Finite Embedding Theorems for Partial Latin Squares, Quasi-groups, and Loops

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In this paper we prove that a finite partial commutative (idempotent commutative) Latin square can be embedded in a finite commutative (idempotent commutative) Latin square. These results are then used to show that the loop varieties defined by any non-empty subset of the identities \( \{x(xy) = y, (yx)x = y\} \) and the quasi-group varieties defined by any non-empty subset of \( \{x^2 = x, x(xy) = y, (yx)x = y\} \), except possibly \( \{x(xy) = y, (yx)x = y\} \), have the strong finite embeddability property. It is then shown that the finitely presented algebras in these varieties are residually finite, Hopfian, and have a solvable word problem.

1. Introduction

By an \( n \times n \) partial Latin square is meant an \( n \times n \) array such that in some subset of the \( n^2 \) cells of the array each of the cells is occupied by an integer from the set \( 1, 2, \ldots, n \) and such that no integer from the set \( 1, 2, \ldots, n \) occurs in any row or column more than once. We shall also refer to an \( n \times n \) partial Latin square as a finite partial Latin square. An \( n \times n \) partial idempotent Latin square is an \( n \times n \) partial Latin square with the additional requirement that cell \((i,i)\) is occupied by \( i, i = 1, 2, \ldots, n \). A partial commutative Latin square is a partial Latin square such that, if cell \((i,j)\) is occupied by \( k \), then so is cell \((j,i)\). Finally, a partial idempotent commutative Latin square is a partial Latin square which is both a partial idempotent and partial commutative Latin square. In [6] it is shown that a finite partial idempotent Latin square can be embedded in a finite idempotent Latin square. The following two questions were subsequently raised by S. K. Stein:

(I) Can a finite partial idempotent commutative Latin square be embedded in a finite idempotent commutative Latin square?
(2) Can a finite partial commutative Latin square be embedded in a finite commutative Latin square?

In this paper we answer both questions in the affirmative. We use these results to show that the loop varieties defined by any non-empty subset of the loop identities \( \{x(xy) = y, (yx)x = y\} \) and the quasi-group varieties defined by any non-empty subset of the quasi-group identities \( \{x^2 = x, x(xy) = y, (yx)x = y\} \), except possibly \( \{x(xy) = y, (yx)x = y\} \), have the \textit{strong} finite embeddability property. Finally, using some recent results of Trevor Evans [4, 5], we show that the finitely presented algebras in the above varieties are residually finite, Hopfian, and have a solvable word problem.

2. Embedding Theorems for Partial Latin Squares

For a more detailed account of the definitions, notations, and ideas used in this section see [6]. The results and terminology used in [6] are crucial to the proof of Theorem 1. We recall here one important definition before stating and proving Theorem 1. By a \textit{partial Steiner triple system} is meant a pair \((S, T)\) where \(T\) is a collection of three element subsets of \(S\) such that any pair of elements of \(S\) occurs in at most one of the triples in \(T\). If every pair of elements of \(S\) occurs in a triple of \(T\), \((S, T)\) is called a \textit{Steiner triple system}.

**Theorem 1.** A finite partial idempotent commutative Latin square can be embedded in a finite idempotent commutative Latin square.

**Proof.** Let \(P\) be a finite partial idempotent commutative Latin square based on \(N = \{1, 2, 3, \ldots, n\}\). Let \((a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t)\) be the occupied cells in \(P\) off the main diagonal with cell \((a_i, b_i)\) occupied by \(c_i\). In [6] it is shown that it is possible to construct a finite partial Steiner triple system \((S_t, T_t)\) with the following properties:

1. \(N \subseteq S_t\),
2. \(T_t\) contains four triples of the form \(\{a_i, b_i, x\}, \{a_i, c_i, y\}, \{b_i, c_i, z\}\) \(\{x, y, z\}\), where \(x, y, z \notin N\),
3. if \(x \in S_t/N\), \(T_t\) contains exactly one triple of the form \(\{a, b, x\}\), where \(a, b \in N\).

Tresh has shown that a finite partial Steiner triple system can be completed to a finite Steiner triple system [8]. Let \((S, T)\) be such a completion of \((S_t, T_t)\) and let \(I\) be the totally symmetric (and hence commutative)
idempotent Latin square associated with \((S, T)\). We now transform \(I\) into a commutative Latin square containing \(P\) in the upper left-hand corner. For each \(i = 1, 2, \ldots, t\), let \(T(a_i, b_i) = \{(a_i, b_i, x), (a_i, c_i, y), (b_i, c_i, z), (x, y, z)\}\) and \(R(a_i, b_i) = \{(a_i, b_i, a), (a_i, y, z), (z, b_i, y)\}\). Now the cells \((a_i, b_i)\) and \((z, y)\) are occupied by \(x\) and the cells \((a_i, y)\) and \((x_i, b_i)\) are occupied by \(c_i\). Then if we interchange the \(x\)'s and the \(c_i\)'s in these four cells the result is a Latin square with cell \((a_i, b_i)\) occupied by \(c_i\). It is shown in [6] that, for \(i \neq j\), \(R(a_i, b_i) \cap R(a_j, b_j) = \varnothing\), so that we may perform the above procedure on each of the rectangles \(R(a_i, b_i)\) simultaneously. This gives a Latin square \(Q\) with \(P\) embedded in the upper left-hand corner. It only remains to show that \(Q\) is idempotent and commutative. \(Q\) is clearly idempotent since none of the cells in any of the rectangles \(R(a_i, b_i)\) belongs to the main diagonal. Since \(P\) was commutative, the cells \((a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t)\) can be paired off so that \((a_i, b_i)\) and \((a_j, b_j)\) are paired where \(a_i = b_j\) and \(b_i = a_j\). Let \((a, b)\) and \((b, a)\) be such a pair where both cells are occupied by \(c\). Then \(T(a, b) = T(b, a) = \{a, b, c, x, a, c, y, b, c, z, x, y, z\}\). Hence \(R(a, b) = \{(a, b), (a, y), (z, b), (y, z)\}\) and \(R(b, a) = \{(b, a), (b, z), (y, a), (z, y)\}\), where the cells \((a, b)\), \((b, a)\), \((y, z)\), and \((z, y)\) are occupied by \(x\) and the cells \((a, y)\), \((y, a)\), \((z, b)\), and \((b, z)\) are occupied by \(c\). Therefore interchanging the \(x\)'s and \(c\)'s in these cells in transforming \(I\) into \(Q\) results in \(Q\) still being commutative. This last statement completes the proof.

**Theorem 2.** A finite partial commutative Latin square can be embedded in a finite commutative Latin square.

**Proof.** Let \(P\) be an \(n \times n\) partial commutative Latin square based on 1, 2, \ldots, \(n\) and let \(I\) be the \(3n \times 3n\) partial idempotent Latin square based on 1, 2, \ldots, 3\(n\) with only the cells on the main diagonal occupied. Remove any symbols occurring on the main diagonal of \(P\) and place the resulting square in the lower right-hand corner of \(I\). Call the resulting square \(I'\). Let \((a_1, a_1), (a_2, a_2), \ldots, (a_t, a_t)\) be the occupied cells of \(P\) on the main diagonal with \(c_i\) the entry in cell \((a_i, b_i), i = 1, 2, \ldots, n\). In \(I'\) place the symbol \(c_i\) in the cells \((2n + a_i, 1 + a_i)\) and \((n + a_i, 2n + a_i)\) and call the resulting square \(I''\). Since \(I''\) is a finite partial idempotent commutative Latin square, by Theorem 1, \(I''\) can be embedded in a finite idempotent commutative Latin square \(S\) based on 1, 2, \ldots, \(s\). Now define a Latin square \(S'\) based on a symbol \(e\) along with 1, 2, \ldots, \(s\) as follows. Cell \((i, i), i = 1, 2, \ldots, s + 1\) is occupied by \(e\), cells \((1, i)\) and \((i, 1)\), \(i = 2, \ldots, s + 1\) are occupied by \(i - 1\), and cell \((i, j), i \neq j, i, j > 1\) is occupied by the symbol in cell \((i - 1, j - 1)\) of \(S\). Then \(S'\) is a commutative Latin square based on \(e, 1, 2, \ldots, s\). Let \(R(a_i, a_i) = \{(2n + a_i + 1, 2n + a_i + 1), (2n + a_i + 1, n + a_i + 1), (n + a_i + 1, 2n + a_i + 1), (n + a_i + 1, n + a_i + 1)\} \).
In $S'$ the cells $(2n + a_i + 1, 2n + a_i + 1)$ and $(n + a_i + 1, n + a_i + 1)$ are occupied by $e$ and the cells $(2n + a_i + 1, n + a_i + 1)$ and $(n + a_i + 1, 2n + a_i + 1)$ are occupied by $c_i$. Therefore, if we interchange the $e$'s and the $c_i$'s in these cells, the result is still a commutative Latin square with cell $(2n + a_i + 1, 2n + a_i + 1)$ occupied by $c_i$. It is clear that the rectangles $R(a_1, a_1), R(a_2, a_2), \ldots, R(a_t, a_t)$ are mutually disjoint. Hence the above procedure applied to each of these rectangles transforms $S'$ into a commutative Latin square containing $P$.

**Theorem 3.** A finite partial commutative Latin square containing the symbol $x$ in every cell on the main diagonal can be embedded in a finite commutative Latin square containing only the symbol $x$ on the main diagonal.

**Proof.** Let $P$ be a partial Latin square containing only $x$ on the main diagonal. Embed $P$ in a commutative Latin square $S$ as in Theorem 2. The only symbol on the main diagonal of $S$ other than $x$ is $e$. Since $S$ is commutative, suitable interchanges of $x$'s and $e$'s in disjoint rectangles as in the proof of Theorem 2 places all $x$'s on the main diagonal.

Theorem 2 can be used to give a new proof of a somewhat weaker form of the following result due to Evans [3], that $n \times n$ partial Latin square can be embedded in a $t \times t$ Latin square for any $t \geq 2n$.

**Theorem 4.** A finite partial Latin square can be embedded in a finite Latin square.

**Proof.** The proof is identical to that of Theorem 2 if we remove the word commutative and complete $I'$ to $S$ using Theorem 1 in [6] instead of Theorem 1 in this paper.

This result is weaker than Evans' result in the sense that Evans' theorem gives the best possible embedding with respect to the size of the containing square.

3. **Embedding Theorems for Quasi-Groups and Loops**

Let $\mathcal{V}$ be a variety of algebras. For the concept of a partial $\mathcal{V}$-algebra see [1] and [2]. An algebra $A$ is a variety $\mathcal{V}$ has the finite embeddability property if every finite partial $\mathcal{V}$-algebra contained in $A$ is embeddable in a finite $\mathcal{V}$-algebra. We say that a variety $\mathcal{V}$ has the strong finite embeddability property if every finite partial $\mathcal{V}$-algebra is embeddable in a finite $\mathcal{V}$-algebra. Clearly strong finite embeddability for a variety implies finite embeddability for the variety. We denote by $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7,$ and $\mathcal{V}_8$ the quasigroup varieties defined by $\{x^2 = x\}$,
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\{xy = yx\}, \{(xy)x = y\}, \{(yx)x = y\}, \{x^2 = x, xy = y\}, \{x^2 = x, x(xy) = y\}, \{x^2 = x, x(xy) = y\}, \{x^2 = x, (xy)x = y\}, \{x^2 = x, (yx)x = y\}, \{x^2 = x, (yx)x = y\}, \{x^2 = x, (yx)x = y\}.

We denote by \(\mathcal{U}_1\), \(\mathcal{U}_2\), \(\mathcal{U}_3\), and \(\mathcal{U}_4\), the loop varieties defined by \(x^2 = 1\), \(xy = yx\), \((x(xy) = y)\), \((yx)x = y\), \((yx)x = y\), \(x^2 = x\), \((yx)x = y\), \((yx)x = y\), and \((x(xy) = y, (yx)x = y)\). We note that \(\mathcal{V}_8\) is the variety of totally symmetric idempotent quasi-groups and \(\mathcal{U}_4\) is the variety of totally symmetric loops.

Now let \(Q\) be a quasi-group and \(\mathcal{N}\) set the set of triples associated with \(Q\) i.e., \((x, y, z) \in \mathcal{N}\) if and only \(x \cdot y = z\) in \(Q\). Let \(\sigma \in S_3\) (the symmetric group on 3 symbols) and let \(\mathcal{N}_2 = \{(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in \mathcal{N}\}\). Then the algebra defined by \(x \cdot y = z\) if and only if \((x, y, z) \in \sigma \mathcal{N}_2\) is a quasi-group denoted by \(\sigma \mathcal{N}_2\). The quasi-groups \(Q\) and \(\sigma Q\) are said to be conjugate [7].

**Theorem 5** (S. K. Stein [7]). If \(Q \in \mathcal{V}_i\), \(i \in \{2, 3, 4\}\), and \(j \in \{2, 3, 4\}\), there is a \(\sigma \in S_3\) such that \(\sigma Q \in \mathcal{V}_j\).

**Corollary 6.** If \(Q_p\) is a partial \(\mathcal{V}_i\)-algebra, \(i \in \{2, 3, 4\}\), and \(j \in \{2, 3, 4\}\), there is a \(\sigma \in S_3\) such that \(\sigma Q_p\) is a partial \(\mathcal{V}_j\)-algebra.

**Lemma 7.** The variety of commutative quasi-groups \(\mathcal{V}_2\) has the strong finite embeddability property.

**Proof.** This is an immediate consequence of Theorem 2.

**Lemma 8.** The quasi-group varieties \(\mathcal{V}_2, \mathcal{V}_3,\) and \(\mathcal{V}_4\) have the strong finite embeddability property.

**Proof.** We already know that \(\mathcal{V}_2\) has the strong finite embeddability property. Let \(Q_p\) be a finite partial \(\mathcal{V}_i\)-algebra, \(i = 3\) or \(4\). By Corollary 6, there is a \(\sigma \in S_3\) such that \(\sigma Q_p\) is a (finite) partial \(\mathcal{V}_j\)-algebra. Let \(F\) be a finite \(\mathcal{V}_2\)-algebra containing \(\sigma Q_p\). Then \(\sigma^{-1} F \in \mathcal{V}_i\) and of course contains \(Q_p\).

**Theorem 9.** The quasi-group varieties \(\mathcal{V}_i, i \in \{1, 2, ..., 8\}\), have the strong finite embeddability property.

**Proof.** We already have this result for \(\mathcal{V}_2, \mathcal{V}_3,\) and \(\mathcal{V}_4\). For \(\mathcal{V}_1\) we recall that finite partial idempotent Latin squares can be finitely embedded in an idempotent Latin square [6]. Christine Treash has shown that \(\mathcal{V}_8\) has the strong finite embeddability property [8]. The fact that \(\mathcal{V}_5, \mathcal{V}_6,\) and \(\mathcal{V}_7\) have the strong finite embeddability property is similar to the proof of Lemma 8 using Theorem 1 and the fact that the quasi-group identity \(x^2 = x\) is invariant under conjugation.
Lemma 10. The variety of loops $\mathcal{U}_1$ has the strong finite embeddability property.

Proof. The proof is an immediate consequence of Theorem 3.

Theorem 11. The loop varieties $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, and $\mathcal{U}_4$ have the strong finite embeddability property.

Proof. We already know that $\mathcal{U}_1$ has the strong finite embeddability and Treash has shown that $\mathcal{U}_4$ has this property [8]. So let $Q_\sigma$ be a finite partial $\mathcal{V}_1$-algebra, $i \in \{2, 3\}$. Now considering $Q_\sigma$ as a quasi-group, that is to say, as a finite partial $\mathcal{V}_1$-algebra, $i \in \{3, 4\}$, one of the permutations $(1, 3), (2, 3)$, which we will denote by $\sigma$, will place $\sigma Q_\sigma$ in the quasi-group variety $\mathcal{V}_2$. It is easily checked that in fact $\sigma Q_\sigma$ is a finite partial loop satisfying the additional identity $x^2 = 1$. Hence $\sigma Q_\sigma$ is a partial $\mathcal{U}_1$-algebra. By Lemma 10 there is a finite $\mathcal{U}_1$-algebra, say $F$, containing this conjugate. Now considering $F$ in $\mathcal{V}_2$, $\sigma^{-1} F \in \mathcal{V}_i$, $i \in \{3, 4\}$, and contains $Q_\sigma$. Again it is easily checked that in fact $\sigma^{-1} F$ is also in $\mathcal{U}_i$, $i \in \{2, 3\}$. This completes the theorem.

4. FINITELY PRESENTED QUASI-GROUPS AND LOOPS

For a more detailed account of the ideas and definitions in what follows the reader is referred to [1], [2], [4], and [5]. Let $V'$ be a variety of algebras. An algebra $A$ in $V'$ is residually finite if, for any $x \neq y$ in $A$, there is a homomorphism $\alpha$ of $A$ onto a finite $V'$-algebra such that $x \alpha \neq y \alpha$. By the word problem for a finitely presented $V'$-algebra $A$ is meant the following: Does there exist a finite recursive process which can be applied to any pair of words in $A$ to decide whether or not they are equivalent? Finally, an algebra is Hopfian if it is not isomorphic to a proper homomorphism image of itself. The following three theorems are due to Evans:

(i) Let $V'$ be any variety of algebras and let $A$ be any finitely generated residually finite algebra in $V'$. Then $A$ is Hopfian [5].

(ii) The finitely presented algebras in a variety $V'$ are residually finite if and only if $V'$ has the finite embeddability property [4].

(iii) If a finitely presented algebra $A$ in a variety $V'$ is residually finite (or, equivalently, if $A$ has the finite embeddability property), the word problem is solvable for $A$ [4].

Theorems 9 and 11 show that the quasi-group varieties $\mathcal{V}_i$, $i \in \{1, 2, \ldots, 8\}$, and the loop varieties $\mathcal{U}_i$, $i \in \{1, 2, 3, 4\}$, have the finite embeddability
property. Hence using (ii), (i), and (iii) in that order gives the following theorem:

**Theorem 12.** The finitely presented algebras in the quasi-group varieties $V_i, i \in \{1, 2, \ldots, 8\}$, and the loop varieties $U_i, i \in \{1, 2, 3, 4\}$, are residually finite, Hopfian, and have a solvable word problem.

5. **Remarks**

Among other things, we have shown that any non-empty subset of the loop identities $\{x(xy) = y, (yx)x = y\}$ and any non-empty subset of the quasi-group identities $\{x^2 = x, x(xy) = y, (yx)x = y\}$, except possibly $\{x(xy)y, (yx)x = y\}$, defines a variety having the strong finite embeddability property. The case of the quasi-group identities $\{(yx)x = y, x(xy) = y\}$ has resisted all attempts by the author for a proof or counterexample. This problem seems closely related to the other problems, but the author has failed to handle it using the techniques in this paper.

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**References**