JOURNAL OF ALGEBRA 150, 308–320 (1992)

Prüfer Domains and Graded Rings

FABIEN DECRUYENAERE*

K.U. Leuven, 3001 Leuven, Belgium

AND

ERIC JESPERS†

Memorial University of Newfoundland, St. John's, Canada A1C5S7

Communicated by Kent R. Fuller

Received July 24, 1990

1. INTRODUCTION

In [1] and [2] D. D. Anderson and D. F. Anderson studied commutative integral domains that are graded by a torsion-free abelian group and that satisfy a certain arithmetical property (such as being a Krull domain, GCD-domain, UFD, ...). Actually for their purposes, it is better to look at these rings as rings graded by a torsion-free cancellative abelian monoid. Their attention was focused on proving that the arithmetical structure is determined by graded information, in particular the structure of the grading monoid. In this way they generalized several (twisted) monoid ring results of (see for example) R. Gilmer [7] and L. Chouinard [3].

An obvious next step is to find a concrete description of torsion-free cancellative abelian monoid graded rings that have a particular arithmetical structure. For example, are such rings automatically twisted semigroup rings? In [13] P. Wauters answered this question for factorial domains in case the grading monoid has no non-trivial units. It turns out that such rings are often polynomial rings. In [4] the authors studied this question for such graded domains which are Dedekind: these rings are either polynomial rings or twisted group rings over a field. In this paper we consider

* The first named author is a research assistant at the National Fund for Scientific Research, Belgium.

† The second named author is supported in part by NSERC-Grant OGP0036631.

0021-8693/92 $5.00

graded domains that are Prüfer domains. We obtain a complete description in case the grading monoid has no non-trivial units. When the grading monoid is a group, which is proved to be the only other possible case, we obtain a complete result in case the ring is of dimension one and has no non-trivial idempotent ideals; such rings are called almost Dedekind domains (cf. [6]).

2. Preliminaries

All semigroups S are abelian, torsion-free, and cancellative, so they are subsemigroups of a torsion-free abelian group. We use the multiplicative notation for semigroups and in case S is a monoid, its identity is denoted by e. However, in the case of the natural numbers $\mathbf{N}$, or the integers $\mathbf{Z}$ or the rationals $\mathbf{Q}$, we use the additive notation and the identity is denoted by 0. The maximal subgroup (or unit group) of a monoid S is denoted by $\mathscr{U}(S)$. For the terminology on semigroups we refer to [7, 8]. All rings R are associative and commutative and contain an identity, denoted by 1. The subgroup of multiplicative invertible elements is denoted by $\mathscr{U}(R)$. A ring R is said to be graded by a monoid S, if R is the direct sum of some additive subgroups R_s, $s \in S$, such that for all s, $t \in S: R_s R_t \subseteq R_{st}$. If S is a group and equality holds for all s, $t \in S$, then R is said to be strongly S-graded. A graded ring R is called a twisted monoid ring if each R_s is a free R_e-module of rank one; i.e., as an R_e-module the ring contains a basis, say $\{\bar{s} \mid s \in S\}$, and there exists a 2-cocycle $\gamma: S \times S \to \mathscr{U}(R)$ such that $\bar{s}\bar{t} = \gamma(s, t)\,\overline{st}$, for all s, $t \in S$. Such a twisted semigroup ring is denoted by $A^t[S]$, where $A = R_e$. In case the cocycle is trivial, then the ring is simply called a monoid ring and is denoted by $A[S]$.

The support of a graded ring R is defined as $\text{Supp}(R) = \{s \in S \mid R_s \neq \{0\}\}$. We shall assume throughout that $\text{Supp}(R) \neq \{e\}$, that is, the support is non-trivial. An element of R_s, $s \in S$, is called a homogeneous element, and $h(R)$ denotes the set of all homogeneous elements of R. When R is a domain, we put $Q^g(R) = R(h(R) \setminus \{0\})^{-1}$, the graded quotient ring of R. Clearly, $Q^g(R)$ is a G-graded domain, where G is the quotient group of S, and it can easily be deduced that $Q^g(R) = Q_e^t[G]$, a twisted group ring over the field $Q_e = (Q^g(R))_e$. The total quotient ring of R is denoted by $Q(R)$. Whenever $T \subseteq S$, we put $R_{[T]} = \bigoplus_{t \in T} R_t$. It is clear that, if T is a monoid ideal of S, then $R_{[T]}$ is an ideal of R. If I is an ideal of R, then $(I)_h$ denotes the ideal generated by all homogeneous elements of I. An ideal I is said to be homogeneous if $I = (I)_h$. Finally, if the symbol $\cong$ is used in connection with graded objects, it always means a graded isomorphism. For further terminology on graded rings we refer to [10], and for monoid rings we refer to [7].

We will use the following equivalent forms of the definition of a Prüfer domain.

PROPOSITION 2.1. *The following are equivalent:*

1. *R is a Prüfer domain;*
2. *the localization R_P is a valuation ring for each prime ideal P of R;*
3. *the localization R_M is a valuation ring for each maximal ideal M of R;*
4. *each non-zero finitely generated ideal of R is invertible;*
5. *R is integrally closed and for each $a, b \in R : (a, b)^2 = (a^2, b^2)$.*

For more information on Prüfer domains we refer to [6]. The next two lemmas play an essential role in our investigations.

LEMMA 2.2. *Let R be an integrally closed domain and let D be a domain, containing R, which is integral over R. If D is a Prüfer domain, then R is Prüfer too.*

Proof. Let P be a prime ideal of R. Then there exists a prime ideal Q of D such that $P = Q \cap R$. It then follows from [6, Proposition 12.7] that $R_P = D_Q \cap Q(R)$. So R_P, as an intersection of a valuation ring and a field, is also a valuation ring. Hence R is Prüfer. ∎

LEMMA 2.3. *Let G be a torsion-free abelian group. If R is a G-graded Prüfer domain, then $S = \mathrm{Supp}(R)$ is a torsion-free cancellative monoid of torsion-free rank one. Moreover, S is isomorphic with either a subgroup of the additive group of rational numbers* **Q** *or a submonoid of* **Q** *without non-trivial units.*

Proof. Since R is a domain and $1 \in R_e$, S is obviously a submonoid of G. Note that, as mentioned in the introduction, we always assume that $S = \mathrm{Supp}(R)$ is non-trivial. Moreover, if $Q = Q^g(R)$, then $Q = Q_e^{\prime}[G]$. Let F be a maximal free subgroup of G. Then $Q_e^{\prime}[G]$ is integral over $Q_e^{\prime}[F] \cong Q_e[X_i \mid i \in I]$, a polynomial ring over Q_e in rank(F) variables. Since a polynomial ring over a field, as a factorial domain, is integrally closed, it follows from Lemma 2.2 that $Q_e^{\prime}[F]$ is a Prüfer domain. It then follows from [6, Proposition 23.5] that $|I| = 1$. Hence G, and thus also S, has torsion-free rank one. Hence S is a submonoid of **Q**, and if S is not a subgroup, it then follows from [7, Theorem 2.9] that S has no non-trivial units. ∎

3. Prüfer Domains: The Monoid Case

We now investigate the case when S has no non-trivial units.

Lemma 3.1. *Let R be a graded Prüfer domain with* $\mathrm{Supp}(R)=S$ *a torsion-free cancellative monoid without non-trivial units. Then R_s is a divisible R_e-module, for each $s\in S\backslash\{e\}$.*

Proof. Let $s\in S\backslash\{e\}$ and $r_s\in R_s\backslash\{0\}$. Choose $0\neq a\in R_e$; then $(a, r_s)^2=(a^2, r_s^2)$. Hence, there exist f, $g\in R$ such that $ar_s=a^2f+r_s^2g$. Comparing homogeneous components in R_s, as s is not invertible and S is cancellative, one obtains $ar_s=a^2f_s$ and thus $r_s=af_s$. This shows that each R_s is a divisible R_e-module. ∎

The next three lemmas deal with the special case that $S=\mathbf{N}$.

Lemma 3.2. *Let R be a graded Prüfer domain with* $\mathrm{Supp}(R)=\mathbf{N}$ *and R_0 a field. Then $M=\sum_{n>0}R_n$ is a homogeneous maximal ideal of R with $\bigcap_{n>0}(MR_M)^n=\{0\}$.*

Proof. Clearly $R/M\cong R_0$ is a field. Assume $q=a^{-1}b\in\bigcap_{n>0}(MR_M)^n$, $a\notin M$. Then, for each $n>0$, there exists $\alpha_n\in R\backslash M$ such that $q\alpha_n\in M^n$, and hence $aq\alpha_n=b\alpha_n\in M^n$. We may assume that for each $n>0$, $(\alpha_n)_0=1$. Now let $b=b_l+b_{l+1}+\cdots+b_{l+k}$; then for each $n>0$, since M^n is homogeneous, $b_l\in M^n$. Thus $\bigcap_{n>0}M^n\neq\{0\}$, while $M^n\subseteq\sum_{k\geqslant n}R_k$; this is a contradiction. ∎

Lemma 3.3. *Let R be a graded domain with* $\mathrm{Supp}(R)=\mathbf{N}$ *and R_0 a field. If R is a Prüfer domain, then $R\cong R_0[X]$, a polynomial ring.*

Proof. Let $0\neq r_1\in R_1$; then $r_1R_M\subseteq MR_M$. Assume $r_1R_M\subseteq M^2R_M$; then there exists $\alpha\in R\backslash M$ such that $r_1\alpha\in M^2$; this is a contradiction since $\alpha_0\neq 0$ and M^2 is homogeneous. Thus $M^2R_M\subset r_1R_M\subseteq MR_M$. It then easily follows that r_1R_M is (MR_M)-primary. From [6, Theorem 17.3], $r_1R_M=M^kR_M$ for some non-zero k. Consequently $r_1R_M=MR_M$. Therefore, if $a\in R_1$, then $a\in r_1R_M$ and there exist $\alpha\in R\backslash M$, $\beta\in R$, such that $\alpha a=\beta r_1$. Comparing the lowest degree components, one gets $\alpha_0a=\beta_0r_1$, with α_0 a unit in R_0. Thus $a=(\beta_0\alpha_0^{-1})\,r_1\in R_0r_1$ and $R_1=R_0r_1$. Now, for $n\geqslant 2$, $r_1^nR_M=(r_1R_M)^n=(MR_M)^n$, from which it follows in an analogous way as above that $R_n=R_0r_1^n$. Consequently $R=\sum_{n\in N}R_0r_1^n$, a polynomial ring over a field. ∎

Proposition 3.4. *Let R be a graded domain with* $\mathrm{Supp}(R)=\mathbf{N}$. *Then R is a Prüfer domain if and only if R_0 is a Prüfer and $R\cong R_0+XQ(R_0)[X]$.*

Proof. Assume R is a Prüfer domain. Clearly R_0 is a domain which is a homomorphic image of R, and thus R_0 is a Prüfer domain. From Lemma 3.1 it follows that $R_{R_0\setminus\{0\}} = Q(R_0) + \sum_{n>0} R_n$. As a homogeneous localization of R, this is again an N-graded Prüfer domain, with component of degree zero a field. Therefore, by Lemma 3.3, $R_n = Q(R_0)\, X^n$, $n > 0$, for some $X \in R_1$. Hence $R = R_0 + \sum_{n>0} Q(R_0)\, X^n$.

For the converse let M be a maximal ideal of R. We prove that R_M is a valuation ring. If $(M)_h = \{0\}$, then $R_M = (Q(R_0)[X, X^{-1}])_{M'}$, where M' is the maximal ideal $MQ(R_0)[X, X^{-1}]$; clearly R_M is then a valuation ring. In case $(M)_h \neq 0$, then one easily verifies that $M = p + XQ(R_0)[X]$, where $p = M \cap R_0$. Replacing R by the localization $R_{R_0\setminus p}$ we may assume that R_0 is a valuation ring (or a field). We have to show that for every $\alpha \in Q(R)$ either $\alpha \in R_M$ or $\alpha^{-1} \in R_M$. For this write $\alpha = (f_0 + f)(g_0 + g)^{-1}$, where f_0, $g_0 \in R_0$ and $f, g \in XQ(R_0)[X]$; we also may assume that not both f_0 and g_0 are zero. In case $f_0 \notin p$, then $\alpha^{-1} \in R_M$ and if $g_0 \notin p$, then $\alpha \in R_M$. Suppose $(g_0) \subseteq (f_0) \subseteq p$; then $\alpha = (1 + f_0^{-1} f)[f_0^{-1}(g_0 + g)]^{-1}$. Since $1 + f_0^{-1} f \in R\setminus M$ and $f_0^{-1}(g_0 + g) \in R$, we get that $\alpha^{-1} \in R_M$. In a similar way, $(f_0) \subseteq (g_0) \subseteq p$ implies that $\alpha \in R_M$. This shows that R_M is a valuation ring. ∎

We now come to the main theorem of this section. Recall (cf. [7]) that a submonoid S of $\mathbf{Q}$ is called a Prüfer monoid if S is the union of an ascending sequence of cyclic submonoids.

THEOREM 3.5. *Let R be a graded domain with* $\mathrm{Supp}(R) = S$ *and* $\mathcal{U}(S) = \{e\}$. *Then R is a Prüfer domain if and only if the following conditions are satisfied*:

1. *S is isomorphic with a Prüfer submonoid of* $\mathbf{Q}$;
2. *R_e is a Prüfer domain*;
3. $R \cong R_e + Q(R_e)^t\, [S\setminus\{e\}]$, *where the latter is considered as a subring of the twisted monoid ring* $Q(R_e)^t\, [S]$.

Proof. For $s \in S$ we denote by $\langle s \rangle$ the submonoid of S generated by s. Assume R is a Prüfer domain. Let $a \in S\setminus\{e\}$. Then $R = R_{[\langle a \rangle]} \oplus \sum_{s \in S\setminus\langle a \rangle} R_s$, a direct sum of $R_{[\langle a \rangle]}$-modules. Consequently each ideal of $R_{[\langle a \rangle]}$ is contracted from R. Therefore, by [7, Theorem 13.1], $R_{[\langle a \rangle]}$ is a Prüfer domain too and N-graded. It then follows from Proposition 3.4 that R_e is a Prüfer domain and that there exists $r_a \in R_a$ such that $R_a = Q(R_e) r_a$. Therefore, $R = R_e + \sum_{a \in S\setminus\{e\}} Q(R_e)\, r_a \cong R_e + Q(R_e)^t\, [S\setminus\{e\}]$.

Further, it follows from Lemma 2.3 that S is a submonoid of $\mathbf{Q}$. Then [7, Theorem 2.9] yields that we may assume that S is a submonoid of $\mathbf{Q}^+$, the non-negative rational numbers. We assert that S is an integrally closed monoid. Indeed, suppose $g^n \in S$, with $e \neq g = s^{-1}t \in G$, where G is the quotient group of S and $s, t \in S$. Note that $g^n \neq e$, since G is torsion-free.

Then $\overline{g^n} \in R = R_0 + Q(R_0)^t [S\backslash\{0\}]$ and $\overline{g^n} = \bar{g}^n u$, where $0 \neq u \in Q(R_0)$. Since R is integrally closed, $\bar{g} \in R$ and therefore $g \in S$; this proves the assertion. Moreover, $S \cap \mathbf{N} \neq \{0\}$; if $0 \neq a \in S \cap \mathbf{N}$, then $\mathbf{Q}^+$ is integral over $a\mathbf{N} \subseteq S$. Consequently $G \cap \mathbf{Q}^+$ is integral over S and so $S = G \cap \mathbf{Q}^+$. It follows from [7, Corollary 2.8] that G is the union of an ascending sequence of cyclic subgroups $grp\langle g_i \rangle$ with $g_i > 0$ for each i. Since $S = G \cap \mathbf{Q}^+$, it follows that $S = \bigcup (grp\langle g_i \rangle \cap \mathbf{Q}^+) = \bigcup \langle g_i \rangle$ and $\langle g_i \rangle \subseteq \langle g_{i+1} \rangle$ for each i. So S is a Prüfer submonoid of $\mathbf{Q}$ and we have proved that the three conditions are necessary.

Conversely, because of Proposition 3.4, the ring R is the direct limit of Prüfer domains, and thus Prüfer. ∎

Remark.

1. It is clear that, in the preceding theorem, R is of dimension one if and only if $R_e = Q(R_e)$, i.e., R_e is a field.

2. We give an example, taken from [2], to show that there exist twisted monoid rings which are Prüfer and which are not monoid rings. So the conditions in the preceding theorem are most accurate. Let S be a Prüfer submonoid of $\mathbf{Q}$ and $R = k[X_s \mid s \in S]$, a polynomial ring over the field k. Clearly R is a unique factorization domain and S-graded with $\deg(X_{s_1}^{n_1} \cdots X_{s_r}^{n_r}) = s_1^{n_1} \cdots s_r^{n_r}$. Then $Q = Q^g(R) = Q_e^t[G]$, where G is a subgroup of $\mathbf{Q}$, is also a unique factorization domain. Since G has torsion-free rank one, $Q \cong Q_e[\mathbf{Z}]^t [G/\mathbf{Z}]$ is integral over $Q_e[\mathbf{Z}]$. Therefore Q is of dimension one, and thus a principal ideal domain and certainly a Prüfer domain. But the cocycle on G is not trivial, since it follows from Gilmer's result [7, Theorem 13.8] that $Q_e[G]$ is not a principal ideal domain.

3. Let R be an S-graded ring, S a monoid without non-trivial units. If R is a Dedekind domain, i.e., a Noetherian Prüfer domain, then it is proved in [4] that R is a polynomial ring over a field. So in that case there is no twist.

4. Prüfer Domains: The Group Case

We have already proved that if R is a G-graded Prüfer domain, then G has torsion-free rank one and is therefore isomorphic with a subgroup of $\mathbf{Q}$. We start with a general lemma.

Lemma 4.1. *If V is a valuation ring with maximal ideal M, then $V/\bigcap_{n=1}^{\infty} M^n$ is either a discrete rank one valuation ring or a field.*

Proof. Put $S = V/\bigcap_{n=1}^{\infty} M^n$. First note that, by [6, Theorem 17.1], $\bigcap_{n=1}^{\infty} M^n$ is a prime ideal of V. If $M = M^2$, then S is a field. If not, then

S has rank one; indeed, if P is a prime ideal of V with $P \subset M$, then $P \subset \bigcap_{n=1}^{\infty} M^n$, because otherwise there exists $n \in \mathbf{N}_0$ such that $M^n \subseteq P$ and so $M = P$. Moreover, it follows from [6, Theorem 17.3(b)] that each $(M/\bigcap_{n=1}^{\infty} M^n)$-primary ideal of S is a power of $M/\bigcap_{n=1}^{\infty} M^n$. Consequently, S is discrete. ▌

We first handle the case when R is $\mathbf{Z}$-graded.

LEMMA 4.2. *Let R be a $\mathbf{Z}$-graded Prüfer domain. If $R_1 R_{-1} = R_0$ (i.e., R is strongly $\mathbf{Z}$-graded), then $R \cong R_0[X, X^{-1}]$ and R_0 is a field.*

Proof. Let M be a maximal ideal of R_0. Then

$$R_{R_0 \setminus M} = (R_0)_M + \sum_{k \neq 0} R_k (R_0)_M$$

is also a Prüfer domain, with part of degree zero a valuation ring. So, to prove the lemma we may assume R_0 is a valuation ring. Let $k \geqslant 1$. Since $R_k R_{-k} = R_0$, write $1 = \sum_{i=1}^{l} a_i b_i$ with $a_i \in R_k$ and $b_i \in R_{-k}$. If $r_k \in R_k$, then $r_k = \sum_{i=1}^{l} a_i (b_i r_k) \in R_0 a_1 + \cdots + R_0 a_l$. So $r_k R_{-k} \subseteq R_0 a_1 R_{-k} + \cdots + R_0 a_l R_{-k}$ and thus, say, $r_k R_{-k} \subseteq R_0 a_1 R_{-k}$. Consequently $r_k \in R_0 a_1 R_k R_{-k} \subseteq R_0 a_1$ and $R_k = R_0 a_1$. So $R \cong R_0^t[\mathbf{Z}] \cong R_0[X, X^{-1}]$ and it follows from [7, Theorem 13.4] that R_0 is a field. ▌

THEOREM 4.3. *Let $R = \sum_{i \in \mathbf{Z}} R_i$ be a $\mathbf{Z}$-graded Prüfer domain with* $\mathrm{Supp}(R) = \mathbf{Z}$. *Assume the following conditions are satisfied:*

1. *R has no homogeneous idempotent maximal ideals;*
2. *the Krull dimension of R_0 is at most one;*
3. *R_0 has no idempotent maximal ideals.*

Then R_0 is a field and thus $R \cong R_0[X, X^{-1}]$.

Proof. Assume R_0 is not a field. As in the proof of the preceding lemma, we may assume that R_0 is a valuation ring with maximal ideal, say M.

First we claim that $R_n R_{-n} \neq R_0$ for all n: indeed, if there exists $n \in \mathbf{N}_0$ such that $R_n R_{-n} = R_0$, then put $T = \sum_{r \in \mathbf{Z}} R_{rn}$. Clearly, R is integral over T. Moreover T is integrally closed: let $\alpha = \alpha_{sn} + \cdots + \alpha_{(s+u)n}$ and $\beta = \beta_{tn} + \cdots + \beta_{(t+v)n}$, elements of T and assume $\alpha^{-1}\beta$ is integral over T. Since $\alpha^{-1}\beta \in Q(R)$ and $\alpha^{-1}\beta$ is integral over R, a Prüfer domain, it follows from Proposition 2.1 that $\alpha^{-1}\beta \in R$. So $\beta = \alpha(r_{k_1} + \cdots + r_{k_l})$ with $r_i \in R$ for $k_1 \leqslant i \leqslant k_l$. Comparing the lowest degree components, one gets $\beta_{t_n} = \alpha_{s_n} r_{k_1}$ and thus $k_1 = (t-s)n$ and $r_{k_1} \in T$. Now $\beta - \alpha r_{(t-s)n} = \alpha(r_{k_2} + \cdots + r_{k_l})$. So, by induction, one obtains $r_{k_i} \in T$ for all i and finally $\alpha^{-1}\beta \in T$. Therefore, by Lemma 2.2, T is Prüfer. Since T is strongly $\mathbf{Z}$-graded, it follows from Lemma 4.2 that R_0 is a field, a contradiction.

Since R_0 is a valuation ring of dimension at most one and has no non-trivial idempotent maximal ideals, it follows that $\bigcap_{n=1}^{\infty} M^n = \{0\}$. Hence, by Lemma 4.1 and the assumptions, $(R_0)_M$ is a discrete rank one valuation ring. Moreover, $R_{R_0 \setminus M}$ has no homogeneous idempotent maximal ideals. Indeed, suppose $P = (P)_h$ is an idempotent maximal ideal of $R_{R_0 \setminus M}$ (note that $P \cap R_0 = M$ and that $P \cap R$ is a homogeneous maximal ideal in R). Then, if $\alpha \in P \cap R$, there exists $q \in R_0 \setminus M$ such that $q\alpha \in (P \cap R)^2$. But $M + qR_0 = R_0$ and thus $M\alpha R + q\alpha R = \alpha R$. Now $M\alpha R \subseteq (P \cap R)^2$ and $q\alpha R \subseteq (P \cap R)^2$, so that $\alpha R \subseteq (P \cap R)^2$. Therefore $P \cap R = (P \cap R)^2$ which is in contradiction with our assumptions.

So we have proved that $R = \sum_{i \in \mathbf{Z}} R_i$ is a Z-graded Prüfer domain with R_0 a discrete rank one valuation ring with non-zero maximal ideal $M = aR_0$ and $R_1 R_{-1} = a^n R_0$, $n \geq 1$ and all $R_n R_{-n} \subseteq aR_0$. Since $\mathrm{Supp}(R) = \mathbf{Z}$ it follows that $Q^g(R) = Q(R_0)[X, X^{-1}]$ and one can choose $X \in R_1$ such that $R_1 R_{-1} = XR_{-1}$. Since R is embeddable in its graded quotient ring, it has the form $R = R_0 + \sum_{n \neq 0} M_n X^n$ where $M_n = \{q \in Q(R_0) \mid qX^n \in R_n\}$. Obviously, all M_n are fractional R_0-ideals. Therefore $M_i = a^{-n_i} R_0$ for all $i \in \mathbf{Z}$, where $n_i \in \mathbf{Z}$. We then obtain that $M_{-1} = R_{-1} X = R_1 R_{-1} = R_0 a^n$ and $M_1 = (R_0 a^n)(M_{-1})^{-1} = R_0$. If we put $-n_{-i} = m_i$ for $i > 0$, then R has the form

$$R = \cdots + R_0 a^{m_3} X^{-3} + R_0 a^{m_2} X^{-2} + R_0 a^n X^{-1} + R_0 + R_0 X + R_0 a^{-n_2} X^2 + R_0 a^{-n_3} X^3 + \cdots.$$

Now put $\mathcal{M} = R_0 a + \sum_{n \neq 0} R_n$. Then obviously $\mathcal{M}$ is a homogeneous maximal ideal of R and so $R_{\mathcal{M}}$ is a valuation ring.

We claim R is a gr-valuation ring, i.e., all homogeneous ideals are linearly ordered. For let I and J be homogeneous ideals of R, then, say, $IR_{\mathcal{M}} \subseteq JR_{\mathcal{M}}$. So, if $i \in h(I)$, then there exists $\alpha \notin \mathcal{M}$ such that $\alpha i \in J$. Since we may assume $\alpha_0 = 1$ and since J is homogeneous, we get $\alpha_0 i = i \in J$ and thus $I \subseteq J$. This proves the claim.

Consequently, either $(X) \subseteq (a)$ or $(a) \subseteq (X)$. But the first inclusion implies $X \in aR_0 X$, a contradiction. So the second inclusion is satisfied and consequently $a \in R_0 a^n$ and $n = 1$. Similarly, for $i > 1$, either $(R_i) \subseteq (a)$ or $(a) \subseteq (R_i)$: the first case is impossible and so the second inclusion is true, whence $R_0 a = R_i R_{-i}$ and $m_i = n_i + 1$. So R has the form

$$R = \cdots + R_0 a^{n_3+1} X^{-3} + R_0 a^{n_2+1} X^{-2} + R_0 a X^{-1} + R_0 + R_0 X + R_0 a^{-n_2} X^2 + R_0 a^{-n_3} X^3 + \cdots. \tag{1}$$

Since R is integrally closed, for each $i > 0$, $(a^{-1}X)^i \notin R_0 a^{-n_i} X^i$, whence

$$n_i \leq i - 1. \tag{2}$$

Moreover, from the inclusions $R_{-1}R_{i+1} \subseteq R_i$ $(i \geqslant 1)$, $R_1 R_i \subseteq R_{i+1}$ $(i \geqslant 1)$, and $R_i R_j \subseteq R_{i+j}$ $(i, j \in \mathbf{Z})$, we get the following inequality for each $i, j > 0$:

$$n_i + n_j \leqslant n_{i+j} \leqslant n_i + n_j + 1. \tag{3}$$

We observe that condition (3) is necessary and sufficient for R, of the form (1), to be a ring.

If all $n_i = 0$, then $R \subseteq R_0[X, X^{-1}]$. Since any overring of a Prüfer domain is Prüfer again (cfr. [6, Theorem 26.1]), it follows from Proposition 3.4 that R_0 is a field, contrary to our assumption. If all $n_i = i - 1$, then it is easy to see that R can be embedded in $R_0[a^{-1}X, (a^{-1}X)^{-1}]$, which as above yields a contradiction. So we may assume there exists $i_0 > 1$ with $0 < n_{i_0} < i_0 - 1$.

We assert that R has gr-Krull dimension one. R is then a gr-valuation ring of gr-dimension one with unique gr-maximal ideal $\mathcal{M}$. Since by assumption $\mathcal{M} \neq \mathcal{M}^2$ and since $\bigcap_{n=1}^{\infty} \mathcal{M}^n$ is a homogeneous prime ideal of R, it follows that $\bigcap_{n=1}^{\infty} \mathcal{M}^n = \{0\}$. Hence, R is a gr-(discrete rank one valuation ring) and by a result of M. Van den Bergh [12, Theorem 3.3] this is a contradiction.

We now prove the assertion. First we claim that if $\alpha \in h(R) \cap \mathcal{M}$ and if $R' = R_{\{\alpha^n \mid n > 0\}}$, then $R' = Q^g(R) = Q(R_0)[X, X^{-1}]$: if $\alpha \in R_0$, then this is clear. If $\alpha \in R_{-1}$, then it follows that $a^{-1}X \in R'$. But then $a^{n_{i_0}+1} X^{-i_0} (a^{-1}X)^{i_0 - 1} = a^{n_{i_0} - i_0 + 2} X^{-1} \in R'$ and so $X^{-1} \in R'$. Since also $a^{-1}X \in R'$, we obtain $a^{-1} \in R'$ and $Q(R_0) \subseteq R'$. Therefore $R' = Q(R_0)[X, X^{-1}]$. If $\alpha \in R_1$, then $X^{-1} \in R'$. But then $(X^{-1})^{i_0} X^{i_0} a^{-n_{i_0}} = a^{-n_{i_0}} \in R'$ and so $a^{-1} \in R'$ and R' is again $Q(R_0)[X, X^{-1}]$. Finally, let $\alpha \in R_i$, $|i| > 1$. By an analogous argument as in the first part of the proof, $R^{(i)} = \sum_{k \in \mathbf{Z}} R_{ki}$ is also a $\mathbf{Z}$-graded Prüfer domain with component of degree one $R_1^{(i)} = R_i$. By the above, $R^{(i)}_{\{\alpha^n \mid n > 0\}} = Q^g(R^{(i)}) \cong Q(R_0)[X, X^{-1}]$. Then certainly $R' = Q(R_0)[X, X^{-1}]$. Now let Q be a homogeneous prime ideal of R with $Q \subset \mathcal{M}$. Then $Q^g(R/Q) \cong R_{h(R \setminus Q)}/Q_{h(R \setminus Q)} = Q(R_0)[X, X^{-1}]/Q_{h(R \setminus Q)}$, where the second equality follows from the foregoing. Since $Q(R_0)[X, X^{-1}]$ is a graded field and $Q_{h(R \setminus Q)}$ is a homogeneous prime ideal of $Q(R_0)[X, X^{-1}]$, it follows that $Q_{h(R \setminus Q)} = \{0\}$ and thus $Q = \{0\}$. Therefore R has gr-Krull dimension one. This ends the proof. ∎

Remark. Note that the ring R defined by (1), with $n_i = i - 1$ for all $i \geqslant 2$ and R_0 a rank one discrete valuation ring with maximal ideal aR_0, is a gr-Prüfer domain (the definition is the graded version of Proposition 2.1). Moreover, its graded quotient ring is Prüfer, but R itself is not Prüfer.

COROLLARY 4.4. *Let R be a G-graded Prüfer domain where* $\operatorname{Supp}(R) = G$ *is a torsion-free abelian group. If the Krull dimension of R_e is* *at most one and if R has no homogeneous idempotent ideals, then $R \cong k^t[G]$, a twisted group ring of a subgroup of Q over a field. Conversely, such a ring is also Prüfer.*

Proof. It follows from Lemma 2.3 that G is isomorphic with a subgroup of Q. Let $g \in G \setminus \{e\}$. Then $R = R_{[grp\langle g\rangle]} \oplus \sum_{h \in G \setminus grp\langle g\rangle} R_h$, a direct sum of $R_{[grp\langle g\rangle]}$-modules. Then by [7, Theorem 13.1], $R_{[grp\langle g\rangle]}$ is Prüfer and thus by Theorem 4.3 R_0 is a field and $R_g = R_e r_g$. Therefore $R \cong k^t[G]$.

Conversely, $k^t[G]$ is a direct limit of polynomial rings $k[X, X^{-1}]$ and is therefore Prüfer. ∎

5. Almost Dedekind Domains

A ring is called an almost Dedekind domain if its localization to each maximal ideal is a discrete rank one valuation ring (the terminology is justified because local Dedekind domains are precisely discrete rank one valuation rings). One can prove that R is almost Dedekind if and only if R is one-dimensional Prüfer without idempotent maximal ideal, or equivalently if R is Prüfer such that $\bigcap_{n=1}^{\infty} A^n = \{0\}$ for each proper ideal A of R (cfr. [6, Theorem 36.5]. From this it is clear that the part of degree zero of a G-graded almost Dedekind domain is also almost Dedekind. We then have

Corollary 5.1. *Let R be a G-graded almost Dedekind domain where G is a torsion-free abelian group. If* Supp(R) *is not a group, then $R \cong k[X]$, a polynomial ring over a field. Otherwise $R \cong k^t[G]$, a twisted group ring of a torsion-free rank one group over a field.*

Proof. Suppose Supp$(R) = S$ and $\mathcal{U}(S) = \{e\}$. Then it follows from Theorem 3.5 and the remark following it that $R \cong k^t[S]$ where S is a Prüfer submonoid of Q. Then $I = S \setminus \{e\}$ is the maximum ideal of S and $M = k^t[I]$ is a maximal ideal of $R = k^t[S]$. Therefore R_M is a discrete rank one valuation ring with defining valuation, say v. Then the mapping $w: \langle S\rangle \to \mathbf{Z}$ defined by $w(g) = v(\bar{g})$ is a group homomorphism, i.e., a valuation on $\langle S\rangle$. One easily verifies that S is the valuation monoid of v. Hence S is a discrete rank one valuation monoid and so, since $\mathcal{U}(S) = \{e\}$, $S = \langle x\rangle$, a cyclic monoid. When Supp(R) is a group, the result immediately follows from Corollary 4.4. ∎

The converse of this corollary is not necessarily valid. For example the group ring $\mathbf{Z}_p[\bigcup \{a/p^n \mid a \in \mathbf{Z},\ n \in \mathbf{N}\}]$, where $\mathbf{Z}_p$ is the field with p elements, is not almost Dedekind by [9, Lemma 27]. We will now determine necessary and sufficient conditions for a group graded ring to be almost Dedekind.

LEMMA 5.2. *Let G be a torsion-free abelian group and H a subgroup of G such that G/H is a torsion group. If R is strongly G-graded and R is almost Dedekind, then $R_{[H]}$ is almost Dedekind.*

Proof. If P is a non-zero prime of $R_{[H]}$, then $PR \cap R_{[H]} = P$. If M is an ideal of R, maximal with respect to $M \cap R_{[H]} = P$, then M is a prime ideal. Since $R_{[H]}$ is integrally closed and R is integral over $R_{[H]}$, it follows from [6, Proposition 12.7] that $(R_{[H]})_P = R_M \cap Q(R_{[H]})$. Because R_M is a discrete rank one valuation ring, it follows from [6, Theorem 19.16] that $(R_{[H]})_P$ is a discrete rank one valuation ring too. ∎

LEMMA 5.3. *Let G be a torsion-free abelian group and H a subgroup of G such that G/H is a torsion group. Suppose $R = R_e^t[G]$ is a twisted group ring over a field R_e and $R_e^t[H]$ is Dedekind. If* $\mathrm{char}(R_e) \nmid |G/H|$, *then R is almost Dedekind.*

Proof. We consider $R_e^t[G]$ as a strongly G-graded ring $R_{[G]}$. Since G/H is a torsion group, R is integral over $R_{[H]}$ and so $\dim(R) = \dim(R_{[H]}) = 1$. Let Q be a non-zero prime ideal of R. Note that if G' is a subgroup of G, then since R is strongly (G/G')-graded, $Q \cap R_{[G']}$ is a nonzero prime of $R_{[G']}$. We will prove R_Q is a Dedekind domain. Let $\{G_i \mid i \in I\}$ be a set of subroups such that for all $i \in I$, G_i/H is finite and $G = \bigcup_{i \in I} G_i$. Obviously, $Q = \bigcup_{i \in I} (Q \cap R_{[G_i]})$ and therefore $R_Q = \bigcup_{i \in I} (R_{[G_i]})_{Q \cap R_{[G_i]}}$; put $R_i = (R_{[G_i]})_{Q \cap R_{[G_i]}}$. Now for each i, $Q(R_e^t[G_i])$ is a finite field extension of $Q(R_e^t[H])$ and $R_e^t[G_i]$ is integral over $R_e^t[H]$. Therefore $R_e^t[G_i]$ is contained in the integral closure of $R_e^t[H]$ in $Q(R_e^t[G_i])$. Since, by [2, Proposition 5.4], $R_e^t[G_i]$ is itself integrally closed in its own quotient field, it is therefore equal to that integral closure and so it is Krull by [5, Proposition 1.3]. Moreover $R_{[G_i]}$ also has dimension one, since it is integral over $R_{[H]}$. Thus $R_{[G_i]}$ is Dedekind and R_Q, as a direct limit of localizations of the $R_{[G_i]}$, is Prüfer of dimension one. So we only have to prove that R_Q is Krull. To do so, we check the two conditions of [5, Proposition 8.6]: let $R_i \subseteq R_j$, $i, j \in I$. First, condition (PDE) is trivially satisfied since all rings R_i have dimension one. Second we assert that, if q is a non-zero prime ideal of R_i, then qR_j is a prime ideal of R_j and therefore a maximal divisorial ideal: to prove this, we first note that $(q \cap R_{[H]}) R_{[G_j]}$ is a semiprime ideal of $R_{[G_j]}$: indeed, $R_{[G_j]}/(q \cap R_{[H]}) R_{[G_j]}$ is strongly (G_j/H)-graded with component of degree e $R_{[H]}/(q \cap R_{[H]}) = R_e^t[H]/(q \cap R_e^t[H])$ a domain; moreover $\mathrm{char}(R_{[H]}/q \cap R_{[H]}) = \mathrm{char}(R_e) \nmid |G_j/H|$. It follows, by a generalization of Maschke's theorem (see, for example, [11, Theorem 4.4]), that $(q \cap R_{[H]}) R_{[G_j]}$ is a semiprime ideal of $R_{[G_j]}$. Moreover $(q \cap R_{[H]}) R_{[G_j]} \neq 0$ and so $(q \cap R_{[H]}) R_j$ is a non-zero semiprime ideal of the discrete rank one valuation ring R_j. Therefore $(q \cap R_{[H]}) R_j$ is the unique maximal ideal of

R_j and since $(q \cap R_{[H]}) R_j \subseteq qR_j \subset R_j$, it follows that qR_j is also equal to the unique maximal ideal of R_j. This proves the assertion. By Fossum's result, R_Q is then a Krull domain. ▌

LEMMA 5.4. *Let H be a torsion-free group and F a free subgroup such that H/F is a p-torsion group, where p is a prime. Assume k is a field of characteristic p. If $k^t[H]$ is almost Dedekind, then $k^t[H]$ is Dedekind.*

Proof. We merely consider $R = k^t[H]$ as a strongly H-graded ring with part of degree zero a field. By [6, Theorem 37.2] we only have to prove that each non-zero element of $k^t[H]$ is contained in only finitely many maximal ideals of $k^t[H]$. Now, if $r \in R$, then, using the Frobenius homomorphism, one can find $n \in \mathbb{N}$ such that $r^{p^n} \in R_{[F]} \cong k[X, X^{-1}]$. So r^{p^n} is contained in only finitely many maximal ideals $M_1, ..., M_l$ of $R_{[F]}$. Now for each i there is only one prime P_i of $k^t[H]$ such that $P_i \cap R_{[F]} = M_i$; for, suppose $P_{i_1} \cap R_{[F]} = P_{i_2} \cap R_{[F]} = M_i$. Then for each $r \in P_{i_1}$ there exists $n \in \mathbb{N}$ such that $r^{p^n} \in R_{[F]} \cap P_{i_1} = R_{[F]} \cap P_{i_2} \subseteq P_{i_2}$ and so $r \in P_{i_2}$. Thus $P_{i_1} = P_{i_2}$ and so r is contained in only finitely many maximal ideals of R. ▌

We can now state the main theorem of this section.

THEOREM 5.5. *Let R be a G-graded ring where G is a torsion-free abelian group. Then R is an almost Dedekind domain if and only if either $R \cong k[X]$, a polynomial ring over a field, or $R \cong k^t[G]$, a twisted group ring of a torsion-free rank one group G over a field, such that the following condition is satisfied: if $\mathrm{char}(k) = p > 0$ and if G_0 is a subgroup of G such that the p-primary component of G/F is G_0/F, then $k^t[G_0]$ is Dedekind.*

Proof. If R is an almost Dedekind domain, then Corollary 5.1 yields the first part of the theorem. Suppose now $R \cong k^t[G]$. Then it follows from Lemma 5.2 that $R_{[G_0]} = k^t[G_0]$ is almost Dedekind and by Lemma 5.4, $k^t[G_0]$ is then Dedekind. The converse immediately follows from Lemma 5.3. ▌

Remark. It follows from Corollary 5.10 in [2] that $k^t[G_0]$ is a Dedekind domain if G_0 has the ascending chain condition on cyclic subgroups. However, the example in the remark following Theorem 3.5 shows that the converse does not hold. However, in case of a group ring $k[G_0]$, i.e., the twist is trivial, it is known that $k[G_0]$ is Dedekind if and only if G_0 has the ascending chain condition on cyclic subgroups, and thus $G_0 \cong \mathbb{Z}$. Hence

COROLLARY 5.6 (R. Gilmer [7]). *A monoid ring $R[S]$ is an almost Dedekind domain if and only if R is a field and one of the following conditions holds:*

1. *S is a cyclic monoid*;

2. *S is a group and* char$(R)=0$;

3. *S is a group*, char$(R)=p>0$, *and* G_0 *is the infinite cyclic group* (*where* G_0 *is defined as in Theorem* 5.5).

References

1. D. F. Anderson, Graded Krull domains, *Comm. Algebra* 7, No. 1 (1979), 79–106.
2. D. D. Anderson and D. F. Anderson, Divisibility properties and graded domains, *Canad. J. Math.* 24 (1982), 196–215.
3. L. G. Chouinard II, Krull semigroups and divisor class groups, *Canad. J. Math.* 33 (1981), 1459–1468.
4. F. Decruyenaere and E. Jespers, Graded rings and Dedekind domains, *Proc. Amer. Math. Soc.*, in press.
5. R. Fossum, "The Divisor Class Group of a Krull Domain," Springer-Verlag, Berlin, 1973.
6. R. Gilmer, "Multiplicative Ideal Theory," Dekker, New York, 1972.
7. R. Gilmer, "Commutative Semigroup Rings," Univ. of Chicago Press, Chicago, IL, 1984.
8. J. M. Howie, "An Introduction to Semigroup Theory," Academic Press, London, 1976.
9. R. Matsuda, *K* semigroup rings and almost Krull semigroups rings, *Math. Japon.* 30, No. 4 (1985), 559–571.
10. C. Nastasescu and F. Van Oystaeyen, "Graded Ring Theory," North-Holland, Amsterdam, 1982.
11. D. S. Passman, "Infinite Crossed Products," Academic Press, San Diego, CA, 1989.
12. M. Van den Bergh, Graded Dedekind domains, *J. Pure Appl. Algebra* 35 (1985), 105–115.
13. P. Wauters, Fractional domains and graded rings, *Comm. Algebra* 17, No. 4 (1989), 827–836.