The Module Factorization of Operators on Hilbert Space

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Submitted by Ky Fan
Received December 31, 1987

1. INTRODUCTION

It is known that solutions of certain operator equations are closely related to the problem of factoring an operator with respect to a nest \( \mathcal{E} \) of projections in a Hilbert space \( H \) \[3\]. A representation \( A = ST \) is a factorization of \( A \) with respect to the nest \( \mathcal{E} \) if \( S \) leaves invariant each member of \( \mathcal{E} \) and \( T \) leaves invariant the orthogonal complement of each member of \( \mathcal{E} \) \[1\]. In infinite dimensions factorizations of this type were first considered by Gohberg and Krein \[4\]. In this paper we generalize such factorizations of \( A \) by introducing the notion of regularity relative to a nest algebra module \( \mathcal{U} \) and then use it to classify them. Moreover we give a necessary and sufficient condition, when \( A \) is a positive operator, to have a factorization of the form \( A = S^*S \) where \( S \in \mathcal{U} \) and \( S^{-1} \in (\mathcal{U}^\perp)^* \). This generalizes the main result in \[3\]. The exposition and structure of this paper follows closely those in \[1, 3\].

Standard terminology and notation will be used throughout this paper (see, for example, \[2\]). The terms Hilbert space, operator, and projection will mean complex Hilbert space, bounded linear operator, and orthogonal projection, respectively. Alg \( \mathcal{E} \) denotes the corresponding nest algebra to the nest \( \mathcal{E} \). If \( E \rightarrow \bar{E} \) is a left continuous order homomorphism of \( \mathcal{E} \), let \( \mathcal{U} = \{ X \in \mathcal{B}(H) : X E = \bar{E} X E \text{ for all } E \in \mathcal{E} \} \). \( \mathcal{U} \) is an Alg \( \mathcal{E} \) module and the set \( \mathcal{U}^\perp = \{ X \in \mathcal{B}(H) : \bar{E} X = \bar{E} X E \text{ for all } E \in \mathcal{E} \} \) is then an Alg \( \mathcal{E}^\perp \) module. By \( A^* \) we mean the adjoint of the operator \( A \) and \( \mathcal{U}^* = \{ A^* : A \in \mathcal{U} \} \).

2. CLASSIFICATION OF MODULE FACTORIZATIONS

In this section we consider factorizations of an operator \( A \) on the Hilbert space \( H \) of the form \( A = ST \) where \( S \in \mathcal{U} \), \( S^{-1} \in (\mathcal{U}^\perp)^* \) and \( T \in \mathcal{U} \), \( T^{-1} \in \mathcal{U}^* \). First we show
PROPOSITION 1. Let δ be a complete nest of projections and \( \mathcal{U} \) an \( \text{Alg } \delta \) module. Suppose that there exists an invertible operator \( T \) such that \( T \) and \( T^{-1} \) belong to \( \mathcal{U} \). Then the order homomorphism \( E \to \bar{E} \) which determines \( \mathcal{U} \) is such that \( E \leq \bar{E} \) for every \( E \in \delta \).

Proof: Since
\[
E = T^{-1}TE = T^{-1}E\bar{E}T = \bar{E}T^{-1}E\bar{E} = \bar{E}T^{-1}TE = \bar{E}E,
\]
it follows that \( \bar{E} \geq E \) for all \( E \in \delta \), which implies that \( \bar{E} \geq E \) for every \( E \in \delta \).

A consequence of the above Proposition 1 is that if \( S \) is an invertible operator such that \( S \in \mathcal{U} \) and \( S^{-1} \in \mathcal{U} \) then \( E \geq \bar{E} \) for every \( E \in \delta \). This shows that we cannot generalize the notion of the regular factorization of an operator (see [1]) relative to the pair of modules \( \mathcal{U}, \mathcal{U} \) because then the homomorphism \( E \to \bar{E} \) becomes the identity and \( \mathcal{U} = \text{Alg } \delta \). But it is possible to have such a generalization when we use the modules \( \mathcal{U} \) and \( (\mathcal{U}^+)^* \). The following example [5] justifies this.

EXAMPLE. Let \( H \) be the Hilbert space \( L_2[0, 1] \) and \( \delta \) the nest
\[
\{E_t : t \in [0, 1], \text{ where } E_t \text{ is the projection on the subspace } L_2[0, t]\}.
\]
Consider a function \( \varphi : [0, 1] \to [0, 1] \) which is onto and differentiable with \( \frac{1}{2} \leq \varphi' \leq 2 \). Then there exists the inverse \( \varphi^{-1} \) of \( \varphi \) and satisfies \( \frac{1}{2} \leq (\varphi^{-1})' \leq 2 \). The correspondence \( E_t \to E_{\varphi(t)} \) is an order homomorphism from \( \delta \) into itself. This homomorphism determines an \( \text{Alg } \delta \) module \( \mathcal{U} \).

Define the operator
\[
A : L_2[0, 1] \to L_2[0, 1], \quad Af = f \circ \varphi^{-1}.
\]
It is easy to see that \( A \) is bounded and invertible. Moreover \( A \in \mathcal{U}, A^{-1} \in (\mathcal{U}^+)^*, \mathcal{U} \neq \text{Alg } \delta, \mathcal{U} \neq (\mathcal{U}^+)^*, \) and \( A \notin \text{Alg } \delta \).

DEFINITION 2. A representation
\[
A = ST
\]
is called a regular factorization of \( A \) with respect to the module \( \mathcal{U} \) if \( S \) and \( T \) are invertible and \( S \in \mathcal{U}, S^{-1} \in (\mathcal{U}^+)^*, T \in \mathcal{U}, \) and \( T^{-1} \in \mathcal{U}^* \). The representation (1) is called a left regular factorization of \( A \) with respect to the module \( \mathcal{U} \) if \( S \) is invertible, \( S \in \mathcal{U}, S^{-1} \in (\mathcal{U}^+)^*, \) and \( T \in \mathcal{U}^\perp \). A right regular factorization is defined similarly.
In the following lemma we prove that a regular factorization is unique up to a multiplicative factor from the algebra $\text{Alg } \mathcal{E} \cap \text{Alg } \mathcal{F}^\perp$.

**Lemma 3.** If $A = S_1T_1$ and $A = S_2T_2$ are two regular factorizations of $A$ with respect to the module $\mathcal{U}$ then there exists an operator $D$ in $\text{Alg } \mathcal{E} \cap \text{Alg } \mathcal{F}^\perp$ such that $S_1 = SD$ and $T_2 = DT_1$.

**Proof.** Let $A = S_1T_1 = S_2T_2$. Then

$$S^{-1}_2S_1 = T_2T^{-1}_1. \quad (2)$$

Since the factorizations are regular we have

(i) $S^{-1}_2SE = S^{-1}_2E S_1E = ES_2^{-1}E S_1E = ES_2^{-1}S_1E$ for every $E \in \mathcal{E}$ and so $S^{-1}_2S_1 \in \text{Alg } \mathcal{E}$.

(ii) $ET_2T^{-1}_1 = ET_2ET_1^{-1} = ET_2ET_1^{-1} \mathcal{E} = ET_2T_1^{-1} \mathcal{E}$ for every $E \in \mathcal{E}$ and so $T_2T^{-1}_1 \in \text{Alg } \mathcal{E}^\perp$.

Therefore from (2) and (i), (ii) we have that $S^{-1}_2S_1, T_2T^{-1}_1$ belong to $\text{Alg } \mathcal{E} \cap \text{Alg } \mathcal{F}^\perp$ and hence there exists an operator $D \in \text{Alg } \mathcal{E} \cap \text{Alg } \mathcal{F}^\perp$ such that $S^{-1}_2S_1 = T_2T^{-1}_1 = D$, and the proof is completed.

In the sequel we examine when a left regular factorization is regular.

**Lemma 4.** Suppose $T$ is an invertible operator in $\mathcal{A}(\mathcal{H})$. Then if $T \in \mathcal{U}$ the following are equivalent.

(i) $T^{-1} \in (\mathcal{U}^\perp)^*$.

(ii) $(\mathcal{E} - \mathcal{F}) T(E - F) (\mathcal{E} - \mathcal{F} \neq 0, E > F)$ is invertible for all $E, F \in \mathcal{E}$.

(iii) $\mathcal{E}TE \ (\mathcal{E} \neq 0)$ is invertible for all $E \in \mathcal{E}$.

(iv) $(I - \mathcal{E}) T(I - E) (\mathcal{E}, E \neq I)$ is invertible for all $E \in \mathcal{E}$.

**Proof.** Suppose (i) holds. Then for $E, F \in \mathcal{E}, E > F, \mathcal{E} \neq \mathcal{F}$ we have $\mathcal{E} > \mathcal{F}$ and since $T^\perp \mathcal{E} = ET^\perp \mathcal{E}$, $(I - F) T^\perp = (I - F) T^\perp (I - \mathcal{F})$, and $(I - \mathcal{F}) T = (I - \mathcal{F}) T(I - F)$ we have

$$(\mathcal{E} - \mathcal{F}) T(E - F)(E - F) T^{-1}(\mathcal{E} - \mathcal{F}) = (\mathcal{E} - \mathcal{F}) T(E - F) T^{-1}(\mathcal{E} - \mathcal{F})$$

$$(I - \mathcal{F}) T(E - F) T^{-1}(I - \mathcal{F}) \mathcal{E} = (I - \mathcal{F}) T E T^{-1}(I - \mathcal{F}) \mathcal{E}$$

$$= (I - \mathcal{F}) T T^{-1} \mathcal{E}$$

$$= \mathcal{E} - \mathcal{F}.$$
Similarly \((E - F) T^{-1}(\bar{E} - \bar{F})(\bar{E} - \bar{F}) T(E - F) = E - F\) and hence \((ii)\) follows. Clearly, since \(0 = 0\), \((ii)\) implies \((iii)\), and we can easily prove that \((i)\) implies \((iv)\). Suppose \((iii)\) holds. Then

\[
\bar{E} T E T^{-1} \bar{E} = T E T^{-1} \bar{E} = \bar{E}
\]

for every \(E \in \mathcal{E}, \bar{E} \neq 0\)

and

\[
T[E T^{-1} \bar{E} + (I - E) T^{-1}(I - \bar{E}) + E T^{-1}(I - \bar{E})] = T E T^{-1} \bar{E} + T(I - E) T^{-1}(I - \bar{E}) + T E T^{-1}(I - \bar{E})
\]

\[
= T E T^{-1} \bar{E} + I - \bar{E} - T E T^{-1} + T E T^{-1} \bar{E} + T E T^{-1} - T E T^{-1} \bar{E} = I.
\]

Therefore

\[
T^{-1} = E T^{-1} \bar{E} + (I - E) T^{-1}(I - \bar{E}) + E T^{-1}(I - \bar{E})
\]

and so

\[
T^{-1} \bar{E} = E T^{-1} \bar{E}
\]

for every \(E \in \mathcal{E}, \bar{E} \neq 0\).

But since the only projection \(E \in \mathcal{E}\) satisfying \(\bar{E} = 0\) is the projection \(E = 0\) \((TE = \bar{E}T = 0TE = 0\) implies \(T^{-1}TE = 0\) and hence \(E = 0\)) we have \(T^{-1} \bar{E} = E T^{-1} \bar{E}\) for all \(E \in \mathcal{E}\) and hence \(T^{-1} \in (\mathcal{M})^*\). Finally with the same argument as above we prove that \((iv)\) implies \((i)\).

**Proposition 5.** Suppose \(A = ST\) is a left regular factorization of \(A\) with respect to the module \(\mathcal{U}\). Then the following are equivalent:

(i) \(A = ST\) is regular.

(ii) \(E A^{-1} \bar{E} (E, \bar{E} \neq 0)\) is invertible for every \(E \in \mathcal{E}\).

(iii) \((I - \bar{E}) A(I - E) (E, \bar{E} \neq I)\) is invertible for every \(E \in \mathcal{E}\).

**Proof.** Let \(A = ST\) be a regular factorization. Then Lemma 3 shows that for every \(E \in \mathcal{E}, \bar{E} \neq 0\) the operators \(E T^{-1} \bar{E}, \bar{E} S^{-1} \bar{E}\) are invertible and for every \(E \in \mathcal{E}, \bar{E} \neq I\) the operators \((I - \bar{E}) S(I - \bar{E}), (I - \bar{E}) T(I - E)\) are also invertible. Therefore from

\[
E A^{-1} \bar{E} = E T^{-1} S^{-1} \bar{E} = (E T^{-1} \bar{E})(\bar{E} S^{-1} \bar{E})
\]

and

\[
(I - \bar{E}) A(I - E) = (I - \bar{E}) ST(I - E)
\]

\[
= (I - \bar{E}) S(I - \bar{E}) T(I - E)
\]

\[
= (I - \bar{E}) S(I - \bar{E})(I - \bar{E}) T(I - E)
\]
we have the invertibility of \( EA^{-1} \) and \( (I - \tilde{E}) A(I - E) \). Suppose (ii) holds. Then

\[
(\tilde{E}T)(EA^{-1}) = ET \tilde{A}^{-1}E = ES^{-1}E
\]

and since the operators \( EA^{-1}, ES^{-1}E \) are invertible we have that \( \tilde{E}T \) is invertible, and so, from Lemma 3, \( T^{-1} \in \mathcal{U}^* \). Therefore the factorization is regular. Similarly (iii) implies (i).

**COROLLARY 6.** If \( A = ST \) is a regular factorization with respect to the module \( \mathcal{U} \) the operators \( I - E + EA^{-1} \) and \( E + (I - \tilde{E}) A(I - E) \) are invertible.

**Proof:** Since from Proposition 4 the inverses of \( EA^{-1} \) and \( (I - \tilde{E}) A(I - E) \) exist we can easily prove that the inverse of \( I - E + EA^{-1} \) is the operator \( I - \tilde{E} + \tilde{E} \tilde{A}E \) and the inverse of \( E + (I - \tilde{E}) A(I - E) \) is the operator \( E + (I - E) A^{-1}(I - \tilde{E}) \).

### 3. The Module Factorization of Positive Operators

In this section we give a necessary and sufficient condition for a positive operator \( A \) to have a factorization of the form \( A = S^*S \) with \( S \in \mathcal{U} \) and \( S^{-1} \in (\mathcal{U}^\perp)^* \). This is a generalization of the main result in [3].

Let \( \mathcal{N} \) be a nest of subspaces of the Hilbert space \( H \), \( \mathcal{E} = \{ E_N : N \in \mathcal{N} \} \) the corresponding nest of projections, and \( \mathcal{U} \) an Alg \( \mathcal{E} \) module determined by the order homomorphism \( E_N \to \tilde{E}_N \).

**THEOREM 7.** Let \( A \) be a positive operator on \( H \). Then \( A = S^*S \) with \( S \in \mathcal{U} \), \( S^{-1} \in (\mathcal{U}^\perp)^* \) if and only if there exists a unitary operator \( U \) such that \( U \tilde{E}_N = E_{A^{1/2}N} U \) for every \( E_N \in \mathcal{E} \).

**Proof.** Suppose that there exists a unitary operator \( U \) such that \( U \tilde{E}_N = E_{A^{1/2}N} U \). Let \( S = U^*A^{1/2} \) and \( \tilde{N} \) be the range of \( \tilde{E}_N \). Then \( A = S^*S \) and \( S(N) = SE_N(H) = U^*A^{1/2}E_N(H) = \tilde{E}_N U^*(H) = \tilde{E}_N(H) = \tilde{N} \) for every \( N \in \mathcal{N} \). Therefore, from \( S(N) = \tilde{N} \), we have \( \tilde{E}_N SE_N = SE_N \) and \( E_N S^{-1} \tilde{E}_N = S^{-1} \tilde{E}_N \) for every \( N \in \mathcal{N} \). Hence \( S \in \mathcal{U} \) and \( S^{-1} \in (\mathcal{U}^\perp)^* \). It is proved in [3, Corollary 3] that the orthogonal projection on \( A^{1/2}E_N(H) \) is \( A^{1/2}(E_NAE_N)^{-1}A^{1/2} \). But

\[
A^{1/2}(E_NAE_N)^{-1}A^{1/2} = A^{1/2}(E_N S^*SE_N)^{-1}A^{1/2}
= A^{1/2}[(E_N S^*\tilde{E}_N)(\tilde{E}_N SE_N)]^{-1}A^{1/2}. \quad (3)
\]
From the fact that $S^{-1} \in (\mathcal{U}^{-1})^*$, Lemma 3 shows that the operators $\tilde{E}_N SE_N$ and $E_N S^* \tilde{E}_N$ are invertible, and hence from (3) we have

$$A^{1/2}(E_N AE_N)^{-1} A^{1/2} = A^{1/2}[(E_N S^{-1} \tilde{E}_N)(\tilde{E}_N S^* - E_N)] A^{1/2}$$

$$= A^{-1/2}A[(E_N S^{-1} \tilde{E}_N)(\tilde{E}_N S^* - E_N)] AA^{-1/2}$$

$$= A^{-1/2}S^*[(SE_N S^{-1} \tilde{E}_N)(\tilde{E}_N S^* - E_N S^*)] SA^{-1/2}$$

$$= A^{-1/2}S^*[(SS^{-1} \tilde{E}_N)(\tilde{E}_N S^* - S^*) S] A^{-1/2}$$

$$= A^{-1/2}S^* \tilde{E}_N SA^{-1/2}.$$

Now if we put $U = A^{-1/2} \tilde{E}_N$ then $U$ is a unitary operator and we have $E_{A^{1/2}N} = U \tilde{E}_N U^*$ or equivalently $U \tilde{E}_N = E_{A^{1/2}N} U$.

REFERENCES

5. N. HADJISAVVAS, personal communication.