Multiplicative semigroup automorphisms of upper triangular matrices over rings

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Abstract

Suppose R is a ring with 1 and C a central subring of R. Let \( T_n(R) \) be the C-algebra of upper triangular \( n \times n \) matrices over R. Recently several authors have shown that if R is sufficiently well behaved, then every C-automorphism of \( T_n(R) \) is the composites of an inner automorphism and an automorphism induced from a C-automorphism of R (see [1-5]). To generalize these results, in this paper we prove that if \( n \geq 2 \) and R is a semi-prime ring or a ring in which all idempotents are central, then \( f: T_n(R) \rightarrow T_n(R) \) (\( T_n(R) \) is only regarded as a multiplicative semigroup) is a multiplicative semigroup automorphism if and only if there exist a nonsingular matrix \( P \) in \( T_n(R) \) and a ring automorphism \( \tau \) of R such that

\[
f(A) = P^{-1} A^t P \quad \forall A = (a_{ij})_{n\times n} \in T_n(R),
\]

where \( A^t = (\tau(a_{ij}))_{n\times n} \). © 1998 Published by Elsevier Science Inc. All rights reserved.

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Let R be a ring with 1 and C a central subring of R, \( T_n(R) \) denotes the C-algebra of upper triangular \( n \times n \) matrices over R. If \( \psi \) is a C-automorphism of R, then \( \psi \) induced a C-automorphism \( \Psi \) of \( T_n(R) \) simply by \( \Psi((a_{ij})_{n\times n}) = (\psi(a_{ij}))_{n\times n} \).
Recently some authors have characterized C-automorphisms of \( T_n(\mathbf{R}) \) (see [1-5]). To mention here only one result, Jondrup [3] has shown that if \( \mathbf{R} \) is a semiprime ring or a ring in which all idempotents are central, then every C-automorphism \( \sigma \) of \( T_n(\mathbf{R}) \) is the composite of an inner automorphism and an automorphism induced from a C-automorphism of \( \mathbf{R} \). In this paper we generalize the mentioned result when \( n \geq 2 \) by weakening the hypothesis, \( \sigma \) is a C-automorphism, to \( \sigma \) is a multiplicative semigroup automorphism \( (T_n(\mathbf{R}) \) is only regarded as a multiplicative semigroup).

We denote the matrix with 1 at the \((i,j)\) position and 0 elsewhere, the \( n \times n \) identity matrix and the zero matrix over \( \mathbf{R} \) by \( E_{ij}, I_n \) and \( O \), respectively. \([1,n]\) denotes the set \( \{1,2,\ldots,n\} \) and \( T^*_n(\mathbf{R}) \) denotes the subgroup of all nonsingular matrices in \( T_n(\mathbf{R}) \).

**Definition.** A matrix \( X \in T_n(\mathbf{R}) \) is said to idempotent if \( X^2 = X \).

For the convenience of writing, we assume throughout that \( n \geq 2 \) and \( f: T_n(\mathbf{R}) \rightarrow T_n(\mathbf{R}) \) is a multiplicative semigroup automorphism in the process of the following proofs.

**Lemma 1.** \( f(O) = O \).

**Proof.** Let \( f(X) = O \) and \( f(O) = Y \). Applying \( f \) to the equation \( OX = O \), we have \( f(O)f(X) = f(O) \), and hence \( Y = O \). \( \square \)

**Lemma 2** (see [3], pp. 209-210). There exist \( P \in T^*_n(\mathbf{R}) \) and \( r \in \mathbf{R} \) such that \( P^{-1}f(rE_{11})P = E_{11} \).

**Lemma 3.** Suppose \( f \) satisfies \( f(rE_{11}) = E_{11} \) for some \( r \in \mathbf{R} \) and \( f(E_{jj}) = E_{jj} \) for any \( j \in [2,n] \). Then \( f(E_{11}) = E_{11} \).

**Proof.** It is easy to see that \( f(E_{jj})f(E_{11}) = f(E_{11})f(E_{jj}) = O \) from \( E_{jj}E_{11} = E_{11}E_{jj} = O \) for any \( j \in [2,n] \). Thus \( f(E_{11}) = dE_{11} \) for some \( d \in \mathbf{R} \). Again applying \( f \) to the equation \( E_{11}(rE_{11}) = rE_{11} \), we have \( f(E_{11})f(rE_{11}) = f(rE_{11}) \), and hence \( d = 1 \). The lemma follows. \( \square \)

**Lemma 4.** Suppose \( h: T_2(\mathbf{R}) \rightarrow T_2(\mathbf{R}) \) is a multiplicative semigroup automorphism. Then there exist a matrix \( P \) in \( T^*_2(\mathbf{R}) \) and a ring automorphism \( \tau \) of \( \mathbf{R} \) such that

\[
h(A) = P^{-1}A^tP \quad \forall A = (a_{ij})_{2 \times 2} \in T_2(\mathbf{R}),
\]

where \( A^t = (\tau(a_{ij}))_{2 \times 2} \).
Proof. Without loss of generality, we may assume that \( h(O) = O \) and \( h(rE_{11}) = E_{11} \) for some \( r \in R \) from Lemmas 1 and 2. Let
\[
h(E_{22}) = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \quad \text{and} \quad h\left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = E_{22}.
\]
Then
\[
E_{11} \begin{pmatrix} b & c \\ 0 & d \end{pmatrix} = 0
\]
and
\[
\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} E_{22} = h^{-1}(dE_{22})
\]
by applying \( h \) to the equation \((rE_{11})E_{22} = 0\) and \( h^{-1} \) to the equation \((E_{22})h(E_{22}) = dE_{22}\), respectively, and hence \( h(E_{22}) = dE_{22} \) and
\[
h\left( \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) = E_{22}.
\]
Let
\[
h(E_{12}) = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix}.
\]
Case (a): Suppose \( R \) is semiprime. Then \( u = w = 0 \) from \( h(E_{12})^2 = O \).

Case (b): Suppose all idempotents of \( R \) are central. Since \( E_{12}E_{22} = E_{12} \) and \( E_{22}E_{12} = O \), \( h(E_{12})h(E_{22}) = h(E_{12}) \) and \( h(E_{22})h(E_{12}) = O \). Thus \( u = w = 0 \).

Summarizing, \( h(E_{12}) = vE_{12} \). Applying \( h^{-1} \) to the equation \( E_{22}(vE_{12}) = O \), we have
\[
\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} E_{12} = O,
\]
and hence \( x = 0 \) (i.e., \( h(E_{22}) = E_{22} \)). Again applying Lemma 3, we obtain \( h(E_{11}) = E_{11} \). Since
\[
\begin{cases} aE_{ij} = E_{ii}(aE_{ij})E_{jj} \\ aE_{12} = (aE_{11})E_{12} = E_{12}(aE_{22}) \end{cases} \quad \forall i, j \in [1, 2], a \in R.
\]
it follows that
\[
\begin{cases} h(aE_{ij}) = h(E_{ii})h(aE_{ij})h(E_{jj}) \\ h(aE_{12}) = h(aE_{11})h(E_{12}) = h(E_{12})h(aE_{22}). \end{cases}
\]
By a direct computation, we have
\[
\begin{cases} h(aE_{ij}) = x_{ii}(a)E_{ij} \\ x_{12}(a) = x_{11}(a)x_{12}(1) = x_{12}(1)x_{22}(a) \end{cases} \quad \forall i, j \in [1, 2], \ a \in R. \quad (1)
\]
where $x_{ij}(a) \in \mathbb{R}$.

Let $h(Y) = E_{12}$. Then $E_{12} = E_{11}E_{12}E_{22} = h(E_{11})h(Y)h(E_{22}) = h(E_{11}YE_{22})$. Hence $h(yE_{12}) = E_{12}$ for some $y \in \mathbb{R}$. Again applying $h(yE_{12}) = h(E_{12})$, we have $1 = x_{12}(1)x_{22}(y) = x_{11}(y)x_{12}(1)$, and thus $x_{12}(1)$ is a unit of $\mathbb{R}$. Let $P = \text{diag}(x_{12}(1), 1)$. Then

$$P^{-1}h(aE_{ij})P = \tau(a)E_{ij} \quad (2)$$

for any $i, j \in [1, 2]$ and $a \in \mathbb{R}$ by applying (1), where $\tau(a) = x_{22}(a)$. Obviously,

$$\tau(1) = 1. \quad (3)$$

For every $A = (a_{ij})_{2 \times 2} \in T_2(\mathbb{R})$, it is easy to see that

$$h(a_{ij}E_{ij}) = h(E_{ii}AE_{jj}) = h(E_{ii})h(A)h(E_{jj}) \quad \forall i < j.$$  

Applying (2) and (3), we have

$$h(A) = P(\tau(a_{ij}))_{2 \times 2}P^{-1} \quad \forall A = (a_{ij})_{2 \times 2} \in T_2(\mathbb{R}). \quad (4)$$

To prove the lemma, we need only prove that $\tau$ is a ring automorphism of $\mathbb{R}$. In fact, let $A_x = xE_{11}$, $A_y = yE_{11}$, $B_x = I_2 + xE_{12}$ and $B_y = I_2 + yE_{12}$ for any $x, y \in \mathbb{R}$. Then

$$\left\{\begin{array}{l}
\tau(xy) = \tau(x)\tau(y) \\
\tau(x + y) = \tau(x) + \tau(y)
\end{array}\right. \quad \forall x, y \in \mathbb{R}$$

by applying $h(A_xA_y) = h(A_x)h(A_y)$, $h(B_xB_y) = h(B_x)h(B_y)$ and Eq. (4), and hence $\tau(0) = 0$. The lemma follows. \[ \square \]

**Theorem.** $f : T_n(\mathbb{R}) \to T_n(\mathbb{R})$ is a multiplicative semigroup automorphism if and only if there exist a matrix $P$ in $T_n(\mathbb{R})$ and a ring automorphism $\tau$ of $\mathbb{R}$ such that

$$f(A) = P^{-1}A'P \quad \forall A = (a_{ij})_{n \times n} \in T_n(\mathbb{R}),$$

where $A' = (\tau(a_{ij}))_{n \times n}$.

**Proof.** The “if” part is obvious. The proof of the “only if” part is by induction on $n$, the size of the upper triangular matrices, $n = 2$ following from Lemma 4, we may assume that the result for $n - 1$ $(n \geq 3)$.

Without loss of generality, we may assume that $f(rE_{11}) = E_{11}$ for some $r \in \mathbb{R}$ from Lemma 2. Since

$$rE_{11} \begin{pmatrix} 0 & O \\ O & Y \end{pmatrix} = O \quad \text{for all } Y \in T_{n-1}(\mathbb{R}),$$

$$f(rE_{11})f \begin{pmatrix} 0 & O \\ O & Y \end{pmatrix} = O,$$

and thus
\[
f\begin{pmatrix} 0 & O \\ O & Y \end{pmatrix} = \begin{pmatrix} 0 & O \\ O & g(Y) \end{pmatrix} = O \quad \forall Y \in T_{n-1}(\mathbb{R}),
\]

where \( g \) is a map from \( T_{n-1}(\mathbb{R}) \) to itself. Obviously, \( g : T_{n-1}(\mathbb{R}) \to T_{n-1}(\mathbb{R}) \) is a multiplicative semigroup automorphism. By inductive hypothesis, there exist \( Q \in T_{n-1}(\mathbb{R}) \) and an automorphism \( \tau \) of \( \mathbb{R} \) such that

\[
g(Y) = QYQ^{-1} \quad \forall Y \in T_{n-1}(\mathbb{R}).
\]

Let

\[
W = \begin{pmatrix} 1 & O \\ O & Q \end{pmatrix}.
\]

Then

\[
\begin{align*}
W^{-1}f(aE_{pq})W &= \tau(a)E_{pq} & \forall p, q \in [2, n], a \in \mathbb{R} \\
W^{-1}f(rE_{11})W &= E_{11}
\end{align*}
\]

Again applying Lemma 3, we have \( W^{-1}f(E_{11})W = E_{11} \).

Since

\[
\begin{align*}
aE_{1i} &= E_{11}(aE_{1i})E_{ii} \\
aE_{1i} &= (aE_{12})E_{2j} & \forall i \in [1, n], j \in [2, n], a \in \mathbb{R}, \\
aE_{12} &= (aE_{11})E_{12} = E_{12}(aE_{22})
\end{align*}
\]

it follows that

\[
\begin{align*}
f(aE_{1i}) &= f(E_{11})f(aE_{1i})f(E_{ii}) \\
f(aE_{1i}) &= f(aE_{12})f(E_{2j}) & \forall i \in [1, n], j \in [2, n], a \in \mathbb{R}. \\
f(aE_{12}) &= f(aE_{11})f(E_{12}) = f(E_{12})f(aE_{22})
\end{align*}
\]

By a direct computation, we have

\[
\begin{align*}
W^{-1}f(aE_{1i})W &= x_1(a)E_{1i} \\
x_i(a) &= x_2(a) & \forall i \in [1, n], j \in [2, n], a \in \mathbb{R}, \\
x_2(a) &= x_1(a)x_2(1) = x_2(1)\tau(a)
\end{align*}
\]

where \( x_i(a) \in \mathbb{R} \).

Let \( f(Y) = E_{12} \). Then \( E_{12} = E_{11}E_{12}E_{22} = f(E_{11})f(Y)f(E_{22}) = f(E_{11}YE_{22}) \).

Hence \( f(yE_{12}) = E_{12} \) for some \( y \in \mathbb{R} \). Again applying \( f(yE_{12}) = f(E_{12})f(yE_{22}) = f(yE_{11})f(E_{12}) \), we have \( 1 = x_2(1)\tau(y) = x_1(y)x_2(1) \), and thus \( x_2(1) \) is a unit of \( \mathbb{R} \). Let \( P = W\text{diag}(x_2(1), 1, \ldots, 1) \). Then

\[
P^{-1}f(aE_{ij})P = \tau(a)E_{ij}
\]

for any \( i, j \in [1, n] \) by applying (5).

For every \( X = (x_{ij})_{n \times n} \in T_n(\mathbb{R}) \), it is easy to see that
Again applying (6), the theorem follows. □

From the theorem, it is easy to see the following three corollaries.

**Corollary 1.** \( \Gamma : T_n(R) \to T_n(R) \) is a ring automorphism if and only if there exist a matrix \( P \) in \( T_n^*(R) \) and a ring automorphism \( \tau \) of \( R \) such that

\[
\Gamma(A) = P^{-1} A' P \quad \forall A = (a_{ij})_{n \times n} \in T_n(R),
\]

where \( A' = (\tau(a_{ij}))_{n \times n} \).

**Corollary 2** (see [3]). Suppose \( \sigma : T_n(R) \to T_n(R) \) is a \( C \)-automorphism. Then \( \sigma \) is the composite of an inner automorphism and an automorphism induced from a \( C \)-automorphism of \( R \).

**Corollary 3** (see [5]). Suppose \( \sigma : T_n(C) \to T_n(C) \) is a \( C \)-algebra automorphism. Then \( \sigma \) is inner.

**References**


