A note on the half-integral multiflow-problem restricted to minor-closed classes of graphs

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Abstract

Seymour (1981) proved that the restriction of the half-integral multiflow-problem to \(K_4\)-free instances is solvable in polynomial time. Middendorf and Pfeiffer (1990) proved the general half-integral multiflow-problem to be NP-complete. Unfortunately, the graphs constructed in their reduction contain arbitrary graphs as minors. We present here a new reduction to prove the NP-completeness of the half-integral multiflow-problem constructing only almost-planar graphs (a graph \(G\) is almost-planar if there exists a vertex \(x \in V(G)\) with \(G - x\) planar). This implies that the restriction of the half-integral multiflow-problem to a given minor-closed class of graphs \(\mathcal{F}\) is NP-complete if \(\mathcal{F}(\mathcal{F})\) (the set of forbidden minors of \(\mathcal{F}\)) does not contain an almost-planar graph. In the present note we also address the half-integral directed-multiflow-problem. We prove that even the restriction to directed planar graphs is NP-complete.

1. Introduction

A comprehensive survey on multiflow-problems can be found in [1]. The starting point of the whole theory is Menger's well-known theorem on the maximum number of edge-disjoint paths connecting two specified vertices \(s\) and \(t\) in a given graph \(G\). A set of edge-disjoint \(s-t\)-paths is usually called an \(s-t\) flow. An \(s-t\) flow may also be considered as a packing of edge-disjoint circuits in the graph arising from \(G\) by adding a suitable number of parallel edges connecting \(s\) and \(t\). We prefer this point of view, especially when dealing with multiflows.

In the present note we study the complexity of the half-integral (directed) multiflow-problem restricted to minor-closed classes of (directed) graphs. Let us recall the basic definitions (concerning the undirected problem).
Definition 1.1. For graphs $S, D$ with $V(D) \subseteq V(S)$ let

$$\mathcal{C}(S,D) = \{ C | C \text{ is cycle in } S \cup D, |E(C) \cap E(D)| = 1 \}.$$ 

- A function $h: \mathcal{C}(S,D) \to \mathbb{R}^+$ is called a $D$-flow in $S$.
- A $D$-flow $h$ in $S$ is $1/k$-integral if $h[\mathcal{C}(S,D)] \subseteq (1/k)\mathbb{N}, k \in \mathbb{N}_0$.

Remark. In this context $S$ and $D$ are usually called supply- and demand-graph, respectively. A $D$-flow in $S$ is also called a multiflow (in $S, D$).

Definition 1.2. Given $S$ with capacity $s: E(S) \to \mathbb{R}^+$ and $D$ with request $d: E(D) \to \mathbb{R}^+$ a $D$-flow $h$ in $S$ is $s, d$-admissible if

$$\forall e \in E(S) \sum_{C \in \mathcal{C}(S,D) \atop e \in E(C)} h(C) \leq s(e)$$

and

$$\forall f \in E(D) \sum_{C \in \mathcal{C}(S,D) \atop f \in E(C)} h(C) \geq d(f).$$

Remark. We also call an $s, d$-admissible $D$-flow in $S$ an admissible multiflow (in $S, D, s, d$) or a solution of $S, D, s, d$.

Problem. $\mathcal{M}_k, k = 1, 2$.

Instance: Graphs $S, D, V(D) \subseteq V(S)$, capacity $s: E(S) \to \mathbb{N}$ and request $d: E(D) \to \mathbb{N}$.

Question: Does there exist a $1/k$-integral admissible multiflow in $S, D, s, d$?

Remark. The restriction of $\mathcal{M}_k$ to instances $S, D, 1, 1$ is usually called the $1/k$-integral edge-disjoint-paths-problem, $k = 1, 2$.

The corresponding directed-multiflow-problems $\mathcal{M}_k, k = 1, 2$, are defined analogously (in terms of suitable packings of directed cycles).

Seymour [5] proved the restriction of the half-integral multiflow-problem to $K_5$-free instances to be solvable in polynomial time. Middendorf and Pfeiffer [3] proved the general half-integral multiflow-problem to be NP-complete. Unfortunately, the graphs constructed in their reduction contain arbitrary graphs as minors. We present here a new reduction to prove the NP-completeness of the half-integral multiflow-problem constructing only almost-planar graphs. This implies that the restriction of the half-integral multiflow-problem to a given minor-closed class of graphs $\mathcal{G}$ is NP-complete if $\mathcal{F}(\mathcal{G})$ (the set of forbidden minors of $\mathcal{G}$) does not contain an almost-planar graph.

We also prove that the restriction of the directed-multiflow-problem to directed planar graphs is NP-complete.
2. The undirected case

As an auxiliary problem we consider here the following problem.

**Problem.** Disjoint-circuits-problem.

*Instance:* Graphs $S, D, V(D) \subseteq V(D)$.

*Question:* Does there exist $h: \mathcal{E}(S, D) \to \{0, 1\}$ satisfying

\[
\forall f \in E(D) \exists C \in \text{supp}(h) f \in E(C) \quad \text{and} \\
\forall v \in V(S) \left| \{ C \in \text{supp}(h) \mid v \in V(C) \} \right| \leq 1
\]

**Definition 2.1.** For a graph $G$ and a function $g: V(G) \to \mathbb{N}$ a $g$-factor of $G$ is a graph $H \subseteq G$ satisfying $\forall v \in V(G)\ deg_H(v) = g(v)$. Usually we identify a function $g: X \to Y$ with a sequence $(g(x))_x$ of their images (for a fixed well-ordering of $X$).

**Definition 2.2.** A solution $h$ of an instance $S, D$ of the disjoint-circuits-problem is a strong solution of $S, D$ if $S - \bigcup_{C \in \text{supp}(h)} E(C)$ has a

\[
\langle \text{degs}_{S \cup D}(v) - (1 + sg(\left| \{ C \in \text{supp}(h) \mid v \in V(C) \} \right|) \rangle_{v \in V(S)}
\]

where $sg: \mathbb{N} \to \{0, 1\}$ denotes the function with

\[
sg(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{otherwise} 
\end{cases}, \quad n \in \mathbb{N}.
\]

**Lemma 2.3.** The restriction of the disjoint circuits-problem to instances $S, D$ satisfying

* $S \cup D$ is planar and bipartite and
* if $S, D$ has a solution at all then there exists a strong solution

is NP-complete.

**Proof.** We are going to reduce planar 3SAT to the problem in question. Recall that planar 3SAT is the restriction of 3SAT to instances $x$ (considered as a set of clauses) with planar graph $G(x)$: the vertex-set of $G(x)$ is the union of $x$ and the set $V$ of variables occurring in $x$ and a pair $v e$ belongs to the edge-set of $G(x)$ if the variable $v$ occurs in the clause $e$. Planar 3SAT was shown in [2] to be NP-complete. Planar 3SAT remains NP-complete even if restricted to instances satisfying

\[
\forall v \in V\ deg_{G(x)}(v) \leq 3 \land \forall e \in x\ deg_{G(x)}(e) = 3.
\]

Without loss of generality, we assume that every variable in $x$ has exactly one negated occurrence.

We built up an instance $S, D$ of the restriction of the disjoint-circuits-problem considered here using copies of the planar bipartite graphs depicted in Fig. 1 (dashed edges in figures are supposed to be demand-edges): for every variable $v$ with unique negated occurrence $v_\perp$, for every clause $e$ with occurrences of variables $v_1, v_2, v_3$ and for every edge $ve \in E(G(x))$ we introduce a copy of $G_v, G_e$ and $G_{ve}$, respectively. To get
the final instance $S, D$ we identify edges of the several copies introduced here as indicated by their denotation in Fig. 1.

By the planarity of the graph $G(\alpha)$ the planarity of $S \cup D$ immediately follows.

Claim. The given instance $\alpha$ of planar $3SAT$ is satisfiable if and only if $S, D$ has a solution (the proof of this claim is straightforward and we omit the details here).

In case of the existence of a solution of $S, D$ it is easy to determine a solution $h$ and a

$\langle \deg_{S \cup D}(v) - (1 + s_{\text{g}}(\{C \in \text{supp}(h) | v \in V(C)\})) \rangle_{v \in V(S\cup D)}$-factor
Fig. 2. The required factor.

in $S - \bigcup_{C \in \text{supp}(h)} E(C)$ as indicated in Fig. 2 (here dashed edges are supposed to belong to $\bigcup_{C \in \text{supp}(h)} E(C)$).

Theorem 2.4. The restriction of $\mathcal{M}_2$ to almost-planar graphs is NP-complete (in the strong sense).

Proof. Let $S, D$ be an instance of the restriction of the disjoint-circuits-problem considered in Lemma 2.3. Without loss of generality, we assume

$$\forall v \in V(D) \deg_D(v) = 1.$$
Starting from \( G := S \cup D \) we construct an almost-planar graph \( G' \) and decompose afterwards \( G' \) into two graphs \( S' \) and \( D' \). Then we define an appropriate capacity and request \( s' \) and \( r' \), respectively. The resulting instance \( S', D', s', d' \) of \( M_2 \) admits a \( \frac{1}{2} \)-integral admissible multiflow if and only if \( S, D \) has a (strong) solution.

Starting from \( G \) add in parallel to every edge \( v_1v_2 = f \in E(D) \) a path \( v_1u_1u_2v_2 \). The resulting graph \( \tilde{G} \) is obviously planar and bipartite again. Denote \( (V(\tilde{G}), E(\tilde{G}) - E(D)) \) by \( \tilde{S} \). Let \( \tilde{G} \subseteq A \ast B, V(\tilde{G}) = A \cup B \) (\( A \ast B \) denotes the complete bipartite graph with color-classes \( A \) and \( B \)). Introduce a new vertex \( x_0 \) adjacent to \( V(\tilde{G}) \) and denote the resulting graph by \( G' \).

Let

\[
S' = (V(\tilde{G}) \cup x_0, (E(\tilde{G}) - E(D)) \cup \{vx_0 | v \in A\})
\]

and

\[
D' = (V(D) \cup B \cup x_0, E(D) \cup \{vx_0 | v \in B\}).
\]

Define \( s' : E(S') \to \mathbb{N} \) and \( d' : E(D') \to \mathbb{N} \) by

\[
s'(e) = \begin{cases} 1 & \text{if } e \in E(G') \\ \deg_{\tilde{G}}(v) - 1 & \text{if } e = vx_0 \end{cases}
\]

and

\[
d'(f) = \begin{cases} 1 & \text{if } f \in E(D) \\ \deg_{\tilde{G}}(v) - 1 & \text{if } e = vx_0 \end{cases}
\]

respectively. Since \( \tilde{G} \) is bipartite the cut \( \partial_{\tilde{G}}(x_0) \) is tight, i.e. \( s'(\partial_{\tilde{S}}(x_0)) = d'(\partial_{\tilde{D}}(x_0)) \).

Let \( h \) be a \( \frac{1}{2} \)-integral admissible multiflow in \( S', D', s', d' \). Denote by \( h_{\mathbb{I}_A} \) the restriction of \( h \) to \( X \subseteq \mathbb{I}(S', D') \). The tightness of \( \partial_{\tilde{G}}(x_0) \) implies

- \( \forall C_1, C_2 \in \text{supp}(h_{\mathbb{I}(S, D)}) \) \( C_1 \neq C_2 \Rightarrow V(C_1) \cap V(C_2) = \emptyset \).
- \( h_{\mathbb{I}(S', D')}[\text{supp}(h_{\mathbb{I}(S, D)})] = \{\frac{1}{2}\} = h_{\mathbb{I}(S', D')}[\text{supp}(h_{\mathbb{I}(S, D)})] \) and
- \( \forall f \in E(D) \exists C \in \text{supp}(h_{\mathbb{I}(S', D')}) f \in E(C) \).

This implies that \( 2h_{\mathbb{I}(S, D)} \) is a solution of \( S, D \).

Let \( h_{\mathbb{I}(S, D)} \to \{0, 1\} \) be a solution of \( S, D \). Without loss of generality we can assume the existence of a

\[
\langle \deg_{S \cup D}(v) - (1 + s_0(\{C \in \text{supp}(h) | v \in V(C)\})) \rangle_{v \in \text{V}(S, D)} \text{-factor } H
\]

in \( S - \bigcup_{C \in \text{supp}(h)} E(C) \).

We extend \( h \) successively to a \( \frac{1}{2} \)-integral multiflow \( h' \) in \( S', D', s', d' \).

\[
h'(C) = \begin{cases} \frac{1}{2} & \text{if } E(C) \cap E(D) = f = v_1v_2 \land V(C) = \{v_1, v_1', v_2, v_2'\} \\ 0 & \text{otherwise} \end{cases}
\]

For \( C \in \mathbb{I}(S', D) - \mathbb{I}(S, D) \) let

\[
h'(C) = \begin{cases} \frac{1}{2} & \text{if } E(C) \cap E(D) = f = v_1v_2 \land V(C) = \{v_1, v_1', v_2, v_2'\} \\ 0 & \text{otherwise} \end{cases}
\]
For $C \in \mathcal{G}(S', D') - \mathcal{G}(S, D)$ let
\[
h'(C) = \begin{cases} 
1 & \text{if } V(C) = \{x_0, u, v\} \land uv \in E(H) \\
\frac{1}{2} & \text{if } V(C) = \{x_0, u, v\} \land v \in \bigcup_{C \in \text{supp}(h)} E(C) . \\
0 & \text{otherwise}
\end{cases}
\]
Obviously,
\[
\forall e \in E(S - x_0) \sum_{C \in \mathcal{G}(S', D')} h'(C) \leq s'(e) = 1
\]
and
\[
\forall f \in E(D) \sum_{C \in \mathcal{G}(S', D')} h'(C) \geq d(f) = 1
\]
holds true. For an edge $e = ux_0$, $v \in A$, we derive the equality
\[
\sum_{C \in \mathcal{G}(S', D')} h'(C) = \text{deg}_{S \cup D}(v) - 1 + (1 - 1 + 1)sg(|\{C \in \text{supp}(h) \mid v \in V(C)\}|)
\]
\[
= \text{deg}_{S \cup D}(v) - 1
\]
\[
= \begin{cases} 
\text{deg}_{S \cup D}(v) - 1 = \text{deg}_S(v) + 1 - 1 = \text{deg}_S(v) - 1 & \text{if } v \in V(D) \\
\text{deg}_{S \cup D}(v) - 1 = \text{deg}_S(v) - 1 = \text{deg}_S(v) - 1 & \text{if } v \notin V(D)
\end{cases}
\]
\[
= \text{deg}_S(v) - 1 = s'(e).
\]
In the same way we derive for an edge $f = vx_0$, $v \in B$, the equality
\[
\sum_{C \in \mathcal{G}(S', D')} h'(C) = d'(f).
\]
This means that $h'$ is indeed a $\frac{1}{2}$-integral admissible multiflow $h'$ in $S', D', s', d'$.

Observe that $\mathcal{F}(\mathcal{G}_{ap})$ contains for example $X + Y$, $X, Y \in \{K_5, K_{3,3}\}$, $K_6$ and $K_{4,4}/e$, $e \in E(K_{4,4})$, where $\mathcal{G}_{ap}$ denotes the class of almost planar graphs. As pointed out earlier the half-integral multiflow-problem for $K_5$-free graphs is polynomially solvable. Let an arbitrary minor-closed class of graphs $\mathcal{G}$ be given. By Middendorf and Pfeiffer [3] the restriction of the half-integral multiflow-problem to $\mathcal{G}$ cannot (assuming $P \neq NP$) be NP-complete in the strong sense if $\mathcal{G}$ doesn’t contain the planar graphs.

Thus, Theorem 1 closes the gap very tightly between polynomially solvable and NP-complete restrictions of the half-integral multiflow-problem to minor-closed classes of graphs.
The main problems to solve to get a complete characterization of all minor-closed classes of graphs for which the half-integral multiflow-problem is solvable at least in pseudo-polynomial time are the following:

**Problem 1.** Does there exist a proper minor-closed subclass of $\mathcal{G}_{sp}$ for which the half-integral multiflow-problem is NP-complete (in the strong sense)?

**Problem 2.** Does there exist an almost-planar graph $G$ such that excluding only $G$ as a minor the restriction of the half-integral multiflow-problem to the resulting minor-closed class of graphs is NP-complete?

### 3. The directed case

Middendorf and Pfeiffer [3] proved the restriction of the integral multiflow-problem to planar instances to be NP-complete (in the strong sense). The analogous statement holds true in the directed case as well as in the undirected one.

**Theorem 3.1.** The restriction of $\overrightarrow{\mathcal{M}}_1$ to instances $\overrightarrow{S}, \overrightarrow{D}, \overrightarrow{s}, \overrightarrow{d}$, with (underlying undirected graph) $S \cup D$ planar is NP-complete.

**Proof.** We reduce the undirected version of this problem to the directed one. Let $S, D, s, d$ be an instance of $\overrightarrow{\mathcal{M}}_1$. We consider $S, D, s, d$ as a graph with possibly parallel edges ($s(e)$ and $d(f)$ denoting the multiplicities of edges $e \in E(S)$ and $f \in E(D)$, respectively). Choose an arbitrary direction of the edges $f \in E(D)$ and replace every edge $e = uv \in E(S)$ by the directed graph $\overrightarrow{S}_{uv} \cup \overrightarrow{D}_{uv}$ depicted in Fig. 3. The resulting directed graph is obviously planar. The correctness of the reduction is based on the following fact: after removing a solution of $\overrightarrow{S}_{uv} \overrightarrow{D}_{uv}$ there exists at most one additional directed $u-v$-(exclusive) or $v-u$-path and such an additional path (of both directions) indeed can be provided by some solution of $\overrightarrow{S}_{uv}, \overrightarrow{D}_{uv}$. □

**Theorem 3.2.** The restriction of $\overrightarrow{\mathcal{M}}_2$ to instances $\overrightarrow{S}, \overrightarrow{D}, \overrightarrow{s}, \overrightarrow{d}$, with $S \cup D$ planar is NP-complete.

**Proof.** By Theorem 3.1 it is sufficient to provide a “half-integral directed edge”. The directed graph $\overrightarrow{S}_{uv} \cup \overrightarrow{D}_{uv}$ depicted in Fig. 4 behaves with respect to half-integral solvability like a directed edge $uv$ of capacity $\frac{1}{2}$. □

By the technique developed in [4] we derive

**Theorem 3.3** (Assume P ≠ NP). Let $\mathcal{G}$ be a minor-closed class of graphs. The restriction of $\overrightarrow{\mathcal{M}}_2$ to instances $\overrightarrow{S}, \overrightarrow{D}, \overrightarrow{s}, \overrightarrow{d}$, with $S \cup D \in \mathcal{G}$ is solvable in pseudo-polynomial time if and only if $\mathcal{F}(\mathcal{G})$ contains a planar graph.
Fig. 3. An instance of $\tilde{G}_1$ equivalent to an undirected edge.

Fig. 4. An instance of $\tilde{G}_2$ equivalent to a directed edge of capacity $\frac{1}{2}$.

References