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Stack words, standard tableaux and Baxter permutations

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Abstract

The origin of this work is based on the enumeration of stack sortable permutations [11,17,18]. The problem, particularly in case of two stacks, exhibits classical objects in combinatorics such as permutations with forbidden subsequences, nonseparable planar maps [4,5], and also standard Young tableaux if we are interested in the movements of stacks. So, we show that the number of $3 \times n$ rectangular standard Young tableaux which avoid two consecutive integers on second row is c_n^2 (where $c_n = (2n)!/(n+1)!n!$) and there is a one-to-one correspondence between the same tableaux which avoid two consecutive integers on the same row and Baxter permutations which are enumerated by $\sum_{m=0}^{n-1} \left[\binom{n+1}{m}\cdot\binom{n+1}{m+1}\cdot\binom{n+1}{m+2}\right] / \left[\binom{n+1}{1}\cdot\binom{n+1}{2}\right]$.

We also give formulas enumerating these objects according to various parameters.

Résumé

L'origine de ce travail est l'énumération des permutations triables par pile [11,17,18]. Ce problème, en particulier dans le cas de deux piles, fait apparaître des objets classiques en combinatoire tels que permutations à motifs exclus, cartes planaires non séparables [4,5], et également les tableaux de Young standard lorsque l'on s'intéresse aux mouvements des piles. Nous montrons ainsi que le nombre de tableaux de Young standard rectangulaires $3 \times n$ n'ayant pas deux entiers consécutifs sur la deuxième ligne est c_n^2 (où $c_n = (2n)!/(n+1)!n!$) et que ces mêmes tableaux n'ayant pas deux entiers consécutifs sur la même ligne sont en correspondance avec les permutations de Baxter et donc au nombre de $\sum_{m=0}^{m-1} \left[\binom{n+1}{m}\cdot\binom{n+1}{m+1}\cdot\binom{n+1}{m+2}\right] / \left[\binom{n+1}{1}\cdot\binom{n+1}{2}\right]$.

Nous obtenons également plusieurs formules correspondant à l'énumération de ces objets suivant divers paramètres.

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0. Introduction

Knuth [11] has been interested in stack sortable permutations. He has characterized them in terms of permutations with forbidden subsequences (that is to say avoiding subsequences of a certain type). In particular, he has shown that the permutations avoiding the pattern 231 (that is to say avoiding subsequences of type 231) constitute the set of one-stack sortable permutations. This set, denoted $S_n(231)$, is sometimes called the set of Catalan permutations because it is enumerated by the numbers $c_n = (2n)!/(n+1)!n!$. Generally speaking, numerous papers [14,17,9] deal with permutations with forbidden subsequences.

We immediately observe in the algorithm which sorts a permutation through one stack that, at any time, the stack can contain only decreasing integers from the top of the stack. Thus, in a way, the stack satisfies 'Hanoï' tower condition refering to the same named problem.

Among possible generalizations of the problem considered by Knuth, West [17,18] has been interested in the enumeration of two-stack sortable permutations, this stack having to satisfy at any time the 'Hanoï' tower condition. He has characterized them in terms of permutations with forbidden subsequences by showing they are the permutations of $S_n(2341, 3\bar{5}241)$: permutations of length n avoiding the patterns 2341 and 3241, the latter being yet permitted when it is itself part of 35241 in the permutation. West has also conjectured that these permutations of length n are enumerated by $2.(3n)!/(n+1)!(2n+1)!$.

D. Zeilberger [20] has analytically proved this conjecture and, more recently, Dulucq et al. [4,5] have found a combinatorial proof of it. For that, they give a correspondence between permutations of $S_n(2341, 3\bar{5}241)$ and rooted nonseparable planar maps with $n+1$ edges, which establishes the result because Tutte [15] has shown that these maps are enumerated by $2.(3n)!/(n+1)!(2n+1)!$. Moreover, [4,5] give many formulas corresponding to the distributions of these permutations of $S_n(2341, 3\bar{5}241)$ according to several parameters (rises, minima).

In this paper, we are especially interested in movements of stacks when we try to sort a permutation. In particular, in case of one stack, its movements can be encoded by a word of a parenthesis system (or Dyck word) and the enumeration of these words gives again the Catalan numbers (in fact, there is an immediate correspondence between these words and the permutations of $S_n(231)$).

If we place k stacks consecutively ($k \geqslant 1$) in order to sort a permutation of length n, all the allowable movements of these stacks (without any condition) are in correspondence with the $(k+1) \times n$ rectangular standard Young tableaux [19] (that is to say of shape $\lambda = (n, n, \ldots, n)$ partition of $(k+1).n$).

Imposing some restrictions on stacks (verifying the 'Hanoï' tower condition for example) is equivalent to impose some restrictions on these standard Young tableaux.

Here, we consider two stacks ($k = 2$) and the objects corresponding to the movements of these stacks are rectangular standard Young tableaux of height 3.

We especially show that the number of $3 \times n$ rectangular standard Young tableaux without two consecutive integers on the second row is c_n^2, the square of the nth Catalan number. This result is close to that of Gouyou-Beauchamps's about the enumeration of standard Young tableaux having at most 4 rows [10].

Moreover, the additional restriction which consists in forbidding, in these rectangular tableaux, two consecutive integers on the same row allows us to make a correspondence with Baxter permutations [1,2,13,16] enumerated by $\sum_{m=0}^{n-1} \left[\binom{n+1}{m}\binom{n+1}{m+1}\binom{n+1}{m+2}\right] / \left[\binom{n+1}{1}\cdot\binom{n+1}{2}\right]$.

We also obtain some more precise results about these enumeration formulas by considering certain distributions of these objects.

The first part of this paper is devoted to define the different objects considered (stack word, standard Young tableaux, twin binary trees) and to state the results obtained. In the second part, we recall the notion of shuffle of two parenthesis systems and the correspondence exhibited by Cori et al. [3] between these words, couples of complete binary trees and alternating Baxter permutations. From this bijection, we prove in the third part the results mentioned above. Finally, in the last part, we present the results obtained according to various parameters.

1. Stack words

Gire considers in her thesis [9] a set of k stacks placed consecutively and regards the movements of the stacks when the identity permutation $\zeta = 12\ldots n$ crosses these k stacks. The words of the language $Y_n^{(k)} = \{f \in \{1,2,\ldots,k+1\}^* : \forall i \in [1,k], \forall f = f'f'', |f'|_i \geqslant |f'|_{i+1}; \forall i \in [1,k+1], |f|_i = n\}$ exactly encode all the movements of k stacks when ζ goes through them (see Fig. 1). This language $Y_n^{(k)}$ also encodes all the rectangular standard Young tableaux of height $k+1$ and length n [19]. We deduce from the hook length formula [7] that $|Y_n^{(k)}| = ((k+1).n)! \prod_{i=0}^{k} \frac{i!}{(n+i)!}$.

If $k = 1$ (only one stack), the words of $Y_n^{(1)}$ are exactly the parenthesis words.

From now on, we consider the following notations and (classical) results.

• The language $P_{z,\bar{z}}$ on the alphabet $\{z, \bar{z}\}$ (corresponding to $Y_{n\geqslant 0}^{(1)}$) is the language of well-formed parenthesis system words (or Dyck words).

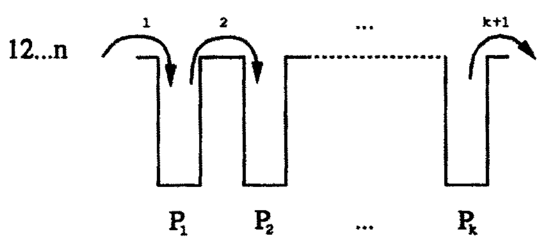

Fig. 1. Stack words.

• Let T_n be the set of complete binary trees with $2n+1$ nodes of whom $n+1$ are leaves.

• The bijection between a word of $P_{z,\bar{z}}$ and a complete binary tree t is defined by the following coding:

$$code(t) = \begin{cases} \varepsilon & \text{if } t \text{ is a node,} \\ z\ code(left\_subtree(t))\ \bar{z}\ code(right\_subtree(t)) & \text{otherwise.} \end{cases}$$

• The number of complete binary trees with $2n+1$ nodes is given by the nth Catalan number $c_n = (2n)!/(n+1)!n!$.

• Let $\mathscr{A} = \{1,2,3\}$ be the alphabet of words associated to two stacks.

• Let $Y = \{f \in \mathscr{A}^* : \forall f = f'f'',\ |f'|_1 \geqslant |f'|_2 \geqslant |f'|_3; |f|_1 = |f|_2 = |f|_3\}$ be the language of words encoding the movements of stacks. Y corresponds to rectangular standard Young tableaux of height 3.

• $Y_n = Y_n^{(2)} = \{f \in Y : |f| = 3n\}$.

Gire has considered different restrictions on the language Y_n corresponding to the extension of the 'Hanoï' tower condition imposed on stacks when a permutation is sorted. Indeed, this condition induces to consider the language $H_n = Y_n \backslash \{f = f'2g2f'' : g \in Y\}$.

Conjecture 1 (Gire [9]). *The number of words of the language $H_n = Y_n \backslash \{f = f'2g\, 2f'' : g \in Y\}$ is*

$$|H_n| = \frac{2^n(3n)!}{(n+1)!(2n+1)!}$$

Gire [9] has also conjectured the two following results that we prove in this paper.

Theorem 1. *The number of words of the language $C_n = Y_n \backslash \{\mathscr{A}^*22\mathscr{A}^*\}$ encoding rectangular standard Young tableaux of height 3 and length n without two consecutive integers on the second row is*

$$|C_n| = c_n^2.$$

Theorem 2. *The number of words of the language $B_n = Y_n \backslash \{\mathscr{A}^*11\mathscr{A}^*, \mathscr{A}^*22\mathscr{A}^*, \mathscr{A}^*33\mathscr{A}^*\}$ encoding rectangular standard Young tableaux of height 3 and length n without two consecutive integers on the same row is*

$$|B_n| = \sum_{m=0}^{n-1} \frac{\binom{n+1}{m} \cdot \binom{n+1}{m+1} \cdot \binom{n+1}{m+2}}{\binom{n+1}{1} \cdot \binom{n+1}{2}}.$$

In order to prove Theorem 2, we consider a class of particular complete binary tree couples: the twin trees.

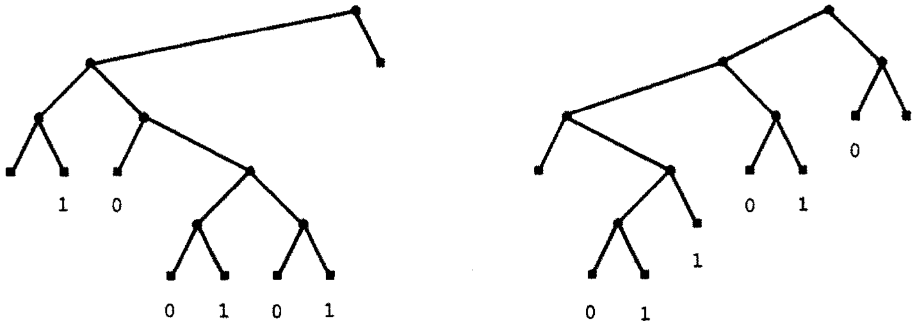

Fig. 2. Two twin trees.

Definition 1. The set of twin trees $Twin_n \subseteq T_n \times T_n$ is

$$Twin_n = \{(t_1, t_2) : t_1, t_2 \in T_n \text{ and } \Theta(code(t_1)) = \Theta^c(code(t_2))\}$$

where Θ is the surjective mapping from $P_{z,\bar{z}}$ to $\{0,1\}^*$ defined by

$$\Theta(z^l \bar{z} w_{l+2} w_{l+3} \dots w_{2n}) = \Theta(\bar{z} w_{l+2}) \Theta(w_{l+2} w_{l+3}) \dots \Theta(w_{2n-1} w_{2n})$$

with

$$\Theta(zz) = \Theta(\bar{z}z) = \varepsilon \text{ (the empty word)}, \quad \Theta(z\bar{z}) = 0, \quad \Theta(\bar{z}\bar{z}) = 1$$

and Θ^c is identical to Θ if we swap the letters 0 and 1.

Note that the mapping Θ labels left (respectively right) leaves of a complete binary tree with the letter 0 (respectively 1) except the two extreme leaves, and that two trees are twins if and only if their labelings are complementary words.

Example 1. Fig. 2 shows two twin trees of $Twin_7$. Indeed, $\Theta(zzz\bar{z}\bar{z}z\bar{z}zz\bar{z}\bar{z}z\bar{z}\bar{z}) = \Theta^c(zzz\bar{z}zz\bar{z}\bar{z}\bar{z}z\bar{z}\bar{z}z\bar{z}) = 100101$.

A consequence of Theorem 2 is the following result.

Corollary 1. *The number of twin trees of* $Twin_n$ *is*

$$|Twin_n| = \sum_{m=0}^{n-1} \frac{\binom{n+1}{m} \cdot \binom{n+1}{m+1} \cdot \binom{n+1}{m+2}}{\binom{n+1}{1} \cdot \binom{n+1}{2}}.$$

2. Shuffle of parenthesis words

Before proving Theorems 1 and 2 and Corollary 1, we recall some definitions and results on Baxter permutations and on shuffle of parenthesis words.

Definition 2. A permutation π of S_n is a Baxter permutation if and only if, for all $p \in [1, n-1]$, π can only be factorized either as $\pi = \pi' p \overset{<}{\pi}\overset{>}{\pi} (p+1)\pi''$ or as $\pi = \pi'(p+1) \overset{>}{\pi}\overset{<}{\pi} p\pi''$ where all the letters of $\overset{<}{\pi}$ (respectively $\overset{>}{\pi}$) are smaller than p (respectively greater than $p+1$). We denote $Baxter_n$ the set of these permutations of S_n.

Note that the two smallest permutations which are not Baxter permutations are 2413 and 3142. The Baxter permutations can be regarded as permutations with forbidden subsequences [9] and $Baxter_n = S_n(25\bar{3}14, 41\bar{3}52)$.

Chung et al. [2] have analytically demonstrated that the number of Baxter permutations of length n is

$$|Baxter_n| = \sum_{m=0}^{n-1} \frac{\binom{n+1}{m}\cdot\binom{n+1}{m+1}\cdot\binom{n+1}{m+2}}{\binom{n+1}{1}\cdot\binom{n+1}{2}}.$$

Viennot [16] has given a combinatorial proof of it in which he obtains the distribution of these permutations according to the number of rises (parameter m). Mallows [13] has analytically found another formula for these permutations according to the number of rises and two other parameters.

We consider the following parameters for permutations of S_n.

- For all $i \in [1, n-1]$, i is called a rise of π if and only if $\pi(i) < \pi(i+1)$.
- For all $i \in [1, n]$, $\pi(i)$ is called a

$$\text{left-to-right maximum element if and only if } \pi(i) > \pi(j) \text{ for all } 1 \leqslant j < i,$$
$$\text{right-to-left maximum element if and only if } \pi(i) > \pi(j) \text{ for all } i < j \leqslant n.$$

We briefly recall the correspondence between (alternating) permutations and increasing (complete) binary trees:

$$incr(u) = \langle incr(v), x, incr(w)\rangle \quad \text{where } u = vxw,\ x = \min\{u_i : u = u_1u_2\ldots u_p\}.$$

This construction is bijective; we just have to project in infix order the labeling of the nodes of the increasing binary tree in order to obtain the permutation.

We now define some other objects.

- Let $\widehat{Baxter}_{2n} = \{\pi \in Baxter_{2n} : \forall i \in [1, n], \pi(2i-1) < \pi(2i) > \pi(2i+1)\}$ be the set of alternating Baxter permutations.
- Let $Shuffle_{2n} = \{\alpha \in P_{a,\bar{a}} \sqcup\!\sqcup\, P_{b,\bar{b}};\ |\alpha| = 2n; \forall \alpha = \alpha' b \alpha'', |\alpha'|_a > |\alpha'|_{\bar{a}}\}$ be the language shuffling two parenthesis languages.

Cori et al. [3] have established a bijection, that we denote Υ, between the language $Shuffle_{2n}$ shuffling two parenthesis languages, the set $\widehat{Baxter}_{2n}$ of alternating Baxter permutations, and the set of all the couples of complete binary trees each of them having $2n+1$ nodes. Thus, these three families of objects are enumerated by ${c_n}^2$, the square of the nth Catalan number.

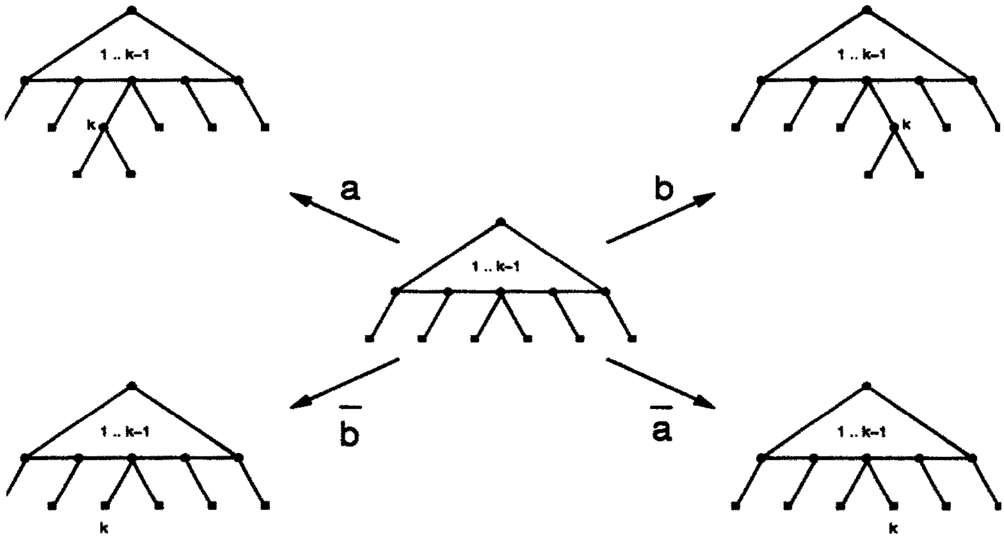

Fig. 3. Operations of the bijection Υ_1.

The first bijection, denoted Υ_1, makes a correspondence between a word α of $\mathit{Shuffle}_{2n}$ and a permutation π of $\widehat{\mathit{Baxter}}_{2n}$. From the complete binary tree reduced to three nodes, that is to say two free leaves and one internal node labeled 1, we successively apply the operations corresponding to the letters $\alpha_2, \alpha_3, \ldots, \alpha_{2n}$ of the word α. These operations, illustrated in Fig. 3, are the following ones:

operation a: label the rightmost free left leaf and add two edges to it
operation b: label the leftmost free right leaf and add two edges to it
operation $\bar{b}$: label the rightmost free left leaf
operation $\bar{a}$: label the leftmost free right leaf

Finally, we obtain an increasing complete binary tree whose infix projection is the permutation π of $\widehat{\mathit{Baxter}}_{2n}$.

The second bijection, denoted Υ_2, consists in building respectively the increasing $\mathit{incr}(\pi)$ and decreasing $\mathit{decr}(\pi)$ binary trees corresponding to π and forgetting their labels. In their paper, Cori et al. give a different construction of the second tree, which is however equivalent to the one described above because of the alternation of these Baxter permutations.

Example 2. The bijection Υ makes a correspondence between the word $a\bar{a}aab\bar{a}\bar{b}b\bar{a}\bar{b}$ of $\mathit{Shuffle}_{10}$, the permutation 4 7 5 6 3 (10) 8 9 1 2 of $\widehat{\mathit{Baxter}}_{10}$ and $(zzz\bar{z}z\bar{z}\bar{z}z\bar{z}\bar{z}, zz\bar{z}z\bar{z}\bar{z}z\bar{z}z\bar{z})$ which encodes a couple of complete binary trees each of them having 11 nodes.

3. Enumeration of stack words

3.1. Standard tableaux without two consecutive integers on the second row

The proof of Theorem 1 is based on the following result.

Lemma 1. *The morphism Φ defined by*

$$\Phi: \quad \begin{array}{c} \mathit{Shuffle}_{2n} \to C_n \\ \alpha = \alpha_1\alpha_2\cdots\alpha_{2n} \mapsto f = \Phi(\alpha_1)\Phi(\alpha_2)\cdots\Phi(\alpha_{2n}) \end{array} \quad \text{with} \begin{cases} \Phi(a) & = 1 \\ \Phi(b) & = 21 \\ \Phi(\bar{a}) & = 23 \\ \Phi(\bar{b}) & = 3 \end{cases}$$

is a bijection between $\mathit{Shuffle}_{2n}$ and C_n (the set of stack words without factor 22).

Proof. We immediately observe that Φ is a mapping from $\mathit{Shuffle}_{2n}$ to C_n and that its converse is clearly defined because $\{1, 21, 23, 3\}$ is a prefix code. It only remains to verify that the conditions imposed to α and to f are the same ones.

- $|\alpha| = 2n, \quad |\alpha|_a = |\alpha|_{\bar{a}}, \quad |\alpha|_b = |\alpha|_{\bar{b}} \Leftrightarrow |f|_1 = |f|_2 = |f|_3 = n$

because

$$|\alpha|_a + |\alpha|_b = n, \quad |\alpha|_b + |\alpha|_{\bar{a}} = n, \qquad |\alpha|_{\bar{a}} + |\alpha|_{\bar{b}} = n.$$

- $\forall \alpha = \alpha'\alpha'', \quad |\alpha'|_a \geqslant |\alpha'|_{\bar{a}}$ and $|\alpha'|_b \geqslant |\alpha'|_{\bar{b}} \Leftrightarrow \forall f = f'f'',$

$$|f'|_1 \geqslant |f'|_2 \geqslant |f'|_3$$

or

$$|\alpha'|_a + |\alpha'|_b \geqslant |\alpha'|_{\bar{a}} + |\alpha'|_b \geqslant |\alpha'|_{\bar{a}} + |\alpha'|_{\bar{b}}.$$

- $\forall \alpha = \alpha' b \alpha'', \qquad |\alpha'|_a > |\alpha'|_{\bar{a}} \Leftrightarrow \forall f = f'21f'', \qquad |f'|_1 > |f'|_2$

since the factor 21 forces a strict excedent of 1 on its left. □

Example 3. Φ associates the words $a\bar{a}aab\bar{a}\bar{b}b\bar{a}\bar{b}$ of $\mathit{Shuffle}_{10}$ and 123112123321233 of C_5.

3.2. Standard tableaux without two consecutive integers on the same row

The proof of Theorem 2 needs two steps. The first one consists in considering the language B_n as a subset of the language C_n and in characterizing the objects obtained by applying the bijections Φ and Υ: thus, the twin trees appear. The second one aims to make a correspondence between twin trees and Baxter permutations.

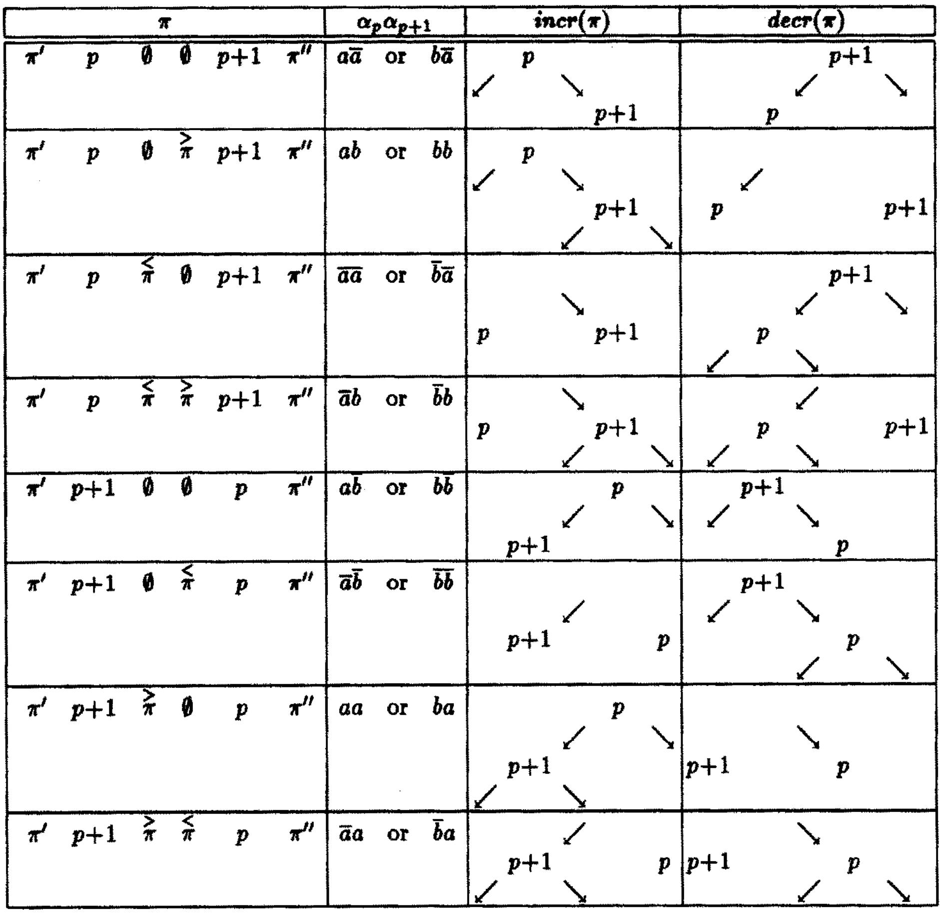

Fig. 4. Relations between $\pi \in \widehat{Baxter}_{2n}$, $\alpha \in Shuffle_{2n}$, $incr(\pi)$ and $decr(\pi)$.

First of all, we need the following proposition.

Proposition 1. *For all permutations $\pi \in \widehat{Baxter}_{2n}$ associated with $\alpha = \alpha_1\alpha_2\ldots\alpha_{2n} \in Shuffle_{2n}$ by Υ, and for all $p \in [1, 2n-1]$, we have the relations displayed on Fig. 4.*

The first column shows the only eight possibilities for factorizing π according to Definition 2. Note that $\overset{<}{\pi}$ and $\overset{>}{\pi}$ are nonempty sequences of consecutive elements respectively smaller than p and greater than $p+1$ in π.

The second column shows the only 16 possibilities of $\{a, \bar{a}, b, \bar{b}\}^2$ for $\alpha_p\alpha_{p+1}$.

The third and fourth columns show respectively the increasing $incr(\pi)$ and decreasing $decr(\pi)$ binary trees of π. See for example the cell corresponding to the first row and the third column, $incr(\pi' p\emptyset\emptyset(p+1)\pi'')$, means that p is the label of an internal node and $p+1$ is the label of a right leaf (whose parent is p).

Proof. All these relations are obvious: we just have to apply the operations of the bijection Υ_1. The only things to verify are for $decr(\pi)$ the cases $\alpha_{p+1} = b$ and $\alpha_{p+1} = a$ which forces p to be the end of a left and right edge, respectively, because we have to avoid $41\bar{3}52$ and $25\bar{3}14$, respectively. □

Lemma 2. *The morphism Φ is a bijection between B_n (set of stack words without factor 11,22,33) and $\widetilde{Shuffle}_{2n} = \{\alpha \in Shuffle_{2n} : |\alpha|_{aa} = |\alpha|_{ba} = |\alpha|_{\bar{a}\bar{b}} = |\alpha|_{\bar{b}\bar{b}} = 0\}$ (language shuffling two parenthesis languages without factor $aa, ba, \bar{a}\bar{b}, \bar{b}\bar{b}$).*

Proof. We easily verify this lemma considering the morphism Φ. □

Definition 3. Let $\widetilde{Baxter}_{2n}$ be the set of alternating Baxter permutations verifying for all $p \in [1, 2n-1]$, if $\pi = \pi'(p+1)\overset{>}{\pi}\overset{<}{\pi}\,p\pi''$ then $\overset{>}{\pi} = \varepsilon \Leftrightarrow \overset{<}{\pi} = \varepsilon$ ($\overset{>}{\pi}$ and $\overset{<}{\pi}$ are either both empty or both nonempty).

Lemma 3. *The bijection Υ makes a correspondence between the language $\widetilde{Shuffle}_{2n}$, the set $\widetilde{Baxter}_{2n}$ and the set $Twin_n$ of twin trees.*

Proof. It is a direct consequence of Proposition 1 by applying the restrictions to the objects in correspondence by Υ.

According to Definition 3, $\widetilde{Baxter}_{2n}$ is effectively the searched set.

The restriction on the couple of complete binary trees leads to characterize the twin trees of $Twin_n$. Thus, for all $p \in [1, 2n-1]$, a left leaf of the increasing binary tree labeled $p+1$ is indexed $2i+1$ in infix order if and only if a right leaf of the decreasing binary tree labeled p has the same index (except for the two extreme leaves, that is to say for all $i \in [1, n-1]$). □

Example 4. Fig. 5 gives the correspondence Φ between the word 123121321213131 232323 of B_7, the word $a\bar{a}ab\bar{b}bb\bar{b}a\bar{b}a\bar{a}\bar{a}\bar{a}$ of $\widetilde{Shuffle}_{14}$, the permutation (11) (12) 3 5 4 (10) 9 (13) 6 8 7 (14) 1 2 of $\widetilde{Baxter}_{14}$ obtained by Υ_1 and the twin trees of $Twin_7$ (already considered in example 1) obtained by Υ_2.

We consider the set $\widetilde{Twin}_n$ of plucked-off twin trees obtained by deleting once and only once all the leaves of (complete) twin trees of $Twin_n$. This process is clearly bijective [11].

Lemma 4. *There is a bijection Ψ between permutations of $Baxter_n$ and couples of twin trees of $\widetilde{Twin}_n$.*

$$\Psi : \quad Baxter_n \to \widetilde{Twin}_n$$
$$\pi \mapsto (t_1, t_2)$$

The mapping Ψ and its converse can be defined in the following way.

- Ψ consists in building the increasing and decreasing binary trees of a permutation π of $Baxter_n$. The two binary trees (t_1, t_2) are respectively these two trees without their labels.

$$1\,2\,3\,1\,2\,1\,3\,2\,1\,2\,1\,3\,1\,3\,1\,2\,3\,2\,3\,2\,3$$

$$\updownarrow$$

$$a\,\bar{a}\,a\,b\,\bar{b}\,b\,b\,\bar{b}\,a\,\bar{b}\,a\,\bar{a}\,\bar{a}\,\bar{a}$$

$$\updownarrow$$

$$11\;\;12\;\;3\;\;5\;\;4\;\;10\;\;9\;\;13\;\;6\;\;8\;\;7\;\;14\;\;1\;\;2$$

$$\updownarrow$$

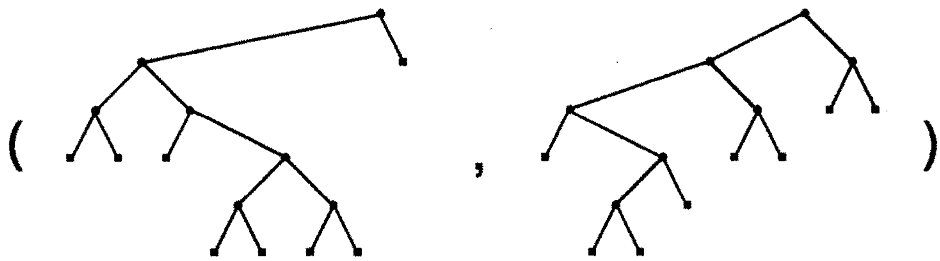

Fig. 5. Restrictions on the bijections Φ and Υ.

• The converse mapping Ψ^{-1} is described by the following algorithm acting on a couple of plucked-off twin trees (t_1, t_2) with n nodes.

For k varying. from n down to 1, repeat the following process:

let i be the rank (infix order) of the root of t_2
label k the node (a leaf) f of rank (infix order) i of t_1
if f is a left leaf
then let s be the last node on the left branch of the right subtree of t_2
 graft the left subtree of t_2 on s (see Fig. 6)
else let s be the last node on the right branch of the left subtree of t_2
 graft the right subtree of t_2 on s
delete the root of t_2
delete the leaf f of t_1

During this algorithm, t_1 has its nodes labeled in increasing order; the permutation π is obtained by projecting in infix order this labeling.

Proof.

• For all permutation, its increasing and decreasing binary trees without their labels and completed with leaves are twins.

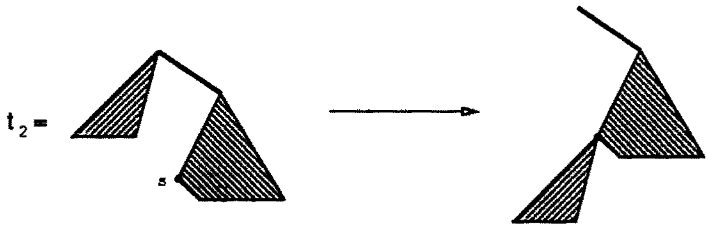

Fig. 6. Graft a left subtree on a node s.

Assume the contrary. Then, there are two leaves (not the first and last ones) with the same infix order number in the completed increasing and decreasing binary trees of a permutation π, having the same left or right direction. Let e be the label of the parent of these leaves corresponding to $\pi(i)$ (e is the label of the ith internal node of the two complete trees). To satisfy the property of increase and decrease of the two trees, $\pi(i-1)$ must simultaneously be smaller and greater than e.

• The converse mapping Ψ^{-1} builds one and only one Baxter permutation π from a couple of plucked-off twin trees $(t_1, t_2) \in \widetilde{Twin}_n$.

— Remark that Ψ^{-1} is well defined.

 * f is a leaf. Otherwise, in order to satisfy the twin property, we would have $i = k = 1$.
 * As f has a parent in t_1, the root of t_2 has an edge of opposite direction.
 * After each step, the new trees are plucked-off twin binary trees.

— Ψ^{-1} reconstructs the increasing labeling of a binary tree, which encodes one and only one permutation.

— The permutation obtained is a Baxter permutation.

At the kth step, for all $k \in [2, n]$, the mapping Ψ^{-1} deletes the element k located just left [resp. right] to a left-to-right [resp. right-to-left] maximum element of the permutation of S_{k-1}, for f a left [resp. right] leaf. However, there is a construction [9] generating once and only once all the Baxter permutations of length n by inserting n into all the active sites of Baxter permutations of length $n-1$; these active sites are precisely the locations just left and right to the respectively left-to-right and right-to-left maximum elements. □

Note that bijections Ψ and Υ_2 are the same when we consider only alternating Baxter permutations. Thus, bijection Ψ generalizes bijection Υ_2 for all Baxter permutations.

Example 5. The two Figs. 7 and 8 display respectively the mappings Ψ and Ψ^{-1} associating the permutation 4 2 3 6 5 7 1 of $Baxter_7$ and twin trees of $\widetilde{Twin}_7$ (corresponding to trees of Example 1).

Fig. 7 illustrates the Baxter permutation, its increasing and decreasing binary trees, and the twin trees.

Fig. 8 shows each step of the preceding algorithm (where the ith nodes of t_1 and t_2 are bracketted, the leaf f of t_1 is labeled k, the node s is in a box and the edges of

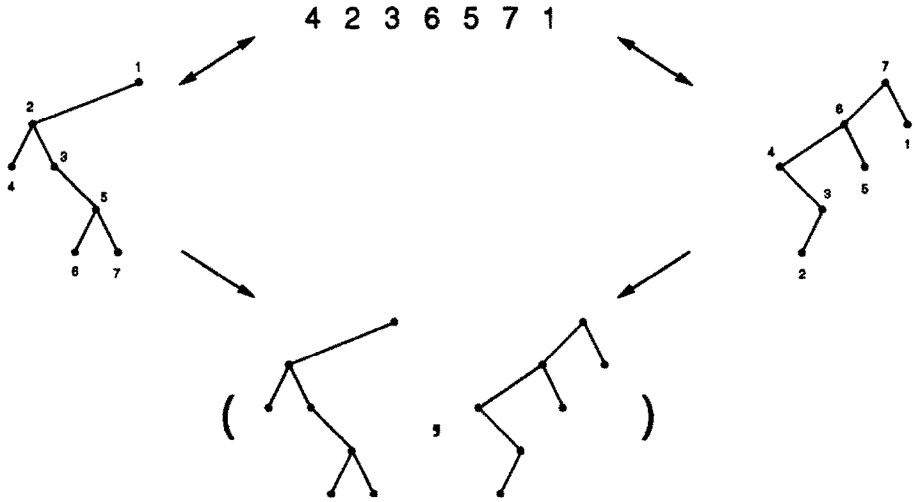

Fig. 7. The bijection Ψ.

t_1 and t_2 to be deleted are in bold type), the increasing binary tree so obtained, and the corresponding Baxter permutation.

4. Refinement of the results

In fact, the enumerating formulas of Theorems 1 and 2 can be specified by considering some distributions of the preceding objects.

4.1. Standard tableaux without two consecutive integers on the second row

Let $T_{n,k}$ be the set of complete binary trees with $2n+1$ nodes and k left leaves, that is to say the set of words of $P_{z,\bar{z}}$ of length $2n$ and having k factors $z\bar{z}$.

It is well known [12] that $|T_{n,k}| = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

From that and from the bijection Φ, we deduce the following distribution giving the number of rectangular standard Young tableaux without two consecutive integers on the second row and having i consecutive integers $(k, k+1)$ on the rows 2 and 3 and having j consecutive integers on the rows 1 and 2.

Corollary 2.

$$\begin{aligned}
&|\{f \in C_n : |f|_{23} = i,\ |f|_{12} = j\}| \\
&\quad = |\{\alpha \in \mathit{Shuffle}_{2n} : |\alpha|_{\bar{a}} = i,\ |\alpha|_{ab} + |\alpha|_{bb} + |\alpha|_{a\bar{a}} + |\alpha|_{b\bar{a}} = j\}| \\
&\quad = |\{(w_1, w_2) \in P_{z,\bar{z}} \times P_{z,\bar{z}} : |w_1| = |w_2| = 2n;\ |w_1|_{\bar{z}\bar{z}} + 1 = i,\ |w_2|_{z\bar{z}} = j\}| \\
&\quad = \frac{1}{n^2}\binom{n}{i}\binom{n}{i-1}\binom{n}{j}\binom{n}{j-1}.
\end{aligned}$$

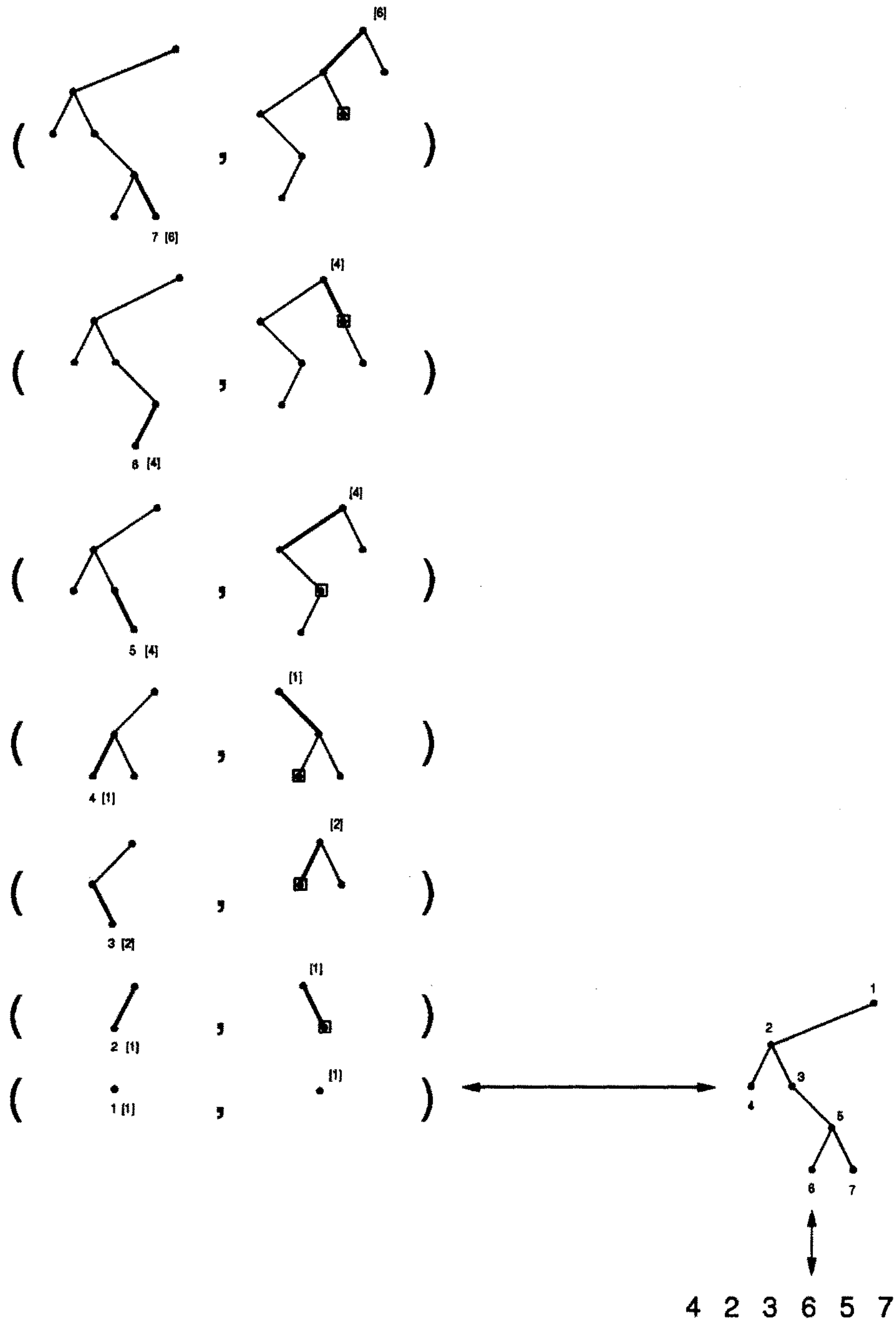

Fig. 8 The bijection Ψ^{-1}.

4.2. Standard tableaux without two consecutive integers on the same row

As some parameters are preserved by the bijections Φ, Υ and Ψ, we obtain the two following results. The first one gives the distribution of rectangular standard Young tableaux without two consecutive integers on the same row and having m consecutive integers $(k, k+1)$ on the rows 1 and 3. The second one comes from the paper of Mallows [13] and from the thesis of Gire [9].

Corollary 3. $|\{f \in B_n : |f|_{13} = m\}| = \left[\binom{n+1}{m}\cdot\binom{n+1}{m+1}\cdot\binom{n+1}{m+2}\right] \Big/ \left[\binom{n+1}{1}\cdot\binom{n+1}{2}\right]$.

Corollary 4.

$$\begin{aligned}
&|\{\pi \in \mathit{Baxter}_n \textit{ having } m \textit{ rises, } g \textit{ left-to-right maximum elements,}\\
&\qquad d \textit{ right-to-left maximum elements}\}|\\
&= |\{(t_1, t_2) \in \mathit{Twin}_n : t_2 \textit{ has } m \textit{ right leaves, } g \textit{ and } d \textit{ edges on its resp.}\\
&\qquad \textit{left and right branch}\}|\\
&= \binom{n+1}{m+1}\frac{g.d}{n.(n+1)}\left[\binom{n-g-1}{n-m-2}\binom{n-d-1}{m-1} - \binom{n-g-1}{n-m-1}\binom{n-d-1}{m}\right].
\end{aligned}$$

To date, we have progressed in our work about Baxter permutations [6]. We have established a new one-to-one correspondence between Baxter's permutations and three nonintersecting paths [8], which unifies the works of Viennot [16] and of Cori et al. [3]. Moreover, we obtain more precise results for the enumeration of (alternating or not) Baxter's permutations according to various parameters and we give a combinatorial interpretation of Mallows's formula [13].

Moreover, we have recently obtained a combinatorial proof of Conjecture 1.

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