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A note on finding clothoids

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Abstract

The clothoid is a spiral that is used as transition curve in highway and railway route design. Although its defining formulas are transcendental functions, recent work has shown that it can be used fairly easily on small computers. A single nonlinear equation must be solved in each of four common arrangements of the clothoid. The purpose of this work is to give starting intervals for which Newton's root finding method applied to those equations is guaranteed to converge.

MSC: 65S05

Keywords: Clothoids; Transition curves

1. Introduction

The clothoid is a spiral that is used as a transition curve in highway and railway route design [2,5,7]. It has the desirable property that the curvature is linearly related to the arc length. Although its defining formulas are transcendental functions, recent work has shown that it can be used fairly easily on small computers [8]. Some authors have avoided working with the transcendental functions by proposing approximations to the clothoid. In [13], the clothoid is approximated by high degree polynomial curves; in [12], the clothoid is approximated by an s-power series, which is a form of polynomial; and in [10], the clothoid is approximated by an arc spline.

A single nonlinear equation must be solved in each of four common arrangements of the clothoid found in [2], while the fifth common arrangement does not require the solution of a nonlinear

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equation. The purpose of this work is to give starting intervals for which Newton's root finding method [3, p. 62] always converges to the solution of the nonlinear equations in the four cases that require them. Previously [8], the bisection method [3, p. 51] was proposed for solving the equations numerically. However, when high accuracy is required, Newton's method can be much more efficient. If a root is known to one significant bit, then about 6 Newton iterations may be sufficient to get full double precision accuracy, while about 52 iterations of the bisection method would be needed for the same accuracy. For most work with the clothoids, the rational approximations given in [6] are convenient. However, clothoids evaluated to high precision with a modification of the routine in [11, p. 255] are used here so that the quadratic convergence of Newton's method can be better demonstrated.

The four common arrangements in which the clothoid transition curves are used are: straight line to circle with a single clothoid, circle to circle with a single clothoid, circle to circle with a pair of clothoids forming an S-shape, and circle to circle with a pair of clothoids forming a C-shape. The clothoids join the line and circles with G^2 continuity; that is, unit tangent vector and curvature are continuous at the point of contact. The four equations are formulated and analyzed in turn. Starting intervals are given for which Newton's root finding method applied to those equations always converges.

The clothoid used here is somewhat restricted in the amount of rotation of the unit tangent vector. The magnitude of rotation in any clothoid is restricted to $\pi/2$ rad. This restriction is helpful in the theory by making the inequalities that arise much easier to work with. This restriction is reasonable because, judging from the highway route location examples in [7, Chapter 6], most of the angular variation in a planned route is in circular arcs and not in the transition spiral.

2. Background

2.1. Convergence of Newton's root finding method

The following theorem gives sufficient conditions for the convergence of Newton's root finding method.

Theorem 1 (Buchanan and Turner [3, p. 64]). *If the function $f(x)$ is twice differentiable and satisfies the following conditions on the interval $[a,b]$:*

(a) *$f(a)$ and $f(b)$ are opposite signs,*
(b) *$f'(x)$ is nonzero in $[a,b]$,*
(c) *$f''(x)$ does not change sign in $[a,b]$,*
(d) *$|f(a)/f'(a)| < b-a$ and $|f(b)/f'(b)| < b-a$,*

then Newton's method converges to the unique root of the equation $f(x)=0$ in $[a,b]$ for any starting value in $[a,b]$.

2.2. Widening the interval containing a root

In proofs of the present application, the function and its derivative are contained in intervals. The range on the function values leads to an interval containing a zero of the function as required in

Theorem 1, but the lack of a precise value of the derivative means this interval has to be widened to successfully meet the conditions of Theorem 1.

Suppose the function $f(x)$ has a root r in the interval (r_L, r_H). Assume $f'(r)$ and $f''(r)$ are non-zero. If $f'(r)$ and $f''(r)$ are not the same sign, reflect $y = f(x)$ across the vertical line through $x = r$; if $f'(r) < 0$, reflect $y = f(x)$ across the X-axis. Now $f(x)$ has the standard form $f(r_L) < 0 < f(r_H)$, $f'(r) > 0$, $f''(r) > 0$.

Theorem 2. *If $f(x)$ changes sign over the interval (r_L, r_H), define the widened interval $[a,b]$, where $a = r_L - h$, $b = r_H + h$, and $h = r_H - r_L$. If $f'(x) > 0$ and $f''(x) > 0$ over the interval $[a,b]$, and if $2h(3f''_{max} - f''_{min}) < f'_{min}$, where f'_{min}, f''_{min}, and f''_{max} are smallest and largest values on $[a,b]$, then Newton's method converges to the unique root of $f(x) = 0$ in (r_L, r_H) for any starting value in $[a,b]$.*

Proof. The hypotheses in this theorem give the first three conditions of Theorem 1; the fourth condition will now be verified by first finding upper bounds on $|f(x)|$ and lower bounds on $f'(x)$ at the endpoints a and b. Notice that $f'(x)$ positive means that $f(a)$ is negative and $f(b)$ is positive.

Expansion by Taylor series gives

$$f(x) = (x-r)f'(r) + \frac{(x-r)^2}{2} f''(\xi_1), \qquad f'(x) = f'(r) + (x-r)f''(\xi_2),$$

where ξ_1 and ξ_2 are between x and r. There are two cases: $x < r$ and $x > r$. If $x < r$,

$$|f(x)| = -f(x) = (r-x)\left[f'(r) - \frac{r-x}{2} f''(\xi_1)\right] \leqslant (r-x)\left[f'(r) - \frac{r-x}{2} f''_{min}\right] \tag{2.1a}$$

and

$$f'(x) \geqslant f'(r) - (r-x)f''_{max}. \tag{2.1b}$$

If $x > r$,

$$|f(x)| \leqslant (x-r)f'(r) + \frac{(x-r)^2}{2} f''_{max} = (x-r)\left[f'(r) + \frac{x-r}{2} f''_{max}\right] \tag{2.1c}$$

and

$$f'(x) \geqslant f'(r) + (x-r)f''_{min}. \tag{2.1d}$$

Notice that $4hf''_{max} \leqslant 2h(2f''_{max} + (f''_{max} - f''_{min}))$ so $2h(3f''_{max} - f''_{min}) < f'_{min}$ implies $4hf''_{max} < f'_{min}$. Differentiation shows that the upper bound in (2.1a) is an increasing function of $(r-x)$ (use the fact that $f'(r) > f'_{min} > 2hf''_{max}$). Further, an upper bound on $r-x$ in (2.1a) and (2.1b) is $2h$, so from (2.1a),

$$|f(a)| < 2h[f'(r) - hf''_{min}]$$

and from (2.1b),

$$f'(a) > f'(r) - 2hf''_{max} > f'_{min} - 2hf''_{max} > 0.$$

An upper bound on $x - r$ in (2.1c) and (2.1d) is $2h$, so from (2.1c),

$$f(b) = |f(b)| < 2h[f'(r) + hf''_{\max}]$$

and from (2.1d),

$$f'(b) > f'(r) + hf''_{\min} > 0.$$

The ratios in the fourth condition of Theorem 1 are now

$$\left|\frac{f(a)}{f'(a)}\right| < \frac{2h[f'(r) - hf''_{\min}]}{f'(r) - 2hf''_{\max}} = h\left[3 - \frac{f'(r) - 2h(3f''_{\max} - f''_{\min})}{f'(r) - 2hf''_{\max}}\right] < 3h = b - a$$

and

$$\left|\frac{f(b)}{f'(b)}\right| < \frac{2h[f'(r) + hf''_{\max}]}{f'(r) + hf''_{\min}} = h\left[3 - \frac{f'(r) - 2hf''_{\max} + 3hf''_{\min}}{f'(r) + hf''_{\min}}\right] < 3h = b - a.$$

With the widened interval, the fourth condition of Theorem 1 is satisfied and Newton's method converges to the unique root of $f(x) = 0$ in (r_L, r_H) for any starting value in $[a, b]$. □

Below it will be shown that it is possible to widen the interval containing a root in each of the four clothoid arrangements treated here.

2.3. Clothoid properties

The clothoid is a transcendental curve that is usually represented in parametric form. Here, the clothoid is expressed in terms of an angle of tangent parameter. The unnumbered equations and inequalities in this subsection are intermediate results that are not referred to elsewhere.

The *standard clothoid* with scaling factor s is

$$\mathbf{H}(\theta) = s\begin{pmatrix} C(\theta) \\ S(\theta) \end{pmatrix}, \quad 0 \leqslant \theta, \tag{2.2}$$

where scaled versions of the Fresnel integrals $C(\theta)$ and $S(\theta)$ are [1, p. 300]

$$C(\theta) = \int_0^\theta \frac{\cos u}{\sqrt{u}}\,\mathrm{d}u, \quad S(\theta) = \int_0^\theta \frac{\sin u}{\sqrt{u}}\,\mathrm{d}u, \qquad 0 \leqslant \theta. \tag{2.3}$$

In applications, the clothoid in (2.2) may be rotated, translated, and reflected across the X-axis. The tangent vector of the clothoid (2.2) is

$$\mathbf{H}'(\theta) = \frac{s}{\sqrt{\theta}}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad 0 \leqslant \theta, \tag{2.4}$$

from which it is clear that the parameter θ is the angle of tangent vector with respect to the X-axis. The curvature of the clothoid is

$$k(\theta) = \frac{\mathbf{H}'(\theta) \times \mathbf{H}''(\theta)}{\|\mathbf{H}'(\theta)\|^3} = \frac{\sqrt{\theta}}{s}, \quad 0 \leqslant \theta. \tag{2.5}$$

For $0 < \theta_0 < \theta_1$, integration by parts on $C(\theta)$ and $S(\theta)$ in (2.3) gives

$$C(\theta_1) - C(\theta_0) = \frac{\sin\theta_1}{\sqrt{\theta_1}} - \frac{\sin\theta_0}{\sqrt{\theta_0}} + \frac{1}{2}\int_{\theta_0}^{\theta_1} \frac{\sin u}{u^{3/2}}\,\mathrm{d}u, \tag{2.6a}$$

$$S(\theta_1) - S(\theta_0) = -\frac{\cos\theta_1}{\sqrt{\theta_1}} + \frac{\cos\theta_0}{\sqrt{\theta_0}} - \frac{1}{2}\int_{\theta_0}^{\theta_1} \frac{\cos u}{u^{3/2}}\,\mathrm{d}u. \tag{2.6b}$$

In order to express the *centre of curvature* of the clothoid, define integrals of modified Fresnel integrals in (2.3) and use integration by parts to give

$$C_{\mathrm{I}}(\theta) = \int_0^{\theta} \frac{C(u)}{2\sqrt{u}}\,\mathrm{d}u = \sqrt{\theta}\,C(\theta) - \sin\theta, \quad 0 \leqslant \theta, \tag{2.7a}$$

$$S_{\mathrm{I}}(\theta) = \int_0^{\theta} \frac{S(u)}{2\sqrt{u}}\,\mathrm{d}u = \sqrt{\theta}\,S(\theta) + \cos\theta - 1, \quad 0 \leqslant \theta. \tag{2.7b}$$

From (2.2), (2.4), (2.5), and (2.7), the centre of the circle of curvature of the clothoid is

$$\begin{aligned}\mathbf{M}(\theta) &= \mathbf{H}(\theta) + \frac{1}{k(\theta)}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\ &= s\begin{pmatrix} C(\theta) \\ S(\theta) \end{pmatrix} + \frac{s}{\sqrt{\theta}}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{s}{\sqrt{\theta}}\begin{pmatrix} C_{\mathrm{I}}(\theta) \\ S_{\mathrm{I}}(\theta) + 1 \end{pmatrix}, \quad 0 < \theta. \end{aligned} \tag{2.8}$$

Using (2.6) and (2.8), the difference between two centres of curvature can be expressed compactly as

$$\mathbf{M}(\theta_1) - \mathbf{M}(\theta_0) = \frac{s}{2}\begin{pmatrix} \displaystyle\int_{\theta_0}^{\theta_1} \frac{\sin u}{u^{3/2}}\,\mathrm{d}u \\ \displaystyle -\int_{\theta_0}^{\theta_1} \frac{\cos u}{u^{3/2}}\,\mathrm{d}u \end{pmatrix}, \quad 0 < \theta_0 < \theta_1. \tag{2.9}$$

Lower and upper *bounds on the clothoid and its integral* will now be developed. When θ is restricted to $0 < \theta < \pi/2$, the following inequalities hold

$$\theta\left(1 - \frac{\theta^2}{6}\right) < \sin\theta < \theta, \qquad 1 - \frac{\theta^2}{2} < \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} < 1. \tag{2.10}$$

A set of inequalities that will be useful later is now developed. Using (2.10) in (2.3),

$$1 - \frac{\theta^2}{10} < \frac{C(\theta)}{2\sqrt{\theta}} < 1, \qquad \frac{\theta}{3}\left(1 - \frac{\theta^2}{14}\right) < \frac{S(\theta)}{2\sqrt{\theta}} < \frac{\theta}{3}, \tag{2.11}$$

and using (2.11) in (2.7) with integration,

$$\theta\left(1 - \frac{\theta^2}{30}\right) < C_{\mathrm{I}}(\theta) < \theta, \qquad \frac{\theta^2}{6}\left(1 - \frac{\theta^2}{28}\right) < S_{\mathrm{I}}(\theta) < \frac{\theta^2}{6}. \tag{2.12}$$

Using differentiation and then integration by parts,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{S(\theta)}{2\sqrt{\theta}}\right)=\frac{2\sqrt{\theta}\sin\theta-S(\theta)}{4\theta\sqrt{\theta}}=\frac{1}{2\theta\sqrt{\theta}}\int_0^{\theta}\sqrt{u}\cos u\,\mathrm{d}u.$$

Lower and upper bounds on this derivative are from (2.10)

$$\frac{1}{3}\left(1-\frac{3\theta^2}{14}\right)<\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{S(\theta)}{2\sqrt{\theta}}\right)<\frac{1}{3}. \tag{2.13}$$

Lower and upper *bounds on two functions used in* Section 6 will now be developed. Define the two functions $c(\theta)$ and $s(\theta)$ in terms of (2.7) as

$$c(\theta)=[C_{\mathrm{I}}(\theta)]^2,\qquad s(\theta)=[S_{\mathrm{I}}(\theta)+1]^2-1. \tag{2.14}$$

Bounds on $c(\theta)$ and $s(\theta)$ are from (2.12)

$$\theta^2\left(1-\frac{\theta^2}{15}+\frac{\theta^4}{900}\right)<c(\theta)<\theta^2,$$

$$\frac{\theta^2}{3}\left(1+\frac{\theta^2}{21}-\frac{\theta^4}{168}+\frac{\theta^6}{9408}\right)<s(\theta)<\frac{\theta^2}{3}\left(1+\frac{\theta^2}{12}\right). \tag{2.15}$$

$c'(\theta)$ and $s'(\theta)$ are

$$c'(\theta)=2C_{\mathrm{I}}(\theta)\frac{C(\theta)}{2\sqrt{\theta}},\qquad s'(\theta)=2(S_{\mathrm{I}}(\theta)+1)\frac{S(\theta)}{2\sqrt{\theta}}. \tag{2.16}$$

Lower bounds on $c'(\theta)$ and $s'(\theta)$ are from (2.11) and (2.12)

$$2\theta\left(1-\frac{2\theta^2}{15}+\frac{\theta^4}{300}\right)<c'(\theta),\qquad \frac{2\theta}{3}\left(1+\frac{2\theta^2}{21}-\frac{\theta^4}{56}+\frac{\theta^6}{2352}\right)<s'(\theta), \tag{2.17}$$

$c''(\theta)$ and $s''(\theta)$ are from (2.16) and (2.7)

$$c''(\theta)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{C_{\mathrm{I}}(\theta)}{\sqrt{\theta}}C(\theta)\right)=\frac{\sin\theta}{\theta}\frac{C(\theta)}{2\sqrt{\theta}}+\frac{C_{\mathrm{I}}(\theta)}{\theta}\cos\theta,$$

$$s''(\theta)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left((S_{\mathrm{I}}(\theta)+1)\frac{S(\theta)}{\sqrt{\theta}}\right)=\frac{S^2(\theta)}{2\theta}+2(S_{\mathrm{I}}(\theta)+1)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{S(\theta)}{2\sqrt{\theta}}\right).$$

Lower and upper bounds on $c''(\theta)$ and $s''(\theta)$ are from (2.10)–(2.13),

$$2\left(1-\frac{2\theta^2}{5}+\frac{\theta^4}{60}\right)<c''(\theta)<2,$$

$$\frac{2}{3}\left(1+\frac{2\theta^2}{7}-\frac{5\theta^4}{56}+\frac{\theta^6}{336}\right)<s''(\theta)<\frac{2}{3}\left(1+\frac{\theta^2}{2}\right). \tag{2.18}$$

2.4. Treatment of the four cases

Three preliminary results are needed: (i) bounds on the unknown θ, which is a root of $f(\theta)=0$, (ii) a lower bound on $f'(\theta)$, and (iii) bounds on $f''(\theta)$. The many constants involved are listed in the appendix as subscripted α, β, γ, δ for the four arrangements that are treated. Numerical values calculated in double precision, but shown to 6 decimals, are also given.

3. A single clothoid from straight line to circle

The first case is the transition from a straight line to a circle of curvature $k_1 > 0$ with a single clothoid (see Fig. 1). A clothoid that joins the straight line to the circle in a G^2 manner is required; both points of contact will be calculated. If there is to be a transition spiral, the circle must not intersect the straight line. Let $Y > 1/k_1$ be the distance from the centre of the circle to the line. Without loss of generality, assume the straight line is the X-axis and the circle is above the X-axis.

Since $k(\theta) = k_1 = \sqrt{\theta}/s$, the scaling of the clothoid is $s = \sqrt{\theta}/k_1$. Equating the y-coordinate of the centre of the circle of curvature to Y in (2.8) gives an equation for θ in terms of the given k_1 and Y [8],

$$\frac{s}{\sqrt{\theta}}(S_{\mathrm{I}}(\theta)+1) = Y, \quad \text{or} \quad \frac{1}{k_1}(S_{\mathrm{I}}(\theta)+1) = Y.$$

The equation for θ is

$$f(\theta) = S_{\mathrm{I}}(\theta) - d^2 = 0, \quad d^2 = Y k_1 - 1 > 0. \tag{3.1}$$

From (2.12),

$$\alpha_1 \theta^2 < S_{\mathrm{I}}(\theta) < \frac{\theta^2}{6},$$

which leads to bounds on θ as

$$\text{(i)} \qquad 6d^2 < \theta^2 < \frac{1}{\alpha_1} d^2 \quad \text{or } \alpha_2 d < \theta < \alpha_3 d. \tag{3.2}$$

θ must be less than $\pi/2$ and this can be forced by restricting d as

$$d < \frac{\pi}{2\alpha_3} = \alpha_4.$$

$f'(\theta)$ is $S(\theta)/(2\sqrt{\theta})$, so from (2.11),

$$\text{(ii)} \qquad \alpha_5 \theta < f'(\theta). \tag{3.3}$$

From (2.13),

$$\text{(iii)} \qquad \alpha_6 < f''(\theta) < \tfrac{1}{3}. \tag{3.4}$$

Eqs. (3.3) and (3.4) show that $f'(\theta) > 0$ and $f''(\theta) > 0$, so the widening technique in Section 2.2 can be applied; from (3.2), $r_{\mathrm{L}} = \alpha_2 d$, $r_{\mathrm{H}} = \alpha_3 d$. Further,

$$h = r_{\mathrm{H}} - r_{\mathrm{L}} = \alpha_7 d, \quad a = r_{\mathrm{L}} - h = \alpha_8 d, \quad b = r_{\mathrm{H}} + h = \alpha_9 d. \tag{3.5}$$

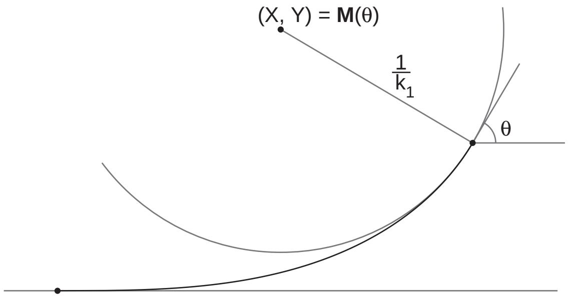

Fig. 1. A single clothoid from a straight line to a circle.

Table 1
Successive iterations of Newton's root solving method applied to $f(\theta)=0$

θ	$f(\theta)$
1.04	1.4×10^{-3}
1.036 74	2.5×10^{-6}
1.036 731 985 9	7.8×10^{-12}
1.036 731 985 880 09	1.3×10^{-16}

This widening means d must be reduced so as to keep θ in range; $b < \pi/2$ implies

$$d < \frac{\pi}{2\alpha_9} = \alpha_{10}. \tag{3.6}$$

Theorem 3. *If Y and k_1 are such that $0 < Yk_1 - 1 = d^2$, where d is restricted as in* (3.6) *and a and b are defined as in* (3.5), *then there is one root of Eq.* (3.1) *in* $[a,b]$. *Furthermore, Newton's method applied to Eq.* (3.1) *converges to that root for any starting value in* $[a,b]$.

Proof. $f(\theta)$ in (3.1) changes sign over $[a,b]$, while $f'(\theta) > 0$ and $f''(\theta) > 0$ over (a,b). From (3.2)–(3.5),

$$2h(3f''_{\max} - f''_{\min}) < 2\alpha_7(1-\alpha_6)d = \alpha_{11}d < \alpha_{12}d = \alpha_5 a < f'_{\min}.$$

Thus, $f(\theta)$ satisfies the conditions of Theorem 2 with interval $[a,b]$, which means Newton's method applied to Eq. (3.1) with a starting value in $[a,b]$ will always converge. □

Once θ is known, the scaling of the clothoid and the contact points of the circle and on the X-axis can be found. The clothoid can then be transformed so that the X-axis falls on the given line and the centre of the clothoid's circle of curvature at θ falls on the centre of the given circle.

Numerical results on an example problem are shown in Fig. 1 using IEEE 754 Standard double precision with $k_1 = 1/145$, and $Y = 170$; d is 0.415 227, which satisfies (3.6), the original interval for θ from (3.2) is (1.017 095, 1.065 107), the widened interval for θ from (3.5) is (0.969084, 1.113 118), and starting at the middle of the above range, the Newton iterations are shown in Table 1.

4. A single clothoid from circle to circle

The second case is the transition from one circle of curvature k_0 to another circle of curvature k_1 with a single clothoid (see Fig. 2). A clothoid that joins a point on one circle to a point on the other circle with G^2 continuity is required. Both points of contact will be calculated and the range of the angle of the tangent vector in the clothoid is limited to $\pi/2$. Without loss of generality, the curvatures of the two circles satisfy $0 < k_0 < k_1$, and let the distance between their centres be D. The clothoid is a spiral and the circles are circles of curvature of that spiral. By Kneser's theorem [4, p. 48], the second circle must be entirely inside the first circle, which means D must satisfy

$$0 < D < \frac{1}{k_0} - \frac{1}{k_1}. \tag{4.1}$$

Consider the two circles as circles of curvatures with curvatures k_0 and k_1 at parameter values θ_0 and θ_1 of a clothoid with scaling factor

$$\sqrt{S} = \sqrt{\frac{s}{(k_1 - k_0)\sqrt{k_0 k_1}}}. \tag{4.2}$$

The equation to be solved will be written in terms of s, but S is used for ease of writing expressions and is replaced by s whenever convenient. The square root of s is used because this results in a first power of s in the nonlinear equation to be solved, which makes differentiation easier. The factor $(k_1 - k_0)\sqrt{k_0 k_1}$ was found by experiment to simplify some expressions. Eq. (2.5) gives the formulas

$$\theta_0 = S k_0^2, \ \ \theta_1 = S k_1^2.$$

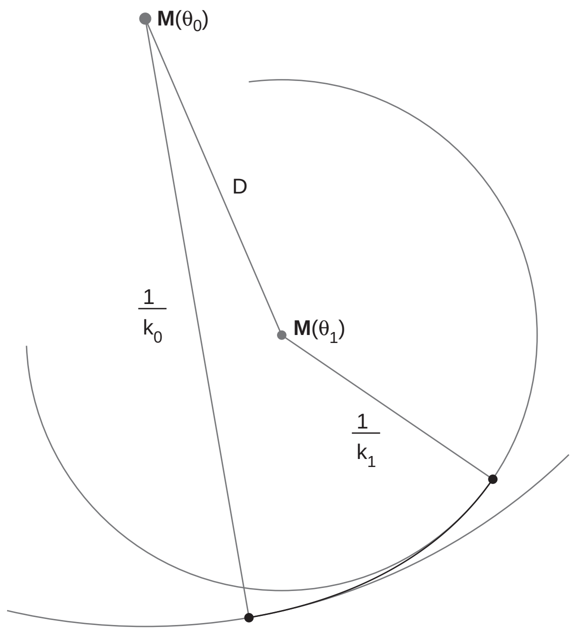

Fig. 2. A single clothoid from one circle to another circle.

The rotation of the angle of tangent $\theta_1 - \theta_0$ over the clothoid is restricted to $\pi/2$ so the ranges for S and s are

$$0 < S < \frac{\pi}{2(k_1^2 - k_0^2)}, \qquad 0 < s < \frac{\pi\sqrt{k_0 k_1}}{2(k_0 + k_1)}. \tag{4.3}$$

The equation to solve for S (or equivalently s) is

$$\sqrt{S}\|\mathbf{M}(Sk_1^2) - \mathbf{M}(Sk_0^2)\| = D. \tag{4.4}$$

Since D is limited by (4.1), Eq. (4.4) can be normalized by dividing by that upper bound. It is convenient to square both sides and subtract from one. Eq. (4.4) is thus transformed into

$$f(s) = E(s) - d^2 = 0, \qquad d^2 = 1 - \frac{k_0^2 k_1^2 D^2}{(k_1 - k_0)^2} > 0, \tag{4.5}$$

where $E(s)$ is defined as

$$E(s) = 1 - S\,\frac{k_0^2 k_1^2}{(k_1 - k_0)^2}\,\|\mathbf{M}(Sk_1^2) - \mathbf{M}(Sk_0^2)\|^2, \tag{4.6}$$

and s is in the range (4.3).

Using (2.9) in (4.6),

$$E(s) = 1 - \frac{Sk_0^2 k_1^2}{4(k_1 - k_0)^2}\left\{\left[\int_{Sk_0^2}^{Sk_1^2} \frac{\sin u}{u^{3/2}}\,\mathrm{d}u\right]^2 + \left[\int_{Sk_0^2}^{Sk_1^2} \frac{\cos u}{u^{3/2}}\,\mathrm{d}u\right]^2\right\}.$$

Replace u by Su^2 in the integrals to give

$$\begin{aligned} E(s) &= 1 - \frac{k_0^2 k_1^2}{(k_1 - k_0)^2}\left\{\left[\int_{k_0}^{k_1} \frac{\sin Su^2}{u^2}\,\mathrm{d}u\right]^2 + \left[\int_{k_0}^{k_1} \frac{\cos Su^2}{u^2}\,\mathrm{d}u\right]^2\right\} \\ &= 1 - \frac{k_0^2 k_1^2}{(k_1 - k_0)^2}\left\{\int_{k_0}^{k_1} \frac{\sin Su^2}{u^2}\,\mathrm{d}u \int_{k_0}^{k_1} \frac{\sin St^2}{t^2}\,\mathrm{d}t + \int_{k_0}^{k_1} \frac{\cos Su^2}{u^2}\,\mathrm{d}u \int_{k_0}^{k_1} \frac{\cos St^2}{t^2}\,\mathrm{d}t\right\} \\ &= 1 - \frac{k_0^2 k_1^2}{(k_1 - k_0)^2} \int_{k_0}^{k_1} \int_{k_0}^{k_1} \frac{\cos S(u^2 - t^2)}{t^2 u^2}\,\mathrm{d}t\,\mathrm{d}u. \end{aligned} \tag{4.7}$$

Identity (A.1) transforms (4.7) so that the argument of the cosine is nonnegative (needed later by the inequalities in Section 2) to give

$$E(s) = 1 - \frac{2k_0^2 k_1^2}{(k_1 - k_0)^2} \int_{k_0}^{k_1} \int_{k_0}^{u} \frac{\cos S(u^2 - t^2)}{t^2 u^2}\,\mathrm{d}t\,\mathrm{d}u.$$

Finally, use of (A.2) gives the formula

$$E(s) = \frac{2k_0^2 k_1^2}{(k_1 - k_0)^2} \int_{k_0}^{k_1} \int_{k_0}^{u} \frac{1 - \cos S(u^2 - t^2)}{t^2 u^2}\,\mathrm{d}t\,\mathrm{d}u. \tag{4.8}$$

Inequalities (2.10) applied to (4.8) with (A.3), (A.5) give bounds on $E(s)$,

$$E(s) > \frac{2k_0^2k_1^2}{(k_1-k_0)^2}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{1}{t^2u^2}\left(\frac{1}{2}[S(u^2-t^2)]^2-\frac{1}{24}[S(u^2-t^2)]^4\right)\mathrm{d}t\,\mathrm{d}u$$

$$> \frac{s^2}{3}-\frac{(k_0+k_1)^2}{84k_0k_1}s^4=\frac{s^2}{3}\left(1-\frac{(k_0+k_1)^2}{28k_0k_1}s^2\right)>\beta_1 s^2, \quad (4.9a)$$

$$E(s) < \frac{2k_0^2k_1^2}{(k_1-k_0)^2}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{1}{t^2u^2}\frac{1}{2}[S(u^2-t^2)]^2\,\mathrm{d}t\,\mathrm{d}u=\frac{1}{3}s^2. \quad (4.9b)$$

Since $E(s)=d^2$, (4.9) gives bounds on s,

(i) $$3d^2 < s^2 < \frac{1}{\beta_1}d^2 \quad \text{or } \beta_2 d < s < \beta_3 d. \quad (4.10)$$

s is limited by (4.3) and this can be forced by restricting d as

$$\beta_3 d < \frac{\pi\sqrt{k_0k_1}}{2(k_0+k_1)} \quad \text{or } d < \beta_4\frac{\sqrt{k_0k_1}}{k_0+k_1}.$$

Differentiating $f(s)$ in (4.5) with respect to s,

$$f'(s)=\frac{2k_0^2k_1^2}{(k_1-k_0)^3\sqrt{k_0k_1}}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{(u^2-t^2)\sin S(u^2-t^2)}{t^2u^2}\,\mathrm{d}t\,\mathrm{d}u. \quad (4.11)$$

Using the lower bound in (2.10) and (A.3), (A.5),

(ii) $$f'(s) > \frac{2k_0^2k_1^2}{(k_1-k_0)^3\sqrt{k_0k_1}}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{(u^2-t^2)}{t^2u^2}\left([S(u^2-t^2)]-\frac{1}{6}[S(u^2-t^2)]^3\right)\mathrm{d}t\,\mathrm{d}u$$

$$> \frac{2}{3}s-\frac{1}{21}\frac{(k_0+k_1)^2}{k_0k_1}s^3=\frac{2}{3}s\left(1-\frac{(k_0+k_1)^2}{14k_0k_1}s^2\right)>\beta_5 s. \quad (4.12)$$

Differentiating (4.11) with respect to s,

$$f''(s)=\frac{2k_0k_1}{(k_1-k_0)^4}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{(u^2-t^2)^2\cos S(u^2-t^2)}{t^2u^2}\,\mathrm{d}t\,\mathrm{d}u.$$

Using the bounds in (2.10)

$$\frac{2k_0k_1}{(k_1-k_0)^4}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{(u^2-t^2)^2}{t^2u^2}\left(1-\frac{1}{2}[S(u^2-t^2)]^2\right)\mathrm{d}t\,\mathrm{d}u$$

$$< f''(s) < \frac{2k_0k_1}{(k_1-k_0)^4}\int_{k_0}^{k_1}\int_{k_0}^{u}\frac{(u^2-t^2)^2}{t^2u^2}\mathrm{d}t\,\mathrm{d}u.$$

Using (A.3) and (A.5),

$$\text{(iii)} \qquad \beta_6 < \frac{2}{3} - \frac{(k_0+k_1)^2}{7k_0k_1}s^2 < f''(s) < \frac{2}{3}. \tag{4.13}$$

Eqs. (4.12) and (4.13) show that $f'(\theta) > 0$ and $f''(\theta) > 0$, so the widening technique in Section 2.2 can be applied; from (4.10), $r_L = \beta_2 d$, $r_H = \beta_3 d$. Further,

$$h = r_H - r_L = \beta_7 d, \quad a = r_L - h = \beta_8 d, \quad b = r_H + h = \beta_9 d. \tag{4.14}$$

This widening means d has to be reduced so as to keep s in range; $b < \pi/2$ implies

$$d < \frac{\pi}{2\beta_9}\frac{\sqrt{k_0k_1}}{k_0+k_1} = \beta_{10}\frac{\sqrt{k_0k_1}}{k_0+k_1}. \tag{4.15}$$

Theorem 4. *If D satisfies (4.1) and d defined in (4.5) satisfies (4.15) and a and b are defined as in (4.14), then there is one root of Eq. (4.5) in $[a,b]$. Furthermore, Newton's method applied to Eq. (4.5) converges to that root for any starting value in $[a,b]$.*

Proof. $f(s)$ in (4.5) changes sign over $[a,b]$, while $f'(s) > 0$ and $f''(s) > 0$ over (a,b). From (4.10), (4.12)–(4.14),

$$2h(3f''_{\max} - f''_{\min}) < 2\beta_7(2-\beta_6)d = \beta_{11}d < \beta_{12}d = \beta_5 a < f'_{\min}.$$

Thus, $f(s)$ satisfies the conditions of widening and consequently satisfies Theorem 1, which means Newton's method applied to Eq. (4.5) with a starting value in $[a,b]$ will always converge. □

Once s is found, calculate θ_0 and θ_1. Now the centres of circles of curvature at θ_0 and θ_1 can be transformed onto the centres of the given circles. This will map the two points on a clothoid to the two circles.

5. A pair of clothoids from circle to circle forming an *S*-curve

The third case is a transition from one circle of curvature $-k_0, k_0 > 0$, to another circle of curvature $k_1 > 0$ with a pair of clothoids forming an *S*-curve (see Fig 3). A clothoid that joins a point on one circle to a point on the other circle with G^2 continuity is required. Both points of contact will be calculated and the range of the angle of the tangent vector in each clothoid is limited to $\pi/2$. Let the distance between the centres of the circles be D. If there is to be an *S*-curve, the circles must be separate from one another, or D must satisfy

$$D > \frac{1}{k_0} + \frac{1}{k_1}. \tag{5.1}$$

Let the scaling factors of the two clothoids be s_0 and s_1. For simplicity of formulas, let the two circles be circles of curvature at the same angle parameter θ in both clothoids. Since $k_0(\theta) = k_0$ and

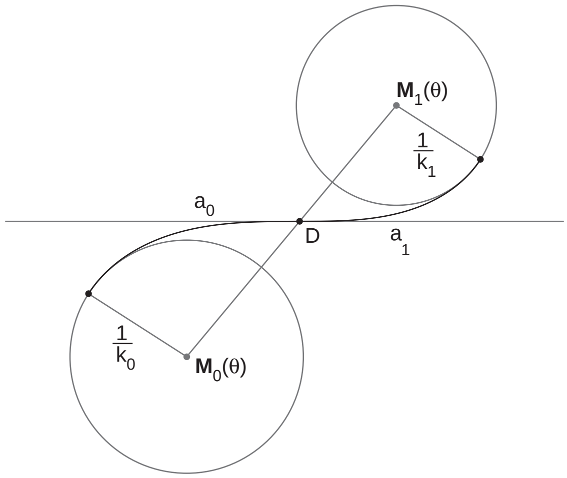

Fig. 3. A pair of clothoids from circle to circle forming an *S*-curve.

$k_1(\theta) = k_1$, (2.5) gives

$$s_0 = \frac{\sqrt{\theta}}{k_0} \quad \text{and} \quad s_1 = \frac{\sqrt{\theta}}{k_1}. \tag{5.2}$$

From (2.8) with modifications due to the geometry, the centres of the circles are

$$\mathbf{M}_0(\theta) = \frac{1}{k_0}\begin{pmatrix} -C_{\mathrm{I}}(\theta) \\ -S_{\mathrm{I}}(\theta) - 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_1(\theta) = \frac{1}{k_1}\begin{pmatrix} C_{\mathrm{I}}(\theta) \\ S_{\mathrm{I}}(\theta) + 1 \end{pmatrix},$$

so the distance between those centres is

$$\|\mathbf{M}_1(\theta) - \mathbf{M}_0(\theta)\| = \left(\frac{1}{k_0} + \frac{1}{k_1}\right)\sqrt{[C_{\mathrm{I}}(\theta)]^2 + [S_{\mathrm{I}}(\theta) + 1]^2}.$$

The equation to solve for θ is

$$\|\mathbf{M}_1(\theta) - \mathbf{M}_0(\theta)\| = D.$$

It is convenient to square, do some normalization and subtract 1. Define $E(\theta)$ using (2.14) as

$$E(\theta) = [C_{\mathrm{I}}(\theta)]^2 + [S_{\mathrm{I}}(\theta) + 1]^2 - 1 = c(\theta) + s(\theta). \tag{5.3}$$

The equation to solve for θ is

$$f(\theta) = E(\theta) - d^2 = 0, \quad d^2 = \frac{{k_0}^2{k_1}^2D^2}{(k_0 + k_1)^2} - 1 > 0. \tag{5.4}$$

With (2.15) in (5.3),

$$\frac{4}{3}\theta^2\left(1 - \frac{4}{105}\theta^2 - \frac{11}{16800}\theta^4 + \frac{1}{37632}\theta^6\right) < E(\theta) < \frac{4}{3}\theta^2\left(1 + \frac{\theta^2}{48}\right).$$

An analysis of the expression in parentheses in the lower bound of the above shows that it is a decreasing function of θ in $(0, \pi/2)$. Lower and upper bounds on $E(\theta)$ are

$$\gamma_1\theta^2 < E(\theta) < \gamma_2\theta^2.$$

(i) $\gamma_1\theta^2 < d^2 < \gamma_2\theta^2 \quad \text{or} \quad \gamma_3 d < \theta < \gamma_4 d$ (5.5)

θ is limited by $\pi/2$ and this can be forced by restricting d as

$$d < \frac{\pi}{2\gamma_4} = \gamma_5.$$

From (2.17),

$$\frac{8\theta}{3}\left(1 - \frac{8\theta^2}{105} - \frac{11\theta^4}{5600} + \frac{\theta^6}{9408}\right) < f'(\theta).$$

The expression in parentheses is a decreasing function of θ in $(0, \pi/2)$, so

(ii) $\gamma_6\theta < f'(\theta).$ (5.6)

From (2.18),

$$\frac{8}{3}\left(1 - \frac{8\theta^2}{35} - \frac{11\theta^4}{1120} + \frac{\theta^6}{1344}\right) < f''(\theta) < \frac{8}{3}\left(1 + \frac{\theta^2}{8}\right).$$

The expression in parentheses in the lower bound is a decreasing function of θ in $(0, \pi/2)$, so

(iii) $\gamma_7 < f''(\theta) < \gamma_8.$ (5.7)

Eqs. (5.6) and (5.7) show that $f'(\theta) > 0$ and $f''(\theta) > 0$, so the widening technique in Section 2.2 can be applied; from (5.5), $r_L = \gamma_3 d$, $r_H = \gamma_4 d$. Further,

$$h = r_H - r_L = \gamma_9 d, \quad a = r_L - h = \gamma_{10} d, \quad b = r_H + h = \gamma_{11} d. \tag{5.8}$$

This widening means d has to be reduced so as to keep θ in range; $b < \pi/2$ implies

$$d < \frac{\pi}{2\gamma_{11}} = \gamma_{12}. \tag{5.9}$$

Theorem 5. *If k_0, k_1, and D are such that (5.1) holds, d as defined in (5.4) is restricted as in (5.9) and a and b are defined in (5.8), then there is one root of equation (5.4) in $[a,b]$. Furthermore, Newton's method applied to Eq. (5.4) converges to that root for any starting value in $[a,b]$.*

Proof. $f(\theta)$ in (5.4) changes sign over $[a,b]$, while $f'(\theta) > 0$ and $f''(\theta) > 0$ over (a,b). From (5.5)–(5.8),

$$2h(3f''_{\max} - f''_{\min}) < 2\gamma_9(3\gamma_8 - \gamma_7)d = \gamma_{13}d < \gamma_{14}d = \gamma_6 a < f'_{\min}.$$

Thus, $f(\theta)$ satisfies the conditions of Theorem 2 with interval $[a,b]$, which means Newton's method applied to Eq. (5.4) with a starting value in $[a,b]$ will always converge. □

Once θ is known, calculate the scaling factors s_0 and s_1 and the centres of the circles of curvature on the clothoids. Now map those centres to the circle centres; the clothoid endpoints will map to points on the circles.

6. A pair of clothoids from circle to circle forming a C-curve

The fourth case is a transition from one circle of curvature $k_0 > 0$ to another circle of curvature $k_1 \geqslant k_0$ with a pair of clothoids that forms a C-curve (see Fig. 4). A clothoid that joins a point on one circle to a point on the other circle with G^2 continuity is required. Both points of contact will be calculated and the range of the angle of the tangent vector in each clothoid is limited to $\pi/2$. Let the distance between their centres be D. If there is to be a C-curve, the circles must be such that one is not enclosed inside the other, or D must satisfy

$$D > \frac{1}{k_0} - \frac{1}{k_1}. \tag{6.1}$$

Let the scaling factors of the two clothoids be s_0 and s_1. As in the previous case, let the two circles be circles of curvature at the same angle parameter θ in both clothoids. Since $k_0(\theta) = k_0$ and $k_1(\theta) = k_1$, (2.5) gives

$$s_0 = \frac{\sqrt{\theta}}{k_0} \quad \text{and} \quad s_1 = \frac{\sqrt{\theta}}{k_1}. \tag{6.2}$$

From (2.8) with modifications due to the geometry, the centres of the circles are

$$\mathbf{M}_0(\theta) = \frac{1}{k_0}\begin{pmatrix} -C_{\mathrm{I}}(\theta) \\ S_{\mathrm{I}}(\theta)+1 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_1(\theta) = \frac{1}{k_1}\begin{pmatrix} C_{\mathrm{I}}(\theta) \\ S_{\mathrm{I}}(\theta)+1 \end{pmatrix},$$

so the distance between those centres is

$$\|\mathbf{M}_1(\theta) - \mathbf{M}_0(\theta)\| = \sqrt{\left[\left(\frac{1}{k_0}+\frac{1}{k_1}\right)C_{\mathrm{I}}(\theta)\right]^2 + \left[\left(\frac{1}{k_1}-\frac{1}{k_0}\right)(S_{\mathrm{I}}(\theta)+1)\right]^2}.$$

The equation to solve for θ is

$$\|\mathbf{M}_1(\theta) - \mathbf{M}_0(\theta)\| = D. \tag{6.3}$$

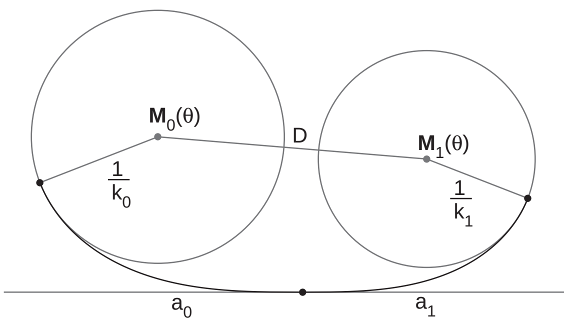

Fig. 4. A pair of clothoids from circle to circle forming a C-curve.

Define r as

$$r=\frac{k_1-k_0}{k_1+k_0},\quad 0\leqslant r<1. \tag{6.4}$$

It is convenient to square both sides of Eq. (6.3), do some normalization, and subtract r^2. Define $E(\theta)$ using (2.14) as

$$E(\theta)=[C_{\rm I}(\theta)]^2+r^2\{[S_{\rm I}(\theta)+1]^2-1\}=c(\theta)+r^2s(\theta). \tag{6.5}$$

The equation to solve for θ is

$$f(\theta)=E(\theta)-d^2=0,\quad d^2=\frac{[k_0k_1D]^2-(k_1-k_0)^2}{(k_1+k_0)^2}>0. \tag{6.6}$$

In (2.15), the expression in parentheses in the lower bound for $c(\theta)$ is decreasing while the expression in parentheses in both the lower and upper bounds for $s(\theta)$ are increasing for θ in $(0,\pi/2)$. These bounds in (6.5) and (6.6) give

$$(\delta_1+\delta_2r^2)\theta^2<d^2<(1+\delta_3r^2)\theta^2, \tag{6.7}$$

(i) $$\frac{d}{\sqrt{1+\delta_3r^2}}<\theta<\frac{d}{\sqrt{\delta_1+\delta_2r^2}}. \tag{6.8}$$

θ is limited by $\pi/2$ and this can be forced by restricting d as

$$d<\frac{\pi}{2}\sqrt{\delta_1+\delta_2r^2}. \tag{6.9}$$

From (2.17), the expression in parentheses in the lower bound for $c'(\theta)$ is decreasing while the expression in parentheses in the lower bound for $s'(\theta)$ is increasing for θ in $(0,\pi/2)$, so

(ii) $$(\delta_4+\delta_5r^2)\theta<f'(\theta). \tag{6.10}$$

From (2.18), the expression in parentheses in the lower bound for $c''(\theta)$ is decreasing. The expression in parentheses in the lower bound for $s''(\theta)$ increases until $\theta=\delta_6$ and then decreases for θ in $(0,\pi/2)$. Checking both endpoints, its minimum value occurs at $\theta=0$, so

(iii) $$\delta_7+\delta_5r^2<f''(\theta)<2+\delta_8r^2. \tag{6.11}$$

Eqs. (6.9) and (6.10) show that $f'(\theta)>0$ and $f''(\theta)>0$, so the widening technique in Section 2.2 can be applied; from (6.8),

$$r_{\rm L}=\frac{d}{\sqrt{1+\delta_3r^2}},\quad r_{\rm H}=\frac{d}{\sqrt{\delta_1+\delta_2r^2}}. \tag{6.12}$$

Further,

$$h=r_{\rm H}-r_{\rm L},\quad a=r_{\rm L}-h,\quad b=r_{\rm H}+h. \tag{6.13}$$

This widening means d has to be reduced so as to keep θ in range; $b < \pi/2$ implies

$$d < \frac{\pi}{2}\frac{\sqrt{\delta_1 + \delta_2 r^2}\sqrt{1 + \delta_3 r^2}}{2\sqrt{1 + \delta_3 r^2} - \sqrt{\delta_1 + \delta_2 r^2}}. \tag{6.14}$$

Theorem 6. *If k_0, k_1, and D are such that (6.1) holds, d as defined in (6.6) is restricted as in (6.14) and a and b are defined as in (6.12), then there is one root of Eq. (6.5) in $[a,b]$. Furthermore, Newton's method applied to Eq. (6.5) converges to that root for any starting value in $[a,b]$.*

Proof. From (6.6) and (6.7), $f(\theta)$ changes sign over $[a,b]$, while $f'(\theta) > 0$ and $f''(\theta) > 0$ over $[a,b]$. Starting from inequality (6.15), one can derive the steps which, in the reverse order, give a proof of (6.15).

Since $\gamma_{13}, \gamma_{14}, \gamma_{15}$, and γ_{16} are all positive, and for $0 \leqslant r < 1$,

$$0 < \delta_{16} + \delta_{13}r^2 + \delta_{14}(1 - r^4) + \delta_{15}(1 - r^6) \quad \text{or} \quad 0 < (\delta_{16} + \delta_{14} + \delta_{15}) + \delta_{13}r^2 - \delta_{14}r^4 - \delta_{15}r^6,$$

$$0 < -(\delta_9 + \delta_{10}r^2)^2\frac{1}{r_L^2} + (\delta_{11} + \delta_{12}r^2)^2\frac{1}{r_H^2} \quad \text{or} \quad (\delta_9 + \delta_{10}r^2)r_H < (\delta_{11} + \delta_{12}r^2)r_L,$$

$$2(r_H - r_L)[(6 - \delta_7) + (3\delta_8 - \delta_5)r^2] < (\delta_4 + \delta_5 r^2)(2r_L - r_H).$$

Thus, from (6.8) and (6.10)–(6.13),

$$2h(3f''_{max} - f''_{min}) < 2h[3(2 + \delta_8 r^2) - (\delta_7 + \delta_5 r^2)] < (\delta_4 + \delta_5 r^2)a < f'_{min}. \tag{6.15}$$

Finally, $f(\theta)$ satisfies the conditions of Theorem 2 with interval $[a,b]$, which means Newton's method applied to Eq. (6.6) with a starting value in $[a,b]$ will always converge. □

Once θ is known, calculate the scaling factors s_0 and s_1 and the centres of the circles of curvature on the clothoids. Now map those centres to the circle centres; the clothoid endpoints will map to points on the circles.

7. Conclusions

The determination of clothoid transition curves is an important part of highway and railway route design. The main result of this paper is that, with some restrictions on the geometric requirements, Newton's root solving method always converges to the unique root in the four common arrangements of clothoids. This result is of use to software developers as they can now confidently incorporate Newton's root solving method into their programs without having to consider the possibility that it might not work.

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Appendix.

Constants used in Section 3:

$$\alpha_1 = \frac{1}{6}\left[1 - \frac{1}{28}\left(\frac{\pi}{2}\right)^2\right] \approx 0.151980,$$

$$\alpha_2 = \sqrt{6} \approx 2.449490,$$

$$\alpha_3 = \frac{1}{\sqrt{\alpha_1}} \approx 2.565117,$$

$$\alpha_4 = \frac{\pi}{2\alpha_3} \approx 0.612368,$$

$$\alpha_5 = \frac{1}{3}\left[1 - \frac{1}{14}\left(\frac{\pi}{2}\right)^2\right] \approx 0.274586,$$

$$\alpha_6 = \frac{1}{3}\left[1 - \frac{3}{14}\left(\frac{\pi}{2}\right)^2\right] \approx 0.157090,$$

$$\alpha_7 = \alpha_3 - \alpha_2 \approx 0.115627,$$

$$\alpha_8 = \alpha_2 - \alpha_7 \approx 2.333863,$$

$$\alpha_9 = \alpha_3 + \alpha_7 \approx 2.680744,$$

$$\alpha_{10} = \frac{\pi}{2\alpha_9} \approx 0.585955,$$

$$\alpha_{11} = 2\alpha_7(1 - \alpha_6) \approx 0.194926,$$

$$\alpha_{12} = \alpha_5\alpha_8 \approx 0.640845 \text{ (larger than } \alpha_{11}).$$

Constants used in Section 4:

$$\beta_1 = \frac{1}{3}\left[1 - \frac{1}{28}\left(\frac{\pi}{2}\right)^2\right] \approx 0.303960,$$

$$\beta_2 = \sqrt{3} \approx 1.732051,$$

$$\beta_3 = \frac{1}{\sqrt{\beta_1}} \approx 1.813811,$$

$$\beta_4 = \frac{\pi}{2\beta_3} \approx 0.866020,$$

$$\beta_5 = \frac{2}{3}\left[1 - \frac{1}{14}\left(\frac{\pi}{2}\right)^2\right] \approx 0.549171,$$

$$\beta_6 = \frac{2}{3}\left[1 - \frac{3}{14}\left(\frac{\pi}{2}\right)^2\right] \approx 0.314181,$$

$$\beta_7 = \beta_3 - \beta_2 \approx 0.081761,$$

$$\beta_8 = \beta_2 - \beta_7 \approx 1.650290,$$

$$\beta_9 = \beta_3 + \beta_7 \approx 1.895572,$$

$$\beta_{10} = \frac{\pi}{2\beta_9} \approx 0.828666,$$

$$\beta_{11} = 2\beta_7(2 - \beta_6) \approx 0.275667,$$

$$\beta_{12} = \beta_5\beta_8 \approx 0.906292 \text{ (larger than } \beta_{11}).$$

Constants used in Section 5:

$$\gamma_1 = \frac{4}{3}\left[1 - \frac{4}{105}\left(\frac{\pi}{2}\right)^2 - \frac{11}{16800}\left(\frac{\pi}{2}\right)^4 + \frac{1}{37632}\left(\frac{\pi}{2}\right)^6\right] \approx 1.203222,$$

$$\gamma_2 = \frac{4}{3}\left[1 + \frac{1}{48}\left(\frac{\pi}{2}\right)^2\right] \approx 1.401872,$$

$$\gamma_3 = \frac{1}{\sqrt{\gamma_2}} \approx 0.844590,$$

$$\gamma_4 = \frac{1}{\sqrt{\gamma_1}} \approx 0.911648,$$

$$\gamma_5 = \frac{\pi}{2\gamma_4} \approx 1.723030,$$

$$\gamma_6 = \frac{8}{3}\left[1 - \frac{8}{105}\left(\frac{\pi}{2}\right)^2 - \frac{11}{5600}\left(\frac{\pi}{2}\right)^4 + \frac{1}{9408}\left(\frac{\pi}{2}\right)^6\right] \approx 2.137721,$$

$$\gamma_7 = \frac{8}{3}\left[1 - \frac{8}{35}\left(\frac{\pi}{2}\right)^2 - \frac{11}{1120}\left(\frac{\pi}{2}\right)^4 + \frac{1}{1344}\left(\frac{\pi}{2}\right)^6\right] \approx 1.033083,$$

$$\gamma_8 = \frac{8}{3}\left[1 + \frac{1}{8}\left(\frac{\pi}{2}\right)^2\right] \approx 3.489134,$$

$$\gamma_9 = \gamma_4 - \gamma_3 \approx 0.067058,$$

$$\gamma_{10} = \gamma_3 - \gamma_9 \approx 0.777532,$$

$$\gamma_{11} = \gamma_4 + \gamma_9 \approx 0.978706,$$

$$\gamma_{12} = \frac{\pi}{2\gamma_{11}} \approx 1.604973,$$

$$\gamma_{13} = 2\gamma_9(3\gamma_8 - \gamma_7) \approx 1.265294,$$

$$\gamma_{14} = \gamma_6\gamma_{10} \approx 1.662146 \text{ (larger than } \gamma_{13}).$$

Constants used in Section 6:

$$\delta_1 = 1 - \frac{1}{15}\left(\frac{\pi}{2}\right)^2 + \frac{1}{900}\left(\frac{\pi}{2}\right)^4 \approx 0.842271,$$

$$\delta_2 = \frac{1}{3},$$

$$\delta_3 = \frac{1}{3}\left[1 + \frac{1}{12}\left(\frac{\pi}{2}\right)^2\right] \approx 0.401872,$$

$$\delta_4 = 2\left[1 - \frac{2}{15}\left(\frac{\pi}{2}\right)^2 + \frac{1}{300}\left(\frac{\pi}{2}\right)^4\right] \approx 1.382613,$$

$$\delta_5 = \frac{2}{3},$$

$$\delta_6 = \sqrt{10 - 2\sqrt{17}} \approx 1.324307,$$

$$\delta_7 = 2\left[1 - \frac{2}{5}\left(\frac{\pi}{2}\right)^2 + \frac{1}{60}\left(\frac{\pi}{2}\right)^4\right] \approx 0.229015,$$

$$\delta_8 = \frac{2}{3}\left[1 + \frac{1}{2}\left(\frac{\pi}{2}\right)^2\right] \approx 1.489134,$$

$$\delta_9 = 12 + \delta_4 - 2\delta_7 \approx 12.924584,$$

$$\delta_{10} = -\delta_5 + 6\delta_8 \approx 8.268136,$$

$$\delta_{11} = 12 + 2\delta_4 - 2\delta_7 \approx 14.307198,$$

$$\delta_{12} = 6\delta_8 \approx 8.934802,$$

$$\delta_{13} = \delta_2\delta_{11}^2 - \delta_3\delta_9^2 + 2\delta_1\delta_{11}\delta_{12} - 2\delta_9\delta_{10} \approx 2.715211 \text{ (positive)},$$

$$\delta_{14} = \delta_{10}^2 - \delta_1\delta_{12}^2 + 2\delta_3\delta_9\delta_{10} - 2\delta_2\delta_{11}\delta_{12} \approx 1.791577 \text{ (positive)},$$

$$\delta_{15} = \delta_3\delta_{10}^2 - \delta_2\delta_{12}^2 \approx 0.862587 \text{ (positive)},$$

$$\delta_{16} = \delta_1\delta_{11}^2 - \delta_9^2 - \delta_{14} - \delta_{15} \approx 2.710408 \text{ (positive)}.$$

Formulas:

1. An identity for double integrals [9]

$$\int_0^a \int_0^a f(t,u)\,\mathrm{d}t\,\mathrm{d}u = \int_0^a \int_0^u [f(t,u) + f(u,t)]\,\mathrm{d}t\,\mathrm{d}u. \tag{A.1}$$

2. Double integral

$$\int_{k_0}^{k_1} \int_{k_0}^{u} \frac{1}{t^2u^2}\,\mathrm{d}t\,\mathrm{d}u = \frac{1}{2}\,\frac{(k_1-k_0)^2}{k_0^2k_1^2}. \tag{A.2}$$

3. Double integral

$$\int_{k_0}^{k_1} \int_{k_0}^{u} \frac{(u^2-t^2)^2}{t^2u^2}\,\mathrm{d}t\,\mathrm{d}u = \frac{1}{3}\,\frac{(k_1-k_0)^4}{k_0k_1}. \tag{A.3}$$

4. Double integral

$$\int_{k_0}^{k_1} \int_{k_0}^{u} \frac{(u^2-t^2)^4}{t^2u^2}\,\mathrm{d}t\,\mathrm{d}u = \frac{1}{105}\,\frac{(k_1-k_0)^6(15k_0^2+26k_0k_1+15k_1^2)}{k_0k_1}$$

$$= \frac{(k_1-k_0)^6((k_0+k_1)^2-4k_0k_1)}{7k_0k_1} \tag{A.4}$$

$$< \frac{(k_1-k_0)^6(k_0+k_1)^2}{7k_0k_1}. \tag{A.5}$$

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