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On a Modified Szasz–Mirakjan-Operator

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Let $C_A[0, \infty)$ be the set of all functions $f \in C[0, \infty)$ satisfying a growth-condition of the form $|f(t)| \leqslant Ae^{mt}$ $(A \in \mathbb{R}^+, m \in \mathbb{N})$. Then for $f \in C_A[0, \infty)$ and $x \in [0, \infty)$ the well-known Szasz–Mirakjan-operator is defined by

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1}$$

It is known (Grof [1]; Hermann [3]) that

THEOREM 1. *$(S_n)_{n \in \mathbb{N}}$ is a sequence of linear positive operators from $C_A[0, \infty)$ into $C[0, \infty)$ with the property*

$$\lim_{n \to \infty} S_n(f; x) = f(x) \text{ for all } f \in C_A[0, \infty),$$

uniformly on every interval $[x_1, x_2]$, $0 \leqslant x_1 < x_2 < \infty$.

The actual construction of the operators S_n requires estimation of infinite series which in a certain sense restricts their usefulness from the computational point of view. Thus the question arises, whether $S_n(f; x)$ cannot be replaced by a finite partial sum provided this will not change essentially the degree of convergence. In connection with this question Grof [2] introduced and examined the operator

$$S_{n,N}(f; x) := e^{-nx} \sum_{k=0}^{N} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{2}$$

for which the following result (cf. Grof [2; p. 114]) is valid.

THEOREM 2. *Let $N(n)$ be a sequence of positive integers with*

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$\lim_{n\to\infty} (N(n)/n) = \infty$. *Then* $(S_{n,N})_{n\in\mathbb{N}}$ *is a sequence of linear positive operators from* $C_A[0, \infty)$ *in* $C[0, \infty)$ *with the property*

$$\lim_{n\to\infty} S_{n,N}(f; x) = f(x) \qquad \textit{for all } f \in C_A[0, \infty) \textit{ and all } x \in [0, \infty).$$

However, Grof does not investigate what happens, if the sequence $N(n)/n$ does not tend to infinity. In particular he gives no answer to the question whether we cannot do without that assumption.

In the present paper we follow a course which is a little different from the one of Grof. Now, for $f \in C_M[0, \infty)$ and $x \in [0, \infty)$ we define

$$S_{n,\delta}(f; x) := e^{-nx} \sum_{k=0}^{[n(x+\delta)]} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{3}$$

where $C_M[0, \infty)$ denotes the set of all functions $f \in C[0, \infty)$ satisfying a growth-condition of the form $|f(t)| \leqslant A + Bt^{2m}$ $(A, B \in \mathbb{R}^+;\ m \in \mathbb{N})$. It is our aim to prove.

THEOREM 3. *Let* $\delta = \delta(n)$ *be a sequence of positive numbers with* $\lim_{n\to\infty} n^{1/2}\,\delta(n) = \infty$. *Then* $(S_{n,\delta})_{n\in\mathbb{N}}$ *is a sequence of positive linear operators from* $C_M[0, \infty)$ *in* $C[0, \infty)$ *with the property*

$$\lim_{n\to\infty} S_{n,\delta}(f; x) = f(x) \qquad \textit{for all } f \in C_M[0, \infty)$$

uniformly on every interval $[x_1, x_2]$, $0 \leqslant x_1 < x_2 < \infty$.

To prove Theorem 3 we need

LEMMA 4 (cf. Rathore [5, pp. 23–25]; Lehnhoff [4]). *Let* $0 \leqslant x_1 < x_2 < \infty$. *Then for every* $m \in \mathbb{N}$ *there exists a positive constant* $C(m, x_1, x_2)$ *such that*

$$S_n((t-x)^{2m}; x) \leqslant \frac{C(m, x_1, x_2)}{n^m} \qquad \textit{uniformly for all } x \in [x_1, x_2].$$

PROOF OF THEOREM 3. For $f \in C_M[0, \infty)$ constants A, $B \in \mathbb{R}^+$ and $m \in \mathbb{N}$ exist with

$$\begin{aligned} |f(t)| &\leqslant A + Bt^{2m} \leqslant A + B2^{2m}\{(t-x)^{2m} + x^{2m}\} \\ &= \underbrace{(A + B(2x)^{2m})}_{=:A_x} + B\, 2^{2m}(t-x)^{2m}. \end{aligned}$$

Thus it follows

$$S_{n,\delta}(f; x) = S_n(f; x) - R_n(f; x)$$

with

$$\begin{aligned}|R_n(f;x)| &\leqslant e^{-nx} \sum_{k=[n(x+\delta)]+1}^{\infty} \frac{(nx)^k}{k!} \left|f\left(\frac{k}{n}\right)\right| \\ &\leqslant e^{-nx} \sum_{k=[n(x+\delta)]+1}^{\infty} \frac{(nx)^k}{k!} \{A_x + B\, 2^{2m}(t-x)^{2m}\} \\ &\leqslant A_x e^{-nx} \sum_{|(k/n)-x|>\delta} \frac{(nx)^k}{k!} + 2^{2m} B S_n((t-x)^{2m}; x) \\ &\leqslant \{\delta^{-2m}(A + B(2x)^{2m}) + B\, 2^{2m}\} \frac{C(m, x_1, x_2)}{n^m} = o(1), \qquad n \to \infty\end{aligned}$$

uniformly on $[x_1, x_2]$, because

$$\lim_{n\to\infty} n^{1/2}\delta(n) = \infty \qquad \Leftrightarrow \qquad (\delta(n))^{-1} = o(n^{1/2}),\ n \to \infty. \ \blacksquare$$

The operators S_n and $S_{n,\delta}$ have the same approximation properties, if and only if

$$R_n(f;x) = S_n(f;x) - S_{n,\delta}(f;x) = o(1/n), \qquad n \to \infty, \tag{4}$$

uniformly on every interval $[x_1, x_2]$, $0 \leqslant x_1 < x_2 < \infty$ for all functions $f \in C_M[0, \infty)$.

Suppose $f \in C_M[0, \infty)$, then constants $A, B \in \mathbb{R}^+$ and $m \in \mathbb{N}$, $m \geqslant 2$, exist such that $|f(t)| \leqslant A + Bt^{2m}$, $t \geqslant 0$. Thus as in the proof of Theorem 3 we obtain

$$|R_n(f;x)| \leqslant (A + B(2x)^{2m})\delta^{-2s} \frac{C(s, x_1, x_2)}{n^s} + B2^{2m} \frac{C(m, x_1, x_2)}{n^m} \tag{5}$$

for every fixed $s \in \mathbb{N}$.

Because of (5) relation (4) holds, if

$$\lim_{n\to\infty} n^{1/2-1/2s}\,\delta(n) = \infty \qquad \text{for any fixed } s \in \mathbb{N}. \tag{6}$$

If $\delta(n) = n^{-\alpha}$ ($\alpha < \frac{1}{2}$) it is easy to verify that relation (6) is valid for every fixed $s \in \mathbb{N}$, $s > 1/(1-2\alpha)$.

Now, let us take the case $\delta(n) \equiv 1$. Then for any $b > 0$ one can consider the corresponding operators of the form (3)

$$\bar{S}_n(f;x) := e^{-nx} \sum_{k=0}^{[n(x+1)]} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{7}$$

as positive linear operators from $C[0, b+1]$ in $C[0, b]$ with the convergence property

$$\lim_{n\to\infty} \|\bar{S}_n(f; \cdot) - f(\cdot)\|_{C[0,b]} = 0 \qquad \text{for all } f \in C[0, b+1].$$

Up to this point we always required δ independent of x. At the end of this paper we briefly deal with the case $\delta(x) = 1 - x$ and the corresponding operators

$$\hat{S}_n(f; x) := e^{-nx} \sum_{k=0}^{n} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad f \in C[0, 1], x \in [0, 1) \tag{8}$$

for which the following theorem holds.

THEOREM 5. *$(\hat{S}_n)_{n\in\mathbb{N}}$ is a sequence of positive linear operators from $C[0, 1]$ in $C[0, 1]$ with the property*

$$\lim_{n\to\infty} \hat{S}_n(f; x) = f(x) \qquad \textit{for all } f \in C[0, 1]$$

uniformly on every compact subinterval of $[0, 1)$.

Proof. Putting

$$\begin{aligned} f^*(x) &:= f(x) \qquad \text{for} \quad 0 \leqslant x \leqslant 1, \\ &:= f(1) \qquad \text{for} \quad x > 1, \end{aligned}$$

we have

$$\hat{S}_n(f; x) = S_n(f^*; x) - f(1) R_n(x)$$

with

$$\begin{aligned} R_n(x) &= e^{-nx} \sum_{k>n} \frac{(nx)^k}{k!} \leqslant e^{-nx} \sum_{|(k/n)-x|>1-x} \frac{(nx)^k}{k!} \\ &\leqslant (1-x)^{-2s} S_n((t-x)^{2s}; x) \\ &\leqslant \frac{C(s, 0, 1)}{(1-x)^{2s} n^s} \qquad \text{for} \quad 0 \leqslant x < 1. \quad \blacksquare \end{aligned}$$

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