}_{r,c} := \frac{(-1)^r}{(r!)^2}\frac{d^r}{dx^r}\left(x^r(1+cx)^r\frac{d^r}{dx^r}\right), \quad r \in \mathbb{N}, \tag{16}$$

and $\mathbf{U}_{0,c} := \mathbf{I}$, where $\mathbf{I}$ denotes the identity operator. This operator was introduced by Derriennic [6] in the case of $r = 1$, $c = -1$ and by Heilmann [9] in the general case. The operator (16) plays an important role in investigations on the Durrmeyer operators (2). One of the most important properties is that the operators $\mathbf{U}_{r,c}$ and $\mathbf{M}_{n,c}$ commute, i.e.

$$\mathbf{U}_{r,c}(\mathbf{M}_{n,c}(f)) = \mathbf{M}_{n,c}(\mathbf{U}_{r,c}(f)) \tag{17}$$

if $f, \mathbf{U}_{r,c}(f) \in L_p(I_c)$. In this general setting it was proved by Heilmann [9].

Our first aim in this section is to show that, if $c \neq 0$, $\mathbf{U}_{r,c}(\mathbf{M}_{n,c})$ can be represented as a linear combination of the operators $\mathbf{M}_{\ell,c}$. The corresponding result for $c = -1$ reads

$$\frac{1}{\binom{n}{r}}\mathbf{U}_{r,-1}(\mathbf{M}_{n,-1}) = \frac{n+1}{r}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}\binom{n+r-\ell}{r-1}\mathbf{M}_{n-\ell,-1}, \tag{18}$$

this is the one-dimensional unweighted case of formula (15) in [4]. This formula was obtained in [4] on the base of spectral properties of the operators $\mathbf{M}_{n,-1}$ and $\mathbf{U}_{r,-1}$. Here we choose another approach which extensively uses properties of the special functions involved. We start with $r = 1$.

**Lemma 3.** *If* $c \neq 0$ *then*

$$\frac{1}{n}\,\mathbf{U}_{1,c}(\mathbf{M}_{n,c}) = \left(-\frac{n}{c}+1\right)(\mathbf{M}_{n,c} - \mathbf{M}_{n+c,c}). \tag{19}$$

**Proof.** If $c = -1$, this is a particular case of (18), and for an arbitrary $c < 0$ it follows by a suitable change of the variables. Now let $c > 0$. We first prove that

$$\frac{1}{n}\,\mathbf{U}_{1,c}(T_{n,c}(x,y)) = \left(-\frac{n}{c}+1\right)T_{n,c}(x,y) + \frac{n}{c}\,T_{n+c,c}(x,y), \quad x,y \in [0,\infty); \tag{20}$$

the differential operator here is taken with respect to the variable $x$. We rewrite (9) as

$$T_{n,c}(x,y) = (c^2xy)^{-\frac{n}{c}}\,z^{\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right) \quad \text{with } z = \frac{c^2xy}{(1+cx)(1+cy)}.$$

Using the formulae [2, (15.2.3)] $\frac{d}{dz}\left\{z^{\alpha}\,{}_2F_1(\alpha,\beta,\gamma,z)\right\} = \alpha z^{\alpha-1}\,{}_2F_1(\alpha+1,\beta,\gamma,z)$ and [2, (15.1.1)] ${}_2F_1(\alpha,\beta,\gamma,z) = {}_2F_1(\beta,\alpha,\gamma,z)$, we find

$$\begin{aligned}\frac{\partial}{\partial x}T_{n,c}(x,y) = &-\frac{n}{cx}\,(c^2xy)^{-\frac{n}{c}}\,z^{\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right)\\ &+\frac{n}{cx(1+cx)}\,(c^2xy)^{-\frac{n}{c}}\,z^{\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c}+1,1,z\right).\end{aligned} \tag{21}$$

Using [2, (15.2.3)] once again, we obtain

$$\begin{aligned}&\frac{1}{n}\,\mathbf{U}_{1,c}(T_{n,c}(x,y))\\ &= \frac{\partial}{\partial x}\left\{\frac{1}{c}(1+cx)(c^2xy)^{-\frac{n}{c}}z^{\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right) - \frac{1}{c}(c^2xy)^{-\frac{n}{c}}z^{\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c}+1,1,z\right)\right\}\\ &= \left(-\frac{n}{c}+1\right)(c^2xy)^{-\frac{n}{c}}\,z^{\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right) - \frac{1}{cx}\,(c^2xy)^{-\frac{n}{c}}\,z^{\frac{n}{c}}\left[\frac{n}{c}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right)\right.\\ &\quad\left.-\,2\frac{n}{c}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c}+1,1,z\right) + \frac{n}{c}\,\frac{1}{1+cx}\;{}_2F_1\left(\frac{n}{c}+1,\frac{n}{c}+1,1,z\right)\right].\end{aligned} \tag{22}$$

By the formula [2, (15.2.15)]

$$(\gamma-\alpha-\beta)\,{}_2F_1(\alpha,\beta,\gamma,z) + \alpha(1-z)\,{}_2F_1(\alpha+1,\beta,\gamma,z) - (\gamma-\beta)\,{}_2F_1(\alpha,\beta-1,\gamma,z) = 0,$$

with $\alpha = \frac{n}{c}$, $\beta = \frac{n}{c}+1$, $\gamma = 1$, we find

$$\frac{n}{c}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right) - 2\frac{n}{c}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c}+1,1,z\right) = -\frac{n}{c}(1-z)\;{}_2F_1\left(\frac{n}{c}+1,\frac{n}{c}+1,1,z\right).$$

Substituting this into (22) and using $1 - z - \frac{1}{1+cx} = \frac{cx}{(1+cx)(1+cy)}$, we finally obtain

$$\begin{aligned}\frac{1}{n}\,\mathbf{U}_{1,c}(T_{n,c}(x,y)) &= \left(-\frac{n}{c}+1\right)[(1+cx)(1+cy)]^{-\frac{n}{c}}\;{}_2F_1\left(\frac{n}{c},\frac{n}{c},1,z\right)\\ &\quad+\frac{n}{c}\,[(1+cx)(1+cy)]^{-\frac{n}{c}-1}\;{}_2F_1\left(\frac{n}{c}+1,\frac{n}{c}+1,1,z\right)\\ &= \left(-\frac{n}{c}+1\right)T_{n,c}(x,y) + \frac{n}{c}\,T_{n+c,c}(x,y),\end{aligned}$$

which is (20).

Now take a function $f \in L_p[0,\infty)$, $1 \leqslant p \leqslant \infty$. Using (20), we write

$$\begin{aligned}
\frac{1}{n}\mathbf{U}_{1,c}(\mathbf{M}_{n,c}(f))(x) &= -(n-c)\frac{1}{n}\frac{\partial}{\partial x}\left\{x(1+cx)\frac{\partial}{\partial x}\left(\int_0^\infty T_{n,c}(x,y)f(y)\,dy\right)\right\}\\
&= -(n-c)\frac{1}{n}\frac{\partial}{\partial x}\left\{x(1+cx)\int_0^\infty \frac{\partial}{\partial x}T_{n,c}(x,y)f(y)\,dy\right\} \qquad (23)\\
&= -(n-c)\int_0^\infty \frac{1}{n}\frac{\partial}{\partial x}\left\{x(1+cx)\frac{\partial}{\partial x}T_{n,c}(x,y)\right\}f(y)\,dy \qquad (24)\\
&= (n-c)\int_0^\infty \frac{1}{n}\mathbf{U}_{1,c}(T_{n,c}(x,y))f(y)\,dy\\
&= (n-c)\int_0^\infty \left(\left(-\frac{n}{c}+1\right)T_{n,c}(x,y)+\frac{n}{c}\,T_{n+c,c}(x,y)\right)f(y)\,dy\\
&= \left(-\frac{n}{c}+1\right)((\mathbf{M}_{n,c}(f))(x)-(\mathbf{M}_{n+c,c}(f))(x)).
\end{aligned}$$

Thus, we obtain (19) formally. It remains to justify the interchanges of integration and differentiation in (23) and (24). We shall use the following corollary from the Lebesgue dominated convergence theorem, e.g. [5]: *Let $\mu$ be a nonnegative measure on a space $X$ and let a function $f : X \times (a,b) \to \mathbb{R}$ be such that the function $x \mapsto f(x,\alpha)$ is integrable for each $\alpha \in (a,b)$. If the function $\alpha \mapsto f(x,\alpha)$ is differentiable for almost all $x$, and there exists an integrable function $\Phi(x)$ such that $\left|\frac{\partial f(x,\alpha)}{\partial\alpha}\right| \leqslant \Phi(x)$ for almost all $x$ and for all $\alpha$, then the function $J(\alpha) := \int_X f(x,\alpha)\,\mu(dx)$ is differentiable, and $J'(\alpha) = \int_X \frac{\partial f(x,\alpha)}{\partial\alpha}\,\mu(dx)$.*

Fix a point $x \in [0,\infty)$. To justify the interchanging of integration and differentiation in (23), we have to prove that the function $\frac{\partial}{\partial x}T_{n,c}(x,y)\,f(y)$ can be bounded by an integrable function of $y$ uniformly in some neighbourhood of $x$. A suitable bound is

$$\left|\frac{\partial}{\partial x}T_{n,c}(x,y)\,f(y)\right| \leqslant \left[\frac{n}{cx}\left(\frac{\Gamma\left(2\frac{n}{c}-1\right)}{\left[\Gamma\left(\frac{n}{c}\right)\right]^2}+\frac{\Gamma\left(2\frac{n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{n}{c}+1\right)}\right)\frac{(1+cx)^{\frac{n}{c}-1}}{(1+cx+cy)^{\frac{n}{c}}}\right]|f(y)|,$$

which is clearly integrable, since $f \in L_p[0,\infty)$ and the expression in square brackets belongs to $L_q[0,\infty)$ for every $q$. This estimate can be obtained from the representation (21) using for the first term the estimate (12), and for the second term a similar estimate which can be obtained in a similar way. To justify the interchanging of integration and differentiation in (24), we have to find a suitable estimate for the function $\left[\left(-\frac{n}{c}+1\right)T_{n,c}(x,y)+\frac{n}{c}\,T_{n+c,c}(x,y)\right]f(y)$. Here we just use (12) twice. □

To find an analogue of (18) for $c \neq -1$ and $r > 1$, we first establish the following recursive relation for the differential operators $\mathbf{U}_{r,c}$.

**Lemma 4.** *Let $c \in \mathbb{R}$ and $r \in \mathbb{N}_0$. It holds*

$$\mathbf{U}_{r+1,c} = \frac{1}{(r+1)^2}\left(\mathbf{U}_{1,c}+cr(r+1)\mathbf{I}\right)\mathbf{U}_{r,c}. \qquad (25)$$

**Proof.** Obviously,

$$\mathbf{U}_{1,c} = -(1+2cx)\frac{d}{dx}-(x+cx^2)\frac{d^2}{dx^2}, \qquad (26)$$

and

$$\mathbf{U}_{\rho,c} = \frac{(-1)^\rho}{(\rho!)^2}\frac{d^\rho}{dx^\rho}\left(\sum_{m=0}^{\rho}\binom{\rho}{m}c^m x^{\rho+m}\frac{d^\rho}{dx^\rho}\right)$$
$$= \frac{(-1)^\rho}{(\rho!)^2}\sum_{k=0}^{\rho}\sum_{m=0}^{\rho}\binom{\rho}{k}\binom{\rho}{m}c^m\frac{(\rho+m)!}{(m+k)!}x^{m+k}\frac{d^{\rho+k}}{dx^{\rho+k}}. \tag{27}$$

Similarly to (27),

$$\frac{d}{dx}(\mathbf{U}_{\rho,c}) = \frac{(-1)^\rho}{(\rho!)^2}\sum_{k=0}^{\rho+1}\sum_{m=0}^{\rho}\binom{\rho+1}{k}\binom{\rho}{m}c^m\frac{(\rho+m)!}{(m+k-1)!}x^{m+k-1}\frac{d^{\rho+k}}{dx^{\rho+k}}, \tag{28}$$

$$\frac{d^2}{dx^2}(\mathbf{U}_{\rho,c}) = \frac{(-1)^\rho}{(\rho!)^2}\sum_{k=0}^{\rho+2}\sum_{m=0}^{\rho}\binom{\rho+2}{k}\binom{\rho}{m}c^m\frac{(\rho+m)!}{(m+k-2)!}x^{m+k-2}\frac{d^{\rho+k}}{dx^{\rho+k}}. \tag{29}$$

Using (26), (27)–(29) with $\rho = r$ and suitable changes of the summation indexes, we obtain by a long but simple calculation for the right-hand side of (25)

$$\frac{1}{(r+1)^2}(\mathbf{U}_{1,c} + cr(r+1)\mathbf{I})\,\mathbf{U}_{r,c}$$
$$= \frac{-1}{(r+1)^2}\left((1+2cx)\frac{d}{dx}(\mathbf{U}_{r,c}) + (x+cx^2)\frac{d^2}{dx^2}(\mathbf{U}_{r,c}) - cr(r+1)\mathbf{U}_{r,c}\right)$$
$$= \frac{(-1)^{r+1}}{((r+1)!)^2}\sum_{k=0}^{r+2}\sum_{m=0}^{r+1}c^m x^{m+k-1}\frac{d^{r+k}}{dx^{r+k}}\left\{\binom{r+1}{k}\binom{r}{m}\frac{(r+m)!}{(m+k-1)!}\right.$$
$$+2\binom{r+1}{k}\binom{r}{m-1}\frac{(r+m-1)!}{(m+k-2)!} + \binom{r+2}{k}\binom{r}{m}\frac{(r+m)!}{(m+k-2)!}$$
$$\left.+\binom{r+2}{k}\binom{r}{m-1}\frac{(r+m-1)!}{(m+k-3)!} - r(r+1)\binom{r}{k}\binom{r}{m-1}\frac{(r+m-1)!}{(m+k-1)!}\right\}$$
$$= \frac{(-1)^{r+1}}{((r+1)!)^2}\sum_{k=0}^{r+2}\sum_{m=0}^{r+1}\binom{r+1}{k-1}\binom{r+1}{m}c^m\frac{(m+r+1)!}{(m+k-1)!}x^{m+k-1}\frac{d^{r+k}}{dx^{r+k}}$$
$$= \frac{(-1)^{r+1}}{((r+1)!)^2}\sum_{k=0}^{r+1}\sum_{m=0}^{r+1}\binom{r+1}{k}\binom{r+1}{m}c^m\frac{(m+r+1)!}{(m+k)!}x^{m+k}\frac{d^{r+k+1}}{dx^{r+k+1}} = \mathbf{U}_{r+1,c},$$

where the last equality follows by (27) for $\rho = r + 1$. □

Now we present the desired linear combination representation of $\mathbf{U}_{r,c}(\mathbf{M}_{n,c})$.

**Lemma 5.** *Let $c \neq 0$ and $n$ be as in Definition 1. It holds*

$$\frac{r!}{n(n+c)\cdots(n+(r-1)c)}\mathbf{U}_{r,c}(\mathbf{M}_{n,c}) = \frac{\left(-\frac{n}{c}+1\right)}{r}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}\binom{-\frac{n}{c}+r-\ell}{r-1}\mathbf{M}_{n+\ell c,c}. \tag{30}$$

**Proof.** We prove the lemma by induction in $r$. The statement for $r = 1$ is just (19). For the induction step $r \to r+1$ we use (25) and the induction hypothesis (30) to obtain

$$\begin{aligned}
&\frac{(r+1)!}{n(n+c)\cdots(n+rc)}\,\mathbf{U}_{r+1,c}(\mathbf{M}_{n,c})\\
&\quad= \frac{(r+1)!}{n(n+c)\cdots(n+rc)}\,\frac{1}{(r+1)^2}\,(\mathbf{U}_{1,c}+cr(r+1)\mathbf{I})\,\mathbf{U}_{r,c}(\mathbf{M}_{n,c})\\
&\quad= \frac{1}{(r+1)(n+rc)}\,(\mathbf{U}_{1,c}+cr(r+1)\mathbf{I})\\
&\qquad\times\left\{\frac{\left(-\frac{n}{c}+1\right)}{r}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\binom{-\frac{n}{c}+r-\ell}{r-1}\mathbf{M}_{n+\ell c,c}\right\}.
\end{aligned}$$

Using (19) for computing of $\mathbf{U}_{1,c}(\mathbf{M}_{n+\ell c,c})$, we continue as

$$\begin{aligned}
&= \frac{\left(-\frac{n}{c}+1\right)}{(r+1)(n+rc)}\left\{\frac{1}{r}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\binom{-\frac{n}{c}+r-\ell}{r-1}(n+\ell c)\left(-\frac{n}{c}-\ell+1\right)(\mathbf{M}_{n+\ell c,c}\right.\\
&\qquad\left.-\mathbf{M}_{n+(\ell+1)c,c}) + c(r+1)\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\binom{-\frac{n}{c}+r-\ell}{r-1}\mathbf{M}_{n+\ell c,c}\right\}\\
&= \frac{\left(-\frac{n}{c}+1\right)}{(r+1)(n+rc)}\sum_{\ell=0}^{r+1}(-1)^{\ell}\mathbf{M}_{n+\ell c,c}\left\{\frac{1}{r}\binom{r}{\ell}\binom{-\frac{n}{c}+r-\ell}{r-1}(n+\ell c)\left(-\frac{n}{c}-\ell+1\right)\right.\\
&\qquad+\frac{1}{r}\binom{r}{\ell-1}\binom{-\frac{n}{c}+r-\ell+1}{r-1}(n+\ell c-c)\left(-\frac{n}{c}-\ell+2\right)\\
&\qquad\left.+c(r+1)\binom{r}{\ell}\binom{-\frac{n}{c}+r-\ell}{r-1}\right\}\\
&= \frac{\left(-\frac{n}{c}+1\right)}{r+1}\sum_{\ell=0}^{r+1}(-1)^{\ell}\binom{r+1}{\ell}\binom{-\frac{n}{c}+r+1-\ell}{r}\mathbf{M}_{n+\ell c,c},
\end{aligned}$$

which is (30) for $r+1$. □

**Remark.** The right-hand side of (30) is in fact a divided difference. The corresponding formula has a simpler form if we consider the modified operator

$$(\mathbf{T}_{n,c}(f))(x) := \int_{I_c} T_{n,c}(x,y)\,f(y)\,dy = \frac{1}{n-c}\,(\mathbf{M}_{n,c}(f))(x).$$

Let

$$[z_0, z_1, \ldots, z_k; g(z)]_z = \sum_{j=0}^{k}\frac{g(z_j)}{\prod\limits_{\substack{m=0\\ m\neq j}}^{k}(z_j-z_m)}$$

denote the $k$th order divided difference of a function $g(z)$ with knots $z_0, z_1, \ldots, z_k$. Then (30) can be rewritten as

$$\begin{aligned}
&\frac{r!}{n(n+c)\cdots(n+(r-1)c)}\mathbf{U}_{r,c}(\mathbf{T}_{n,c})\\
&\quad= [n, n+c, \ldots, n+rc; (z-c)\cdots(z-rc)\mathbf{T}_{z,c}]_z. \qquad (31)
\end{aligned}$$

Now let us consider the case $c = 0$. Passing formally to the limit as $c \to 0$ in (31), we obtain the following representation for $\mathbf{U}_{r,c}(\mathbf{T}_{n,c})$.

**Lemma 6.** *It holds*

$$\frac{r!}{n^r}\,\mathbf{U}_{r,0}(\mathbf{T}_{n,0}) = \frac{1}{r!}\,\frac{\partial^r}{\partial n^r}\,(n^r\,\mathbf{T}_{n,0}). \tag{32}$$

Let us prove this statement. First we consider the case $r = 1$ in a separate lemma.

**Lemma 7.** *It holds*

$$\frac{1}{n}\,\mathbf{U}_{1,0}(\mathbf{T}_{n,0}) = \frac{\partial}{\partial n}(n\mathbf{T}_{n,0}). \tag{33}$$

**Proof.** We first prove that

$$\frac{1}{n}\,\mathbf{U}_{1,0}(T_{n,0}(x,y)) = \frac{\partial}{\partial n}(nT_{n,0}(x,y)), \tag{34}$$

the differential operator $\mathbf{U}_{1,0}$ here is taken with respect to $x$. We rewrite (10) as

$$T_{n,0}(x,y) = e^{-n(x+y)}I_0(z), \quad z = 2n\sqrt{xy}.$$

Using the formula $I_0'(z) = I_1(z)$ [2, (9.6.27)], we find

$$\frac{\partial}{\partial x}T_{n,0}(x,y) = -n\,T_{n,0}(x,y) + \frac{1}{2x}\,e^{-n(x+y)}zI_1(z).$$

Applying the formula $\frac{1}{z}\frac{d}{dz}\{zI_1(z)\} = I_0(z)$ [2, (9.6.28)], we obtain for the left-hand side of (34)

$$\begin{aligned}\frac{1}{n}\mathbf{U}_{1,0}(T_{n,0}(x,y)) &= \frac{\partial}{\partial x}\left\{x\,T_{n,0}(x,y) - \frac{1}{2n}\,e^{-n(x+y)}zI_1(z)\right\}\\ &= T_{n,0}(x,y) - n(x+y)\,T_{n,0}(x,y) + e^{-n(x+y)}\,2n\sqrt{xy}\,I_1(2n\sqrt{xy}).\end{aligned} \tag{35}$$

For the right-hand side of (34) we have

$$\begin{aligned}\frac{\partial}{\partial n}(nT_{n,0}(x,y)) &= T_{n,0}(x,y) + n\,\frac{\partial}{\partial n}\left\{e^{-n(x+y)}\,I_0(z)\right\}\\ &= T_{n,0}(x,y) - n(x+y)\,e^{-n(x+y)}\,I_0(z) + ne^{-n(x+y)}\,2\sqrt{xy}\,I_1(z),\end{aligned}$$

which coincides with (35), and (34) is proved. Now let $f \in L_p[0,\infty)$, $1 \leqslant p \leqslant \infty$. Using (34), we write

$$\begin{aligned}\frac{1}{n}\,\mathbf{U}_{1,0}(\mathbf{T}_{n,0}(f))(x) &= -\frac{1}{n}\frac{\partial}{\partial x}\left\{x\frac{\partial}{\partial x}\left(\int_0^\infty T_{n,0}(x,y)f(y)\,dy\right)\right\}\\ &= -\frac{1}{n}\frac{\partial}{\partial x}\left\{x\int_0^\infty \frac{\partial}{\partial x}T_{n,0}(x,y)f(y)\,dy\right\} && (36)\\ &= -\int_0^\infty \frac{1}{n}\frac{\partial}{\partial x}\left\{x\frac{\partial}{\partial x}T_{n,0}(x,y)\right\} f(y)\,dy && (37)\\ &= \int_0^\infty \frac{1}{n}\mathbf{U}_{1,0}(T_{n,0}(x,y))f(y)\,dy = \int_0^\infty \frac{\partial}{\partial n}(nT_{n,0}(x,y))f(y)\,dy\\ &= \frac{\partial}{\partial n}\left\{n\int_0^\infty T_{n,0}(x,y)f(y)\,dy\right\} = \frac{\partial}{\partial n}(n\mathbf{T}_{n,0}(f)(x)). && (38)\end{aligned}$$

We justify the interchanging of integration and differentiation in (36)–(38) as in Lemma 3. For (36), we need to prove that the function

$$\frac{\partial}{\partial x} T_{n,0}(x,y) f(y) = \left[ -n\, T_{n,0}(x,y) + \frac{1}{2x}\, e^{-n(x+y)} 2n\sqrt{xy}\, I_1(2n\sqrt{xy}) \right] f(y)$$

can be bounded by an integrable function of $y$ uniformly in some neighbourhood of $x$. For (37) and (38), we need to estimate the function

$$\frac{1}{n} \mathbf{U}_{1,0}(T_{n,0}(x,y)) f(y) = \left[ (1-n(x+y))\, T_{n,0}(x,y) + e^{-n(x+y)} 2n\sqrt{xy} I_1(2n\sqrt{xy}) \right] f(y).$$

As in Lemma 3, we can show that the expressions in square brackets belong to $L_q[0,\infty)$ for each $q$. In both cases we use for the first terms the estimate (13) and for the second terms the estimate

$$e^{-n(x+y)}\, I_1(2n\sqrt{xy}) \leqslant n\sqrt{xy}\, e^{-n(\sqrt{x}-\sqrt{y})^2},$$

which can be again obtained from the integral representation of the function $I_1$ [2, (9.6.18)]. □

**Proof of Lemma 6.** We prove (32) by induction in $r$. The statement for $r = 1$ is just proved (33). For the induction step $r \to r+1$ we use the recursion (25), the induction hypothesis (32) and (33)

$$\begin{aligned}
&\frac{(r+1)!}{n^{r+1}} \mathbf{U}_{r+1,0}(\mathbf{T}_{n,0}) \\
&= \frac{(r+1)!}{n^{r+1}} \frac{1}{(r+1)^2} \mathbf{U}_{1,0}(\mathbf{U}_{r,0}(\mathbf{T}_{n,0})) \\
&= \frac{1}{(r+1)\,n} \mathbf{U}_{1,0}\left( \frac{1}{r!} \frac{\partial^r}{\partial n^r} (n^r\, \mathbf{T}_{n,0}) \right) = \frac{1}{(r+1)!\,n} \frac{\partial^r}{\partial n^r} \left( n^{r+1} \frac{\partial}{\partial n} (n\mathbf{T}_{n,0}) \right) \\
&= \frac{1}{(r+1)!\,n} \frac{\partial^r}{\partial n^r} \left( n^{r+1}\, \mathbf{T}_{n,0} + n^{r+2} \frac{\partial}{\partial n} (\mathbf{T}_{n,0}) \right) \\
&= \frac{1}{(r+1)!\,n} \sum_{\ell=0}^{r} \binom{r}{\ell} \left\{ \frac{\partial^{r-\ell}}{\partial n^{r-\ell}} (n^{r+1}) \frac{\partial^\ell}{\partial n^\ell} (\mathbf{T}_{n,0}) + \frac{\partial^{r-\ell}}{\partial n^{r-\ell}} (n^{r+2}) \frac{\partial^{\ell+1}}{\partial n^{\ell+1}} (\mathbf{T}_{n,0}) \right\} \\
&= \frac{1}{(r+1)!} \sum_{\ell=0}^{r} \binom{r}{\ell} \left\{ \frac{(r+1)!}{(\ell+1)!} n^\ell \frac{\partial^\ell}{\partial n^\ell} (\mathbf{T}_{n,0}) + \frac{(r+2)!}{(\ell+2)!} n^{\ell+1} \frac{\partial^{\ell+1}}{\partial n^{\ell+1}} (\mathbf{T}_{n,0}) \right\} \\
&= \frac{1}{(r+1)!} \sum_{\ell=0}^{r+1} \left\{ \binom{r}{\ell} \frac{(r+1)!}{(\ell+1)!} + \binom{r}{\ell-1} \frac{(r+2)!}{(\ell+1)!} \right\} n^\ell \frac{\partial^\ell}{\partial n^\ell} (\mathbf{T}_{n,0}) \\
&= \frac{1}{(r+1)!} \sum_{\ell=0}^{r+1} \binom{r+1}{\ell} \frac{(r+1)!}{\ell!} n^\ell \frac{\partial^\ell}{\partial n^\ell} (\mathbf{T}_{n,0}) \\
&= \frac{1}{(r+1)!} \sum_{\ell=0}^{r+1} \binom{r+1}{\ell} \frac{\partial^{r+1-\ell}}{\partial n^{r+1-\ell}} (n^{r+1}) \frac{\partial^\ell}{\partial n^\ell} (\mathbf{T}_{n,0}) \\
&= \frac{1}{(r+1)!} \frac{\partial^{r+1}}{\partial n^{r+1}} (n^{r+1}\, \mathbf{T}_{n,0}). \qquad \square
\end{aligned}$$

Now we are going to prove an inequality of Bernstein type, which gives an upper bound for the $L_p$-norm of $\mathbf{U}_{r,c}(\mathbf{M}_{n,c}(f))$ for $f \in L_p(I_c)$, $1 \leqslant p \leqslant \infty$. In the case $c \neq 0$ we need one more lemma. We introduce the 'difference' operator

$$\mathbf{T}_{n,c}^{(r)}(f) := (-1)^r \Delta_c^r \mathbf{T}_{n,c}(x,y) = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \mathbf{T}_{n+\ell c,c}(f), \quad r \in \mathbb{N}, \quad c \neq 0. \tag{39}$$

Note that in the case of $c = -1$ this operator differs slightly from those introduced in [4].

In the following lemma we give a representation of $\mathbf{U}_{r,c}(\mathbf{T}_{n,c})$, $c \neq 0$, as a linear combination of the operators (39).

**Lemma 8.** *Let $c \neq 0$ and $n$ be as in Definition 1. It holds*

$$\frac{r!}{n(n+c)\cdots(n+(r-1)c)} \mathbf{U}_{r,c}(\mathbf{T}_{n,c}) = \sum_{\ell=0}^{r} \binom{r}{\ell} \binom{-\frac{n}{c}}{\ell} \mathbf{T}_{n,c}^{(\ell)}. \tag{40}$$

**Proof.** Using the formula $\binom{-\frac{n}{c}}{\ell}\binom{\ell}{\lambda} = \binom{-\frac{n}{c}}{\lambda}\binom{-\frac{n}{c}-\lambda}{\ell-\lambda}$, we evaluate the right-hand side as

$$\begin{aligned}
&\sum_{\ell=0}^{r} \binom{r}{\ell} \binom{-\frac{n}{c}}{\ell} \mathbf{T}_{n,c}^{(\ell)} \\
&\quad = \sum_{\ell=0}^{r} \binom{r}{\ell} \binom{-\frac{n}{c}}{\ell} \sum_{\lambda=0}^{\ell} (-1)^\lambda \binom{\ell}{\lambda} \frac{1}{n+c\lambda-c} \mathbf{M}_{n+c\lambda,c} \\
&\quad = \sum_{\lambda=0}^{r} (-1)^\lambda \binom{-\frac{n}{c}}{\lambda} \frac{1}{n+c\lambda-c} \mathbf{M}_{n+c\lambda,c} \sum_{\ell=\lambda}^{r} \binom{r}{r-\ell} \binom{-\frac{n}{c}-\lambda}{\ell-\lambda} \\
&\quad = -\frac{1}{c} \sum_{\lambda=0}^{r} (-1)^\lambda \binom{-\frac{n}{c}}{\lambda} \frac{1}{(-\frac{n}{c}-\lambda+1)} \mathbf{M}_{n+c\lambda,c} \sum_{\ell=0}^{r-\lambda} \binom{r}{r-\lambda-\ell} \binom{-\frac{n}{c}-\lambda}{\ell} \\
&\quad = -\frac{1}{c} \sum_{\lambda=0}^{r} (-1)^\lambda \binom{-\frac{n}{c}}{\lambda} \frac{1}{(-\frac{n}{c}-\lambda+1)} \binom{-\frac{n}{c}+r-\lambda}{r-\lambda} \mathbf{M}_{n+c\lambda,c} \\
&\quad = \frac{1}{n-c} \frac{(-\frac{n}{c}+1)}{r} \sum_{\lambda=0}^{r} (-1)^\lambda \binom{r}{\lambda} \binom{-\frac{n}{c}+r-\lambda}{r-1} \mathbf{M}_{n+c\lambda,c},
\end{aligned}$$

which is equal to $\frac{r!}{n(n+c)\cdots(n+(r-1)c)} \mathbf{U}_{r,c}(\mathbf{T}_{n,c})$ by (30). $\square$

Now we are in position to establish the main result of this section.

**Theorem 2.** *Let $c \in \mathbb{R}$ and $n$ be as in Definition 1. For $r \in \mathbb{N}_0$ and $1 \leqslant p \leqslant \infty$ it holds*

$$\|\mathbf{U}_{r,c}(\mathbf{M}_{n,c}(f))\|_p \leqslant \frac{2^r n(n+c)\cdots(n+(r-1)c)}{r!} \|f\|_p.$$

**Proof.** As usual (e.g. [4]), it is enough to prove the inequality for $p = \infty$.

Let us first consider the case $c \neq 0$. Recall that if $c \geqslant 0$ then the function $T_{n,c}$ is completely monotonic with respect to $n$ (Theorem 1). This implies that

$$(-1)^r \Delta_c^r T_{n,c} = \sum_{l=0}^{r} (-1)^\ell \binom{r}{\ell} T_{n+\ell c,c} \geqslant 0, \quad c > 0,$$

which shows that the operators $\mathbf{T}_{n,c}^{(r)}$ are positive. If $c < 0$, we obtain from (15)

$$\Delta_c^r T_{n,c} = (-1)^r \sum_{l=0}^{r} (-1)^\ell \binom{r}{\ell} T_{n+\ell c,c} \geqslant 0, \quad c < 0,$$

and thus the operators $(-1)^r \mathbf{T}_{n,c}^{(r)}$ are positive. In both cases we have

$$\|\mathbf{T}_{n,c}^{(\ell)}\|_{C(I_c)\to C(I_c)} = \|\mathbf{T}_{n,c}^{(\ell)}(\mathbf{1})\|_\infty,$$

where $\mathbf{1}$ denotes the function constant equal one. By (39) we find

$$\mathbf{T}_{n,c}^{(\ell)}(\mathbf{1}) = \sum_{\lambda=0}^{\ell} (-1)^\lambda \binom{\ell}{\lambda} \mathbf{T}_{n+\lambda c,c}(\mathbf{1}) = \sum_{\lambda=0}^{\ell} (-1)^\lambda \binom{\ell}{\lambda} \frac{1}{n+\lambda c - c} \mathbf{1}.$$

The last sum here can be calculated as follows:

$$\begin{aligned}
&\sum_{\lambda=0}^{\ell} (-1)^\lambda \binom{\ell}{\lambda} \frac{1}{n+\lambda c - c} \\
&\quad = -\frac{1}{c} \sum_{\lambda=0}^{\ell} (-1)^\lambda \binom{\ell}{\lambda} \frac{1}{-\frac{n}{c} - \lambda + 1} \frac{\left(-\frac{n}{c}+1\right)\left(-\frac{n}{c}\right)\cdots\left(-\frac{n}{c}-\ell+1\right)}{\left(-\frac{n}{c}+1\right)\left(-\frac{n}{c}\right)\cdots\left(-\frac{n}{c}-\ell+1\right)} \\
&\quad = -\frac{1}{c\left(-\frac{n}{c}+1\right)\binom{-\frac{n}{c}}{\ell}} \sum_{\lambda=0}^{\ell} (-1)^\lambda \binom{-\frac{n}{c}+1}{\lambda} \binom{-\frac{n}{c}-\lambda}{\ell-\lambda} \\
&\quad = \frac{1}{(n-c)\binom{-\frac{n}{c}}{\ell}} (-1)^\ell \sum_{\lambda=0}^{\ell} \binom{-\frac{n}{c}+1}{\lambda} \binom{\frac{n}{c}+\ell-1}{\ell-\lambda} = (-1)^\ell \frac{1}{(n-c)\binom{-\frac{n}{c}}{\ell}},
\end{aligned}$$

and

$$\|\mathbf{T}_{n,c}^{(\ell)}\|_{C(I_c)\to C(I_c)} = \frac{1}{(n-c)\left|\binom{-\frac{n}{c}}{\ell}\right|}. \tag{41}$$

Now for an arbitrary function $f \in C(I_c)$ we write using (40) and (41)

$$\begin{aligned}
&\frac{r!}{n(n+c)\ldots(n+(r-1)c)} \|\mathbf{U}_{r,c}(\mathbf{M}_{n,c}(f))\|_\infty \\
&\quad \leqslant (n-c) \sum_{\ell=0}^{r} \binom{r}{\ell} \left|\binom{-\frac{n}{c}}{\ell}\right| \|\mathbf{T}_{n,c}^{(\ell)}\|_{C(I_c)\to C(I_c)} \|f\|_\infty \\
&\quad = \sum_{\ell=0}^{r} \binom{r}{\ell} \|f\|_\infty = 2^r \|f\|_\infty,
\end{aligned}$$

which proves the theorem for $c \neq 0$.

Now let us consider the case $c = 0$. Recalling that $\mathbf{M}_{n,0} = n\,\mathbf{T}_{n,0}$ and using the Leibniz formula, we rewrite (32) as

$$\frac{r!}{n^r}\,\mathbf{U}_{r,0}(\mathbf{M}_{n,0}) = \frac{n}{r!}\,\frac{\partial^r}{\partial n^r}\,(n^r\,\mathbf{T}_{n,0}) = n\,\sum_{\ell=0}^{r}\binom{r}{\ell}\frac{n^\ell}{\ell!}\,\frac{\partial^\ell}{\partial n^\ell}\,\mathbf{T}_{n,0}.$$

Since the function $T_{n,0}(x, y)$ is completely monotonic with respect to $n$ (Theorem 1), i.e. $(-1)^\ell\,\frac{\partial^\ell}{\partial n^\ell}\,T_{n,0}(x, y) \geqslant 0$, we conclude that the operators $(-1)^\ell\,\frac{\partial^\ell}{\partial n^\ell}\,\mathbf{T}_{n,0}$ are positive. Thus,

$$\left\|\frac{\partial^\ell}{\partial n^\ell}\,\mathbf{T}_{n,0}\right\|_{C(I_0)\to C(I_0)} = \left\|\frac{\partial^\ell}{\partial n^\ell}\,\mathbf{T}_{n,0}(\mathbf{1})\right\|_\infty = \left|\frac{\partial^\ell}{\partial n^\ell}\,\frac{1}{n}\right| = \frac{\ell!}{n^{\ell+1}}.$$

Consequently, for a function $f \in C(I_0)$ we obtain

$$\begin{aligned}\frac{r!}{n^r}\,\|\mathbf{U}_{r,0}(\mathbf{M}_{n,0}(f))\|_\infty &\leqslant n\,\sum_{\ell=0}^{r}\binom{r}{\ell}\frac{n^\ell}{\ell!}\,\left\|\frac{\partial^\ell}{\partial n^\ell}\,\mathbf{T}_{n,0}\right\|_{C(I_0)\to C(I_0)}\,\|f\|_\infty\\ &= \sum_{\ell=0}^{r}\binom{r}{\ell}\,\|f\|_\infty = 2^r\,\|f\|_\infty,\end{aligned}$$

and the theorem is completely proved. □

## 4. The quasi-interpolants

In this section we apply results obtained in Sections 2 and 3 to studying the natural quasi-interpolants of the operators (2) which are defined as follows.

**Definition 2.** Let $c \in \mathbb{R}$ and $n$ be as in Definition 1. Let $r \in \mathbb{N}_0$, and in the case when $c < 0$ we suppose additionally that $0 \leqslant r \leqslant -\frac{n}{c}$. Define

$$\mathbf{Q}_{n,c}^{(r)}(f) := \sum_{k=0}^{r}\frac{k!}{n(n+c)\cdots(n+c(k-1))}\mathbf{U}_{k,c}(\mathbf{M}_{n,c}(f)) \tag{42}$$

for $f \in L_p(I_c)$, $1 \leqslant p \leqslant \infty$.

This construction was introduced in the case of $c = -1$ by Jetter and Stöckler [11] (in a more general situation on multidimensional simplices in spaces with Jacobi weights) with the aim to accelerate the convergence. The quasi-interpolants $\mathbf{Q}_{n,-1}^{(r)}$ are quite well studied by Jetter, Stöckler and the author [3,4]. The construction of Jetter and Stöckler (in the one-dimensional unweighted case) was extended to $c \geqslant 0$ by Abel [1].

The first simple application of the Bernstein inequality (Theorem 2) is the statement that the quasi-interpolants (42) are uniformly bounded.

**Theorem 3.** *Under the assumptions of Definition* 2 *we have*

$$\|\mathbf{Q}_{n,c}^{(r)}(f)\|_p \leqslant (2^{r+1}-1)\,\|f\|_p.$$

The proof is obvious.

Next we show that in the case $c \neq 0$ the quasi-interpolants (42) can be represented as linear combinations of the operators (2). It follows from this statement, in particular, that the quasi-interpolants are well-defined for all $f \in L_p(I_c)$, $1 \leqslant p \leqslant \infty$.

**Theorem 4.** *Let $c \neq 0$ and $n$ and $r$ be as in Definition 2. The quasi-interpolants $\mathbf{Q}_{n,c}^{(r)}$ have the following representation as a linear combination of the Durrmeyer operators (2):*

$$\mathbf{Q}_{n,c}^{(r)} = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \binom{-\frac{n}{c} + r - \ell + 1}{r} \mathbf{M}_{n+\ell c,c}. \tag{43}$$

**Proof.** We insert the representation (30) into the definition (42) to obtain

$$\begin{aligned}
\mathbf{Q}_{n,c}^{(r)} &= \sum_{k=0}^{r} \sum_{\ell=0}^{k} \frac{\left(-\frac{n}{c}+1\right)}{k} (-1)^\ell \binom{k}{\ell} \binom{-\frac{n}{c} + k - \ell}{k-1} \mathbf{M}_{n+\ell c,c} \\
&= \sum_{\ell=0}^{r} (-1)^\ell \, \mathbf{M}_{n+\ell c,c} \frac{\left(-\frac{n}{c}+1\right)}{\ell} \sum_{k=\ell}^{r} \binom{-\frac{n}{c} + k - \ell}{k-\ell} \binom{-\frac{n}{c}}{\ell - 1} \\
&= \sum_{\ell=0}^{r} (-1)^\ell \, \mathbf{M}_{n+\ell c,c} \frac{\left(-\frac{n}{c}+1\right)}{\ell} \binom{-\frac{n}{c} + r - \ell + 1}{r-\ell} \binom{-\frac{n}{c}}{\ell - 1} \\
&= \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \binom{-\frac{n}{c} + r - \ell + 1}{r} \mathbf{M}_{n+\ell c,c}. \qquad \square
\end{aligned}$$

**Remark 1.** The linear combination (43) is again a divided difference. Namely,

$$\mathbf{Q}_{n,c}^{(r)} = [n, n+c, \ldots, n+rc; (z-2c)\cdots(z-(r+1)c)\mathbf{M}_{z,c}]_z. \tag{44}$$

**Remark 2.** The coefficients in the representation (43) are formally a particular case of coefficients of linear combinations considered by Heilmann in [9]. However, Heilmann required that the sum of absolute values of the coefficients is bounded uniformly in $n$. This condition is not satisfied in our case.

**Remark 3.** Following Heilmann [9], it is not difficult to show on the base of the representation (43), that the quasi-interpolants $\mathbf{Q}_{n,c}^{(r)}$, $c > 0$, reproduce polynomials of degree at most $r$. This was also proved by Abel [1] for all $c$ directly.

If $c \to 0$, (44) turns into the following formula.

**Theorem 5.** *It holds*

$$\mathbf{Q}_{n,0}^{(r)} = \frac{1}{r!} \frac{\partial^r}{\partial n^r} (n^r \mathbf{M}_{n,0}). \tag{45}$$

**Proof.** Using $\mathbf{M}_{n,0} = n\,\mathbf{T}_{n,0}$, we rewrite (32) as

$$\frac{r!}{n^r} \mathbf{U}_{r,0}(\mathbf{M}_{n,0}) = \frac{n}{r!} \frac{\partial^r}{\partial n^r} (n^{r-1} \mathbf{M}_{n,0}).$$

Substituting this into the definition (42) and using the Leibniz formula twice, we obtain

$$\begin{aligned}\mathbf{Q}_{n,0}^{(r)} &= \sum_{k=0}^{r} \frac{n}{k!} \frac{\partial^k}{\partial n^k} (n^{k-1} \mathbf{M}_{n,0}) = \sum_{k=0}^{r} \sum_{\ell=0}^{k} \binom{k-1}{\ell-1} \frac{n^\ell}{\ell!} \frac{\partial^\ell}{\partial n^\ell} (\mathbf{M}_{n,0}) \\ &= \sum_{\ell=0}^{r} \frac{n^\ell}{\ell!} \frac{\partial^\ell}{\partial n^\ell} (\mathbf{M}_{n,0}) \sum_{k=\ell}^{r} \binom{k-1}{\ell-1} = \sum_{\ell=0}^{r} \frac{n^\ell}{\ell!} \frac{\partial^\ell}{\partial n^\ell} (\mathbf{M}_{n,0}) \binom{r}{r-\ell} \\ &= \frac{1}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{\partial^{r-\ell}}{\partial n^{r-\ell}} (n^r) \frac{\partial^\ell}{\partial n^\ell} (\mathbf{M}_{n,0}) = \frac{1}{r!} \frac{\partial^r}{\partial n^r} (n^r \mathbf{M}_{n,0}). \qquad \square\end{aligned}$$

It follows from (7), (17), (6) and Theorem 3 that

$$\|\mathbf{Q}_{n,c}^{(r)}(f) - f\|_p \to 0, \quad n \to \infty \quad \text{for } f \in L_p[0,\infty), \quad 1 \leqslant p < \infty. \tag{46}$$

Indeed, if $f \in L_p[0,\infty)$ is such that $\mathbf{U}_{k,c}(f) \in L_p[0,\infty)$ for $0 \leqslant k \leqslant r$, then the convergence follows from the estimate

$$\|f - \mathbf{Q}_{n,c}^{(r)}(f)\|_p \leqslant \|f - \mathbf{M}_{n,c}(f)\|_p + \sum_{k=1}^{r} \frac{k!}{n(n+c)\cdots(n+c(k-1))} \|\mathbf{U}_{k,c}(f)\|_p.$$

For an arbitrary $f \in L_p[0,\infty)$, we approximate $f$ by a smooth function and use the fact that the quasi-interpolants $\mathbf{Q}_{n,c}^{(r)}$ are bounded uniformly in $n$.

The results obtained above allow us to prove a direct result for the quasi-interpolants $\mathbf{Q}_{n,r}^{(r)}$, $c > 0$, in the spaces $L_p[0,\infty)$, $1 \leqslant p < \infty$, in terms of a new K-functional. The arguments are more or less standard. First we show the following estimate of Jackson–Favard type for smooth $f$.

**Theorem 6.** *Let $c > 0, n > c, r \in \mathbb{N}_0, 1 \leqslant p < \infty$ and let $f \in L_p[0,\infty)$ such that $\mathbf{U}_{r+1,c}(f) \in L_p[0,\infty)$. Then*

$$f - \mathbf{Q}_{n,c}^{(r)}(f) = \sum_{\ell=0}^{\infty} \frac{(r+1)!c(r+1)}{(n+(\ell-1)c)(n+\ell c)\cdots(n+(\ell+r)c)} \mathbf{M}_{n+\ell c,c}(\mathbf{U}_{r+1,c}(f)), \tag{47}$$

*with convergence in norm. Moreover,*

$$\|f - \mathbf{Q}_{n,c}^{(r)}(f)\|_p \leqslant \frac{(r+1)!}{(n-c)n\cdots(n+(r-1)c)} \|\mathbf{U}_{r+1,c}(f)\|_p. \tag{48}$$

**Proof.** Using (43), we write

$$\begin{aligned}&\mathbf{Q}_{n+c,c}^{(r)} - \mathbf{Q}_{n,c}^{(r)} \\ &= \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \binom{-\frac{n}{c}+r-\ell}{r} \mathbf{M}_{n+c+\ell c,c} - \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \binom{-\frac{n}{c}+r-\ell+1}{r} \mathbf{M}_{n+\ell c,c} \\ &= -\sum_{\ell=0}^{r+1} (-1)^\ell \binom{r+1}{\ell} \binom{-\frac{n}{c}+r-\ell+1}{r} \mathbf{M}_{n+\ell c,c} \\ &= \frac{(r+1)!c(r+1)}{(n-c)n\cdots(n+rc)} \mathbf{U}_{r+1,c}(\mathbf{M}_{n,c}),\end{aligned}$$

the last equality follows by Lemma 5. Now take $f \in L_p[0,\infty)$, $1 \leqslant p < \infty$, with $\mathbf{U}_{r+1,c}(f) \in L_p[0,\infty)$, and consider the series

$$\sum_{\ell=0}^{\infty}\left(\mathbf{Q}^{(r)}_{n+(\ell+1)c,c}(f)-\mathbf{Q}^{(r)}_{n+\ell c,c}(f)\right) = \sum_{\ell=0}^{\infty}\frac{(r+1)!c(r+1)}{(n+(\ell-1)c)\cdots(n+(\ell+r)c)}\,\mathbf{M}_{n+\ell c,c}(\mathbf{U}_{r+1,c}(f)),$$

we use the fact that the operators $\mathbf{M}_{n+\ell c,c}$ and $\mathbf{U}_{r+1,c}$ commute. Since the operators $\mathbf{M}_{n+\ell c,c}$ are contractions in $L_p$, the series converges in norm. On the other hand, the sum of the series is $f-\mathbf{Q}^{(r)}_{n,c}(f)$, to see it apply a telescoping argument and the statement about convergence (46). Thus, (47) is proved.

The Jackson–Favard statement (48) follows from (47). We estimate

$$\begin{aligned}\|f-\mathbf{Q}^{(r)}_{n,c}(f)\|_p &\leqslant \|\mathbf{U}_{r+1,c}(f)\|_p\,(r+1)!\sum_{\ell=0}^{\infty}\frac{c(r+1)}{(n+(\ell-1)c)\cdots(n+(\ell+r)c)}\\ &= \|\mathbf{U}_{r+1,c}(f)\|_p\,(r+1)!\sum_{\ell=0}^{\infty}\left(\frac{1}{(n+(\ell-1)c)\cdots(n+(\ell+r-1)c)}\right.\\ &\qquad\left.-\frac{1}{(n+\ell c)\cdots(n+(\ell+r)c)}\right)\\ &= \|\mathbf{U}_{r+1,c}(f)\|_p\frac{(r+1)!}{(n-c)n\cdots(n+(r-1)c)}. \qquad \square\end{aligned}$$

For $c > 0$ and $r \in \mathbb{N}$, we introduce the K-functional

$$K_{r,c,p}(f,t) := \inf\{\|f-g\|_p + t\|\mathbf{U}_{r,c}(g)\|_p : g \in L_p[0,\infty), \mathbf{U}_{r,c}(g) \in L_p[0,\infty)\},\quad t > 0.$$

**Theorem 7.** *Let $c > 0, n > c, r \in \mathbb{N}_0$. For $f \in L_p[0,\infty)$, $1 \leqslant p < \infty$, we have*

$$\|f-\mathbf{Q}^{(r)}_{n,c}(f)\|_p \leqslant 2^{r+1}\,K_{r+1,c,p}\left(f, \frac{(r+1)!}{2^{r+1}(n-c)\cdots(n+(r-1)c)}\right).$$

**Proof.** The statement follows from Theorem 3 and Theorem 6 via standard arguments. Take $g \in L_p[0,\infty)$ with $\mathbf{U}_{r+1,c}(g) \in L_p[0,\infty)$, then

$$f-\mathbf{Q}^{(r)}_{n,c}(f) = (f-g) + (g-\mathbf{Q}^{(r)}_{n,c}(g)) + \mathbf{Q}^{(r)}_{n,c}(g-f)$$

and

$$\begin{aligned}\|f-\mathbf{Q}^{(r)}_{n,c}(f)\|_p &\leqslant \|f-g\|_p + \frac{(r+1)!}{(n-c)\cdots(n+(r-1)c)}\|\mathbf{U}_{r+1,c}(g)\|_p\\ &\quad +(2^{r+1}-1)\|f-g\|_p\\ &= 2^{r+1}\left(\|f-g\|_p + \frac{(r+1)!}{2^{r+1}(n-c)\cdots(n+(r-1)c)}\|\mathbf{U}_{r+1,c}(g)\|_p\right).\end{aligned}$$

Taking the infimum over $g$, we obtain the result. □

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