Tiling space with notched cubes

James H. Schmerl

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA

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Abstract

Stein (1990) discovered \((n-1)!\) lattice tilings of \(\mathbb{R}^n\) by translates of the notched \(n\)-cube which are inequivalent under translation. We show that there are no other inequivalent tilings of \(\mathbb{R}^n\) by translates of the notched cube.

0. Introduction

A notched cube of dimension \(n\) is obtained from a unit \(n\)-cube by deleting a rectangular box from one of its corners. To be precise, let \(\mathbb{R}^n\) be \(n\)-dimensional Euclidean space, let \(I^n = \{(x_1,x_2,...,x_n)\in\mathbb{R}^n: 0\leq x_i \leq 1\text{ for each } i = 1,2,...,n\}\) be the unit \(n\)-cube, and let \(a=(a_1,a_2,...,a_n)\) be a point in the interior of \(I^n\). Then the \(a\)-notched \(n\)-cube is the polytope,

\[
K = \{(x_1,x_2,...,x_n)\in I^n: 0\leq x_i \leq a_i \text{ for some } i\}.
\]

The reader is warned that our notation is at variance with [1]: our \(a_i\) is Stein's \(1-a_i\). Stein [1] showed that \(\mathbb{R}^n\) can be tiled by translates of the \(a\)-notched \(n\)-cube. Moreover, he explicitly exhibited \((n-1)!\) inequivalent lattice tilings. He remarked that, for \(n = 1,2,3\), there are precisely \((n-1)!\) inequivalent tilings, and left open whether, for \(n \geq 4\), there are additional tilings. Our purpose here is to show that in all dimensions there are no tilings other than those discovered by Stein.

Theorem. For each \(n\) there are precisely \((n-1)!\) inequivalent tilings of \(\mathbb{R}^n\) by translates of the \(a\)-notched \(n\)-cube.

Thus, there are no tilings (up to equivalence) other than the tilings constructed by Stein, and therefore all tilings are lattice tilings. The Stein tilings will be described in Section 2. Conventions and basic definitions will be presented in Section 1, and the proof of the Theorem will make up Section 3.
1. Conventions and basic definitions

We let $n$ denote an integer such that $n \geq 2$, and let $[n] = \{1, 2, 3, \ldots, n\}$. Points in $\mathbb{R}^n$ will be denoted in bold-face. For example, if $x \in \mathbb{R}^n$, then $x = (x_1, x_2, \ldots, x_n)$; or if $g_i \in \mathbb{R}^n$, then $g_i = (g_{i1}, g_{i2}, \ldots, g_{in})$. We let $\mathbf{0}$ denote the vector $(0, 0, \ldots, 0)$. For a subset $X \subseteq \mathbb{R}^n$, we let $\text{Int}(X)$ be the interior of $X$. We always let $a$ denote a point in $\text{Int}(I^n)$, and $K$ denote the $a$-notched $n$-cube $\{x \in I^n: 0 < x_i < a_i \text{ for some } i \in [n]\}$.

A tiling of $\mathbb{R}^n$ by translates of $K$ (or, simply: tiling) is a set $\mathcal{T}$ such that the following hold:

1. each $T \in \mathcal{T}$ is a translate of $K$ (i.e. $T = K + h$ for some $h \in \mathbb{R}^n$);
2. if $T_1, T_2 \in \mathcal{T}$ are distinct, then $\text{Int}(T_1 \cap T_2) = \emptyset$ (or, equivalently, $T_1 \cap \text{Int}(T_2) = \emptyset$);
3. for each $x \in \mathbb{R}^n$ there is $T \in \mathcal{T}$ such that $x \in T$.

The elements of $\mathcal{T}$ will be referred to as tiles. Two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent if there is some $x \in \mathbb{R}^n$ such that $\mathcal{T}_2 = \{T + x: T \in \mathcal{T}_1\}$. Each tiling is equivalent to exactly one in which $K$ is a tile. We will always let $\mathcal{T}$ denote a tiling in which $K$ is a tile.

We will let $\text{Sym}(n)$ be the set of all permutations of $[n]$, and we let $\text{Cyc}(n)$ be the subset of $\text{Sym}(n)$ consisting of all cyclic permutations.

2. The Stein tilings

All the tilings that Stein constructed are lattice tilings, and each is determined by a cyclic permutation of $[n]$. Let $\gamma \in \text{Cyc}(n)$. For each $i \in [n]$, let $g_i \in \mathbb{R}^n$ be such that for each $j \in [n]$

$$g_{ij} = \begin{cases} 
1 & \text{if } j = i, \\
 a_j - 1 & \text{if } j = \gamma(i), \\
0 & \text{if } j \notin \{i, \gamma(i)\}.
\end{cases}$$

Notice that the vectors $g_1, g_2, \ldots, g_n$ form a basis for $\mathbb{R}^n$; in fact, as Stein [1] notes, the determinant of the $n \times n$ matrix whose $i$th row is $g_i$ is $1 - (1 - a_1)(1 - a_2) \cdots (1 - a_n)$, which is also the volume of $K$. Let $L'$ be the lattice spanned by these vectors. Note that $a = g_1^1 + g_2^2 + \cdots + g_n^n \in L'$. Stein proved that

$$\mathcal{T}' = \{K + x: x \in L'\}$$

is a tiling. These are the Stein tilings. Obviously, each Stein tiling is a lattice tiling. Stein gave an indirect proof that distinct cycles in $\text{Cyc}(n)$ determine distinct tilings. A simpler argument follows. Suppose $\gamma, \delta \in \text{Cyc}(n)$ and $L' = L''$. For $i \in [n]$ and sufficiently small $\varepsilon > 0$, $(\varepsilon, \varepsilon, \ldots, \varepsilon, 1 + \varepsilon, \varepsilon, \ldots, \varepsilon) \in \text{Int}(K + g_i^i) \cap \text{Int}(K + g_i'')$ (where the $i$th coordinate is $1 + \varepsilon$), so by condition (2) in the definition of tiling $g_i'' = g_i^i$, implying $\gamma(i) = \delta(i)$.

Thus, there are $(n - 1)!$ distinct (and even inequivalent) Stein tilings.
3. The proof of the theorem

We begin with a technical lemma which will be useful on several occasions.

**Lemma 1.** Let \( r \) be the largest integer such that \( ra \in \mathbb{I}^* \). Then \( \mathbb{I}^* \subseteq \mathbb{K} \cup (\mathbb{K} + a) \cup (\mathbb{K} + 2a) \cup \ldots \cup (\mathbb{K} + ra) \)

**Proof.** Consider \( x \in \mathbb{I}^* \). Let \( m \) be the largest integer such that \( ma \leq x \), for all \( i \in [n] \). Clearly, \( 0 \leq m \leq r \) and \( x - ma \in \mathbb{K} \). Thus, \( x \in \mathbb{K} + ma \). \( \square \)

What makes Lemma 1 useful is that all of the tiles occurring in its statement are actually tiles of \( \mathcal{F} \). This is a consequence of the next lemma.

**Lemma 2.** Let \( T \) be a translate of \( \mathbb{K} \). Then \( T \) is a tile of \( \mathcal{F} \) iff \( T + a \) is a tile of \( \mathcal{F} \).

**Proof.** Suppose that \( T \) is a tile of \( \mathcal{F} \) and, without loss of generality, let \( T = \mathbb{K} \). The point \( a \) is on the boundary of \( \mathbb{K} \), so there must be some \( x \neq \emptyset \) such that \( au \in \mathbb{K} + x \in \mathcal{F} \).

We will show that \( x = a \), so suppose \( x 
eq a \). Let \( b \in \mathbb{K} \) be such that \( a = b + x \). Then \( b \neq \emptyset \) since \( x \neq a \). Therefore \( b_j > 0 \) for some \( j \in [n] \). Let \( c \) be such that for a sufficiently small \( \varepsilon > 0 \),

\[
    c_i = \begin{cases} 
        b_i & \text{if } i \neq j, \\
        b_i - \varepsilon & \text{if } i = j.
    \end{cases}
\]

Then \( c \in \mathbb{K} \), and \( c + x = a + (c - b) \subseteq \text{Int}(\mathbb{K}) \), so that \( c + x \subseteq (\mathbb{K} + x) \cap \text{Int}(\mathbb{K}) \). Hence, \( (\mathbb{K} + x) \cap \text{Int}(\mathbb{K}) \neq \emptyset \), implying \( x = 0 \), which is a contradiction.

Conversely, suppose that \( T + a \) is a tile of \( \mathcal{F} \) and, without loss of generality, let \( T + a = \mathbb{K} \). We will show that \( T = \mathbb{K} - a \) is also a tile. The point \( 0 \) is on the boundary of \( \mathbb{K} \), so there must be some tile \( \mathbb{K} - b \neq \mathbb{K} \) of \( \mathcal{F} \) such that \( 0 \in \mathbb{K} - b \). Thus, \( 0 \neq b \subseteq \mathbb{K} \). Let \( c \) be such that for a sufficiently small \( \varepsilon > 0 \), \( c_i = \max(a_i, b_i) + \varepsilon \) for each \( i \in [n] \). One easily checks that \( c - a, c - b \subseteq \text{Int}(\mathbb{K}) \). Therefore, \( c \subseteq \text{Int}(\mathbb{K} + a) \cap \text{Int}(\mathbb{K} + b) \neq \emptyset \), and thus also \( \text{Int}(\mathbb{K} - b + a) \cap \text{Int}(\mathbb{K}) \neq \emptyset \). Since \( \mathbb{K} - b \) is a tile of \( \mathcal{F} \), by the first part of this proof \( \mathbb{K} - b + a \) is a tile. Hence \( b = a \), and so \( \mathbb{K} - a \) is a tile. \( \square \)

Let \( \mathcal{F} \) be a tiling. If \( T \) is a tile of \( \mathcal{F} \) and \( T = \mathbb{K} + x \), then let \( T^* = \mathbb{K} - x \). Let \( \mathcal{F}^* = \{ T^* : T \in \mathcal{F} \} \); it will be proved in the next lemma that \( \mathcal{F}^* \) is also a tiling, which can be regarded as the tiling dual to \( \mathcal{F} \). Since the Stein tilings are lattice tilings, it is immediate that \( (\mathcal{F}^*)^* = \mathcal{F} \).

**Lemma 3.** If \( \mathcal{F} \) is a tiling, then so is \( \mathcal{F}^* \).

**Proof.** Condition (1) in the definition of tiling is trivial.

We will prove condition (2) in the definition of tiling. Consider arbitrary tiles \( \mathbb{K} + x \) and \( \mathbb{K} + y \) of \( \mathcal{F}^* \), and suppose \( u \in \text{Int}(\mathbb{K} + x) \cap \text{Int}(\mathbb{K} + y) \). Then \( K - y \) and \( K - x \) are
tiles of $\mathcal{T}$, and $u - x - y \in \text{Int}(K - y) \cap \text{Int}(K - x)$. Hence, $K - y = K - x$, so that $K + x = K + y$.

We now prove condition (3). Consider arbitrary $x \in \mathbb{R}^n$, and let $y$ be such that $-x + (1, 1, \ldots, 1) \in K + y \in \mathcal{T}$. Then $-x - y \in K - (1, 1, \ldots, 1) \subseteq I^* - (1, 1, \ldots, 1) - I^*$, so that $x + y \in I^*$. Using Lemma 1, let $m$ be an integer such that $x + y \in K + ma$. Then $x \in K + ma - y = (K - ma + y)^*$. By Lemma 2, $K - ma + y \in \mathcal{T}$, so that $(K - ma + y)^* \in \mathcal{T}^*$.

**Lemma 4.** Let $s \in [n - 1]$ be such that for each $i \in [s - 1]$, $T_i = K + (a_1, a_2, \ldots, a_{i-1}, a_i, 0, 0, \ldots, 0, 1)$ is a tile in $\mathcal{T}$. Then there is $r$ such that $s \leq r < n$ and $K + (a_1, a_2, \ldots, a_{s-1}, 0, 0, \ldots, 0, a_r - 1, 0, \ldots, 1)$ is a tile of $\mathcal{T}$.

**Remark.** In the case that $s = 1$, the hypothesis is vacuously satisfied and the conclusion is that $K + (0, 0, \ldots, a_r - 1, 0, \ldots, 1)$ is a tile of $\mathcal{T}$ for some $r \in [n - 1]$.

By permuting coordinates, we see that this case of the lemma implies that for each $i \in [n]$ there is $r \neq i$ such that $K + g$ is a tile of $\mathcal{T}$, where

$$g_m = \begin{cases} 1 & \text{if } m = i \\ a_r - 1 & \text{if } m = r \\ 0 & \text{if } m \neq i, r. \end{cases}$$

**Proof of Lemma 4.** Let $b = (a_1, a_2, \ldots, a_{s-1}, 0, 0, \ldots, 1)$. Then for all sufficiently small $\varepsilon > 0$, $b + (\varepsilon, \varepsilon, \ldots, \varepsilon) \notin K$, $T_1, \ldots, T_{s-1}$. Therefore, there is $r$ such that $K + r$ is a tile which is distinct from $K$, $T_1, \ldots, T_{s-1}$ and for all sufficiently small $\varepsilon > 0$, $b + (\varepsilon, \varepsilon, \ldots, \varepsilon) \in K + t$. Hence, also $b \in K + t$. Clearly,

$$a_i - 1 < t_i \leq a_i \quad \text{for } i < s;$$

$$-1 < t_s \leq 0 \quad \text{for } s \leq i < n;$$

$$0 < t_s \leq 1.$$

But, in fact, $t_s = 1$, as otherwise for sufficiently small $\varepsilon > 0$, $b + (\varepsilon, \varepsilon, \ldots, \varepsilon) \in \text{Int}(K) \cap \text{Int}(K + t)$. Also, $t_i = a_i$ for $i < s$. For, if not, then consider some $i$ for which $1 < i < s$ and $t_i < a_i$. Then for sufficiently small $\varepsilon > 0$

$$(a_1 + \varepsilon, a_2 + \varepsilon, \ldots, a_{i-1} + \varepsilon, a_i - \varepsilon, a_{i+1} + \varepsilon, \ldots, a_{s-1} + \varepsilon, \ldots, \varepsilon, 1 + \varepsilon)$$

$$\in \text{Int}(K + t) \cap \text{Int}(T_i),$$

a contradiction.

Next, we show that there is an integer $m$ such that $s \leq m < n$ and $t_m \leq a_m - 1$. Suppose not; i.e., suppose that for all $m$ if $s \leq m < n$, then $t_m > a_m - 1$. Then for sufficiently small $\varepsilon > 0$,

$$(a_1 + \varepsilon, a_2 + \varepsilon, \ldots, a_{n-1} + \varepsilon, 1 + \varepsilon) \in \text{Int}(K + t) \cap \text{Int}(K + a).$$
By Lemma 2, \( t_s = a_n \), which is a contradiction since \( a_n < 1 = t_s \). For notational simplicity, and without loss of generality, let us assume that \( t_s < a_n - 1 \), and let \( T_s = K + t \).

Our object is to show that \( t = (a_1, a_2, \ldots, a_{s-1}, a_s - 1, 0, 0, \ldots, 0, 1) \). Suppose, to the contrary, that this is not so; that is, either \( t_s < a_n - 1 \) or \( t_s < 0 \) where \( s < i < n \).

Let \( y = t + (0, 0, \ldots, 0, 1, \ldots, 1, 0) \); that is, \( y = (a_1, a_2, \ldots, a_{s-1}, t_s + 1, \ldots, t_n + 1, 1) \). Let \( d \in \mathbb{R}^n \) be such that \( d_i = 1 \) if \( 1 < i < s \), or if \( i = s \) and \( t_s < a_n - 1 \), or if \( s < i < n \) and \( t_i < 0 \), or if \( i = n \); \( d_i = -1 \) otherwise. Then for all sufficiently small \( \varepsilon > 0 \), \( y + \varepsilon d \in K, T_s, T, \ldots, T_s \). Therefore, there is \( u \) such that \( K + u \) is a tile which is distinct from \( K, T_1, T_2, \ldots, T_s \) and for all sufficiently small \( \varepsilon > 0 \), \( y + \varepsilon d \in K + u \). Hence, also \( y \in K + u \). Clearly,

\[
\begin{align*}
a_i - 1 &< u_i < a_i & \text{for } i < s; \\
a_{s-1} - 1 &< u_s < a_s; \\
0 &< u_i < 1 & \text{for } s < i < n; \\
0 &< u_n < 1.
\end{align*}
\]

But, in fact, \( u_n = 1 \), as otherwise for sufficiently small \( \varepsilon > 0 \), \((a_1 + \varepsilon, a_2 + \varepsilon, \ldots, a_{s-1} + \varepsilon, a_s - \varepsilon, 1 - \varepsilon, \ldots, 1 - \varepsilon, 1 + \varepsilon) \in \text{Int}(K) \cap \text{Int}(I^u + u) \). By Lemma 1, \( \text{Int}(K) \cap (K + u + ma) \neq \emptyset \) for some integer \( m > 0 \). By Lemma 2, \( K + u + ma \) is a tile of \( \mathcal{F} \), so \( u = -ma \). However, \( u_{n-r} > 0 \) and \( -ma_n < 0 \), so we have a contradiction.

Also, \( u_i = a_i \) for \( i < s \). For, if not, then consider some \( i \) for which \( 1 \leq i < s \) and \( u_i < a_i \). Then for sufficiently small \( \varepsilon > 0 \),

\[
(a_1 + \varepsilon, \ldots, a_{i-1} + \varepsilon, a_i - \varepsilon, a_{i+1} + \varepsilon, \ldots, a_{s-1} + \varepsilon, a_s - \varepsilon, 1 - \varepsilon, \ldots, 1 - \varepsilon, 1 + \varepsilon) \in \text{Int}(K + u) \cap \text{Int}(T_s),
\]

a contradiction.

We will finish the proof by showing that \( t = u \), thereby contradicting that \( K + u \neq T_s \).

It is easily checked that \( u \in I^u + t \), so \( t + (1, 1, \ldots, 1) \in I^u + u \), and then by Lemma 1, \( t + (1, 1, \ldots, 1) \in K + u + ma \) for some integer \( m > 0 \). Now consider arbitrary \( x \) that is sufficiently close to \((1, 1, \ldots, 1)\). Let \( h \) be the largest integer such that \( x(a_i) < x_i \) for all \( i \in [n] \). Clearly, \( h \geq 1 \) and \( h = r - 1 \) or \( h = r \) (where \( r \) is from Lemma 1). Then \( x - ha \in K \) so \( x \in K + ha \). Hence, \((1, 1, \ldots, 1) \in \text{Int}((K + (r - 1)a) \cup (K + ra)) \), so that \( t + (1, 1, \ldots, 1) \in \text{Int}((K + t + (r - 1)a) \cup (K + t + ra)) \). Clearly then, for \( h = r - 1 \) or \( h = r \), \((K + u + ma) \cap \text{Int}(K + t + ha) \neq \emptyset \). Since, by Lemma 2, both \( K + t + ha \) and \( K + u + ma \) are tiles of \( \mathcal{F} \), it follows that \( t + ha = u + ma \). But \( t_n = u_n = 1 \) and \( a_n > 0 \), so that \( h = m \) and \( t = u \). \( \square \)

Lemma 4 has a dual version.

**Lemma 5.** Let \( s \in [n-1] \) be such that for each \( i \in [s-1], T_i = K + (0, 0, \ldots, 0, 1, a_i + 1, a_{i+2}, \ldots, a_n, a_{s-1}, a_{s-1}) \) is a tile of \( \mathcal{F} \). Then there is \( r \) such that \( s \leq r < n \) and \( K + (0, 0, \ldots, 0, a_s, a_{s+1}, \ldots, a_{r-1}, 1, a_{r+1}, \ldots, a_n, a_n - 1) \) is a tile of \( \mathcal{F} \).

**Proof.** Apply Lemma 4 to the tiles \( T_i^+ + a \) of \( \mathcal{F}^+ \), using Lemma 3 to guarantee that \( \mathcal{F}^+ \) is a tiling and Lemma 2 to guarantee that each \( T_i^+ + a \) is a tile of \( \mathcal{F}^+ \). \( \square \)
Considered as a polytope, \( K \) has \( 3n \) faces, which are of three types. We now give names to these faces. Let \( i \in [n] \).

1. The \( i \)-th small face of \( K \) is the set \( \{ x \in K : x_i = a_i \text{ and } x_j \geq a_j \text{ for all } j \in [n] \} \).
2. The \( i \)-th notched face of \( K \) is the set \( \{ x \in K : x_i = 1 \} \).
3. The \( i \)-th big face of \( K \) is the set \( \{ x \in K : x_i = 0 \} \).

These definitions of faces extended naturally to translates of \( K \). If \( F \) is a face of \( K \) of a certain type, then \( F + x \) is a face of \( K + x \) of the same type.

Suppose \( T_1 \) and \( T_2 \) are distinct tiles of \( T \) and \( F \) is a face of \( T_1 \). We will say that \( T_2 \) is adjacent to \( T_1 \) at \( F \) if there is some \( x \in T_2 \) which is in \( F \) but is in no other face of \( T_1 \). If \( T_2 \) is adjacent to \( T_1 \) at some face, then \( T_2 \) is adjacent to \( T_1 \). Adjacency is a symmetric relation; that is, if \( T_2 \) is adjacent to \( T_1 \) at \( F_1 \), then there is a face \( F_2 \) of \( T_2 \) such that \( T_1 \) is adjacent to \( T_2 \) at \( F_2 \). Thus, with this definition of adjacency, \( T \) can be considered to be a graph.

The next lemma is quite clear. Even much more general statements are true; however, this is all that will be needed.

**Lemma 6.** Each \( T^i \) is a connected graph.

Next we will analyze adjacency in the Stein tilings. That is, for each face \( F \) of \( K \), we will determine which tiles of \( T^i \) are adjacent to \( K \) at \( F \). For \( F \) a small face, the result is quite general and follows easily from Lemma 2.

**Corollary 7.** Let \( T_1, T_2 \) be tiles of \( T \) and let \( F \) be a small face of \( T_1 \). Then \( T_2 \) is adjacent to \( T_1 \) at \( F \) iff \( T_2 = T_1 + a \).

Consider the Stein tiling \( T^i \), and let \( F \) be a small face of \( K \). By Corollary 7, \( K + a \) is the only tile adjacent to \( K \) at \( F \). To consider the other two types of faces, fix some \( i \in [n] \). For each \( j \in [n-1] \), let

\[
h_{ij}^r = \sum_{k=0}^{r-1} g_{\gamma^r(i)}^k
\]

The vectors \( h_{ij}^r \), which of course are in \( L' \), can be described explicitly. For each \( m \in [n] \),

\[
h_{ijm}^r = \begin{cases} 
1 & \text{if } m = i; \\
a_m & \text{if } m = \gamma^r(i), \text{ where } 1 \leq r < j; \\
a_m - 1 & \text{if } m = \gamma^i(i); \\
0 & \text{if } m = \gamma^r(i), \text{ where } j < r < n.
\end{cases}
\]

**Lemma 8.** Consider the Stein tiling \( T^i \), and let \( i \in [n] \).

1. Suppose \( F \) is the \( i \)-th notched face of \( K \). Then there are \( n-1 \) tiles adjacent to \( K \) at \( F \), namely \( K + h_{ij}^r \) for \( j \in [n-1] \).
(2) Suppose $F$ is the $i$-th big face of $K$. Then there are $n$ tiles adjacent to $K$ at $F$, namely $K-a$ and $K-h_{ij}$ for $j \in [n-1]$.

**Proof.** (1) Let $F$ be the $i$-th notched face of $K$. To see that $K + h_{ij}$ is adjacent to $K$ at $F$, consider sufficiently small $\varepsilon > 0$, and then let $x \in \mathbb{R}^n$ be such that

$$x_m = \max(\varepsilon, h_{ij}).$$

Clearly, $x \in K + h_{ij}$ and $x \notin F$ but $x$ is in no other face of $K$.

Conversely, suppose $K + u$ is adjacent to $K$ at $F$, and let $x \in K + u$ be such that $x \notin F$ but $x$ is in no other face of $K$. Then $x_i = 1$ and $x_m < a_m$ for some $m \in [n]$. Let $j$ be such that $\gamma(j) = m$. Then $x \in K + h_{ij}$. Moreover, without loss of generality, we can assume that $x \in \text{Int}(K \cup (K + h_{ij}))$, so that $(K + u) \cap \text{Int}(K + h_{ij}) \neq \emptyset$ so that either $u = 0$ or $u = h_{ij}$. As $u \neq 0$, we conclude that $u = h_{ij}$.

The proof of (2) is similar. $\square$

Suppose $T$ is a tile of $\mathcal{S}$, $F$ is a face of $T$ and $\gamma \in \text{Cyc}(n)$. We will say that $T$ locally resembles $\mathcal{S}^\gamma$ at $F$ if for every tile $T + x$ which is adjacent to $T$ at $F$,

$$K + x \in \mathcal{S}^\gamma.$$ 

In the light of Lemma 8, there is an alternate characterization of local resemblance at notched faces and big faces. If $F$ is the $i$-th notched face of $T$, then $T$ locally resembles $\mathcal{S}^\gamma$ at $F$ iff $T + h_{ij}$ is a tile of $\mathcal{S}$ for each $j \in [n-1]$. If $F$ is the $i$-th big face of $T$, then $T$ locally resembles $\mathcal{S}^\gamma$ at $F$ iff $T - h_{ij}$ is a tile of $\mathcal{S}$ for each $j \in [n-1]$.

In case $F$ is a small face of $T$, Lemma 2 implies that local resemblance is quite trivial: for each $\gamma \in \text{Cyc}(n)$, $T$ locally resembles $\mathcal{S}^\gamma$ at $F$. Thus, the following lemma is interesting only when $F$ is either a notched face or a big face.

**Lemma 9.** Suppose $T$ is a tile of $\mathcal{S}$ and $F$ is a face of $T$. Then there is $\gamma \in \text{Cyc}(n)$ such that $T$ locally resembles $\mathcal{S}^\gamma$ at $F$.

**Proof.** As mentioned previously, for $F$, a small face, the result follows immediately from Lemma 2. In fact, if $F$ is a small face and $\gamma \in \text{Cyc}(n)$ is arbitrary, then $T$ locally resembles $\mathcal{S}^\gamma$ at $F$.

Next, we suppose that $F$ is a notched face, and without loss of generality, we can assume $T = K$ and that $F$ is the $n$th notched face. Applying Lemma 4, $n - 1$ times, possibly permuting coordinates if needed, we obtain a permutation $\pi \in \text{Sym}(n-1)$ such that for each $j \in [n-1] K + h_{ij}$ is a tile of $\mathcal{S}$ where

$$h_{jm} = \begin{cases} 
-1 & \text{if } m = \pi(r), \text{ where } 1 \leq r < j; \\
0 & \text{if } m = \pi(j); \\
1 & \text{if } m = n.
\end{cases}$$
Let $y \in \text{Cyc}(n)$ be such that $y'(n) = \pi(r)$ for $r \in [n-1]$. Then, clearly, $h_j = h_{ij}$ and each $K + h_j$ is adjacent to $K$ at $F$. Thus, by Lemma 8(1), $K$ locally resembles $\mathcal{S}_\pi$ at $F$.

Finally, we suppose that $F$ is a big face, and without loss of generality, we can assume that $T = K + a$ and that $F$ is the $n$th big face. Applying Lemma 5, $n-1$ times, possibly permuting coordinates if needed, we obtain a permutation $\pi \in \text{Sym}(n-1)$ such that for each $j \in [n-1]$ $K + h_j$ is a tile of $\mathcal{S}$ where

$$h_{jm} = \begin{cases} 
0 & \text{if } m = \pi(r), \text{ where } 1 \leq r < j; \\
1 & \text{if } m = \pi(j); \\
a_m & \text{if } m = \pi(r), \text{ where } j < r < n; \\
a_{n-1} & \text{if } m = n.
\end{cases}$$

Let $y \in \text{Cyc}(n)$ be such that $y'(n) = \pi(r)$ for $r \in [n-1]$. Then, clearly, $h_j = a - h_{ij}$, and each $K + h_j$ is adjacent to $K + a$ at $F$. Thus by Lemma 8(2), $K + a$ locally resembles $\mathcal{S}_\pi$ at $F$. \( \square \)

Lemma 9 does not preclude the possibility that for distinct faces $F$ and $G$ of $T$ (neither of which is small) there are distinct cycles $\gamma, \delta \in \text{Cyc}(n)$ such that $T$ locally resembles $\mathcal{S}_\gamma$ at $F$ and $\mathcal{S}_\delta$ at $G$. We will show in Lemma 11 that this is impossible. First we need a combinatorial lemma.

**Lemma 10.** For each $i \in [n]$ let $\pi_i \in \text{Sym}(n)$ be such that $\pi_i(1) = i$. Suppose that whenever $i, j, k \in [n]$ and $i \neq j$, then either $\pi_j^{-1}(i) \geq \pi_j^{-1}(k)$ or $\pi_i^{-1}(j) \geq \pi_i^{-1}(k)$. Then there is $y \in \text{Cyc}(n)$ such that $\pi_i(k) = y_{\pi_i^{-1}(k)}(i)$ for each $i, k \in [n]$.

**Proof.** Let $\pi_i \in \text{Sym}(n)$ for $i \in [n]$ satisfy the hypothesis of the lemma. We will make use of the following immediate consequence of these hypotheses.

If $a, b, c \in [n]$ are distinct and $\pi_a^{-1}(b) < \pi_c^{-1}(c)$, then $\pi_b^{-1}(c) < \pi_b^{-1}(a)$. \( (*) \)

Now let $y \in \text{Cyc}(n)$ be the cycle $(1, \pi_2(2), \pi_3(3), \ldots, \pi_n(n))$; that is, $y_i = \pi_i(k)$ for each $k \in [n]$. To prove the lemma, we will prove by induction on $m \in [n]$ the following: if $\pi_i(m) = i$, then $\pi_i(k) = y_{\pi_i^{-1}(k)}(i)$ for all $k \in [n]$.

The basis step in which $m = 1$ is true by definition. We proceed with the inductive step. Suppose $m \in [n-1]$, $\pi_i(m) = i$, $\pi_i(m + 1) = j$, and $\pi_i(k) = y_{\pi_i^{-1}(k)}(i)$ for all $k \in [n]$. Thus, $j = \gamma(i)$. We will prove that $\pi_i(k) = y_{\pi_i^{-1}(k)}(j)$ for all $k \in [n]$.

Trivially, $\pi_i(1) = j = \gamma(1) = \gamma(1)$, so that $\pi_i(1) = \pi_i(2)$.

We next show that $\pi_i(1) = \gamma(1)$. Observe that $\gamma^{-1}(j) = \gamma^{-1}(j) = i$, so it suffices to show $\pi_i(n) = i$. Notice that $\pi_i(2) = \gamma(1) = j$, so for all $k \in [n]$, if $k \neq i, j$, then $\pi_i^{-1}(j) < \pi_i^{-1}(k)$. Therefore, it follows that $\pi_i^{-1}(1) \geq \pi_i^{-1}(k)$ for all $k \in [n]$, so that $\pi_i^{-1}(1) = n$. Thus, $\pi_i(n) = i = \pi_i(1)$. \( \square \)
It remains to prove that $\pi_j(k) = \gamma^{k-1}(j)$ whenever $1 < k < n$. Since $\gamma^{k-1}(j) - \gamma^{k-1}(j) = \gamma^{k}(i) - \gamma^{k}(i) - \pi_i(k+1)$, it suffices to prove that $\pi_j(k) = \pi_i(k+1)$ whenever $1 < k < n$. Thus, since $\pi_i(1) = \pi_i(2)$, it suffices to prove the following: if $2 < \pi_i^{-1}(r) < \pi_i^{-1}(s)$, then $\pi_j^{-1}(r) < \pi_j^{-1}(s)$.

We will prove this statement by applying ($\ast$) four times. So assume

$$2 < \pi_i^{-1}(r) < \pi_i^{-1}(s).$$

Therefore, $i, j, r, s$ are distinct, and we have

$$\pi_j^{-1}(r) < \pi_j^{-1}(i) = n.$$ 

Applying ($\ast$) to (1) yields

$$\pi_r^{-1}(s) < \pi_i^{-1}(i),$$

and applying ($\ast$) to (2) yields

$$\pi_r^{-1}(i) < \pi_j^{-1}(j).$$

From (3) and (4) we get

$$\pi_r^{-1}(s) < \pi_j^{-1}(j),$$

to which we apply ($\ast$) twice in succession to obtain that $\pi_j^{-1}(r) < \pi_j^{-1}(s)$. \qed

**Lemma 11.** Suppose $T$ is a tile of $F$, and $F$ and $G$ are faces of $T$ neither of which is small. Suppose $\gamma, \delta \in \text{Sym}(n)$ are such that $T$ locally resembles $F^{\gamma}$ at $F$ and $T$ locally resembles $F^{\delta}$ at $G$. Then $\gamma = \delta$.

**Proof.** This proof has three cases, depending on the types of faces of $F$ and $G$.

**Case 1:** $F$ and $G$ are both notched faces. For each $i \in [n]$ let $F_i$ be the $i$th notched face, and by Lemma 9 let $\gamma_i \in \text{Cyc}(n)$ be such that $T$ locally resembles $F^{\gamma_i}$ at $F_i$. We will prove that all the $\gamma_i$ are the same.

Let $\pi_i \in \text{Sym}(n)$ be such that for each $k \in [n],$

$$\pi_i(k) = \gamma_i^{k-1}(i).$$

To obtain a contradiction, assume that not all the $\gamma_i$ are the same. Then by Lemma 10 there are $i, j, k \in [n]$ such that $i \neq j$ and

$$\pi_j^{-1}(i) < \pi_j^{-1}(k)$$

and

$$\pi_i^{-1}(j) < \pi_i^{-1}(k).$$

Let $r = n_i^{-1}(k) - 1$ and $s = n_j^{-1}(k) - 1$, so that $r, s \in [n-1]$. Let $u = h_r^i$ and $v = h_s^j$. Following Lemma 8(1), $T + u$ is a tile of $F$ which is adjacent to $T$ at $F_i$, and $T + v$ is a tile of $F$ which is adjacent to $T$ at $F_j$.

It is easy to verify that the following hold.

$$u_i = 1 \quad \text{and} \quad v_i = a_i; \quad u_j = a_j \quad \text{and} \quad v_j = 1; \quad u_k = v_k = a_k - 1.$$
Therefore, for each \( m \in [n] \), \( |u_m - v_m| < 1 \) and \( |u_k - v_k| = 0 \). It follows that there are \( b, c \in \text{Int}(K) \) such that \( u + b = v + c \). Therefore, \( \text{Int}(K + u) \cap \text{Int}(K + v) \neq \emptyset \), so that also \( \text{Int}(T + u) \cap \text{Int}(T + v) \neq \emptyset \). But \( T + u \) and \( T + v \) are both tiles of \( \mathcal{T} \); hence, \( u = v \), which is clearly a contradiction. This completes Case 1.

**Case 2:** \( F \) and \( G \) are both big faces. For each \( i \in [n] \) let \( F_i \) be the \( i \)th big face, and let \( \gamma_i \in \text{Cyc}(n) \) be such that \( T \) locally resembles \( \mathcal{T}^{\gamma_i} \) at \( F_i \). We will prove that all the \( \gamma_i \) are the same.

Let \( \pi \in \text{Sym}(n) \) be such that for each \( k \in [n] \),

\[
\pi_i(k) = \gamma_i^{-1}(i).
\]

To obtain a contradiction, assume that not all the \( \gamma_i \) are the same. Then by Lemma 10 there are \( i, j, k \in [n] \) such that \( i \neq j \) and

\[
\pi_j^{-1}(i) < \pi_j^{-1}(k)
\]

and

\[
\pi_i^{-1}(j) < \pi_i^{-1}(k).
\]

As in Case 1, let \( r = \pi_i^{-1}(k) - 1 \), \( s = \pi_j^{-1}(k) - 1 \), \( u = h'_u \), and \( v = h'_v \). Following Lemma 8(2), \( T - u \) is a tile in \( \mathcal{T} \) which is adjacent to \( T \) at \( F_i \), and \( T - v \) is a tile in \( \mathcal{T} \) which is adjacent to \( T \) at \( F_j \).

Let \( b, c \) be as in Case 1; thus, \( b, c \in \text{Int}(K) \) and \( u + b = v + c \). Therefore, \( c - u = b - v \in \text{Int}(K - u) \cap \text{Int}(K - v) \neq \emptyset \), so that also \( \text{Int}(T - u) \cap \text{Int}(T - v) \neq \emptyset \). But \( T - u \) and \( T - v \) are both tiles of \( \mathcal{T} \); hence, \( u = v \), which is clearly a contradiction. This completes Case 2.

**Case 3:** \( F \) is a notched face and \( G \) is a big face. Let \( \gamma, \delta \in \text{Cyc}(n) \) be such that \( T \) locally resembles \( \mathcal{T}^{\gamma} \) at \( F \), and \( T \) locally resembles \( \mathcal{T}^{\delta} \) at \( G \). We wish to show that \( \gamma = \delta \).

Suppose, to the contrary, that \( i \in [n] \) and \( \gamma(i) \neq \delta(i) \). Let \( j = \gamma(i) \) and \( t = \delta^{-1}(j) \), and let \( k \in [n] \) be such that \( \delta(k) = i \). Clearly, \( i, j, t \) are distinct and \( k \neq n - 1 \). Let \( u = h'_u \) and \( v = h'_v \). By Lemmas 8(1) and 9 and by Case 1 of this proof, \( T + u \) is a tile of \( \mathcal{T} \). By Lemmas 8(2) and 9 and by Case 2 of this proof, \( T - v \) is a tile of \( \mathcal{T} \).

It is easy to verify that the following hold.

\[
\begin{align*}
u_i &= a_i - 1; \\
u_j &= a_j - 1; \\
u_j &= a_j - 1; \\
u_i &= 0 \quad \text{if } r \neq i, j; \\
u_r &= 0.
\end{align*}
\]

Let \( b = (\varepsilon, \varepsilon, \ldots, \varepsilon) \) for sufficiently small \( \varepsilon > 0 \), and let \( c = u + v + b \). Then, \( c_i \in \{a_i + \varepsilon\} \) for each \( r \in [n] \), and \( c_i = \varepsilon \). Clearly, \( b, c \in \text{Int}(K) \). Therefore, \( b + u = c - v \in \text{Int}(K + u) \cap \text{Int}(K - v) \neq \emptyset \), so that also \( \text{Int}(T + u) \cap \text{Int}(T - v) \neq \emptyset \). But \( T + u \) and \( T - v \) are both tiles of \( \mathcal{T} \); hence, \( u = v \), which is clearly a contradiction. This completes Case 3. \( \square \)
We will say that a tile $T$ of $\mathcal{F}$ locally resembles $\mathcal{F}^Y$ if for all faces $F$ of $T$, $T$ locally resembles $\mathcal{F}^Y$ at $F$. Lemma 11 implies that for each tile $T$ there is a unique $y \in \text{Cyc}(n)$ such that $T$ locally resembles $\mathcal{F}^Y$.

The following proposition whose straightforward proof we leave to the reader will be useful in the proof of the lemma following it.

**Proposition 12.** If $i, r \in [n]$ and $j, k, s \in [n - 1]$ are such that $j = k + s$ and $r = \gamma^k(i)$, then $h_{ij}^r = h_{ik}^r + h_{is}^r$.

**Lemma 13.** Suppose that $T_1, T_2$ are adjacent tiles of $\mathcal{F}$ and that $T_1$ locally resembles $\mathcal{F}^Y$. Then $T_2$ locally resembles $\mathcal{F}^Y$.

**Proof.** Let $F$ be a face of $T_1$ at which $T_2$ is adjacent to $T_1$. If $F$ is a small face, then $T_2 = T_1 + a$ by Corollary 7. Again by Corollary 7, $T$ is adjacent to $T_2$ iff $T - a$ is adjacent to $T_1$. Thus $T_2$ and the tiles adjacent to it are obtained from $T_1$ and the tiles adjacent to it by translation by $a$. Consequently, then $T_2$ locally resembles $\mathcal{F}^Y$.

Now suppose $F$ is the $i$th notched face to $T_1$. By Lemmas 8(1) and 9, $T_2 = T_1 + h_{ij}^r$ for some $j \in [n - 1]$. Thus, $T_1 = T_2 - h_{ij}^r$, so $T_1$ is adjacent to $T_2$ at the $i$th big face of $T_2$. Let $G$ be the $i$th big face of $T_2$. It is clear, by Lemma 11, that it suffices to show that $T_2$ locally resembles $\mathcal{F}^Y$ at $G$. This will be accomplished by showing that for each $k \in [n - 1]$, $T_2 - h_{ik}^r$ is a tile of $\mathcal{F}$. If $k = j$, then $T_2 - h_{ik}^r = T_1 \in \mathcal{F}$. If $k \neq j$, then we will show that $T_2 - h_{ik}^r$ is adjacent to $T_1$. First suppose $k < j$, and let $s = j - k$ and $r = \gamma^k(i)$. By Proposition 12, $T_2 - h_{ik}^r = T_1 - h_{ij}^r$ is adjacent to $T_1$. Next, suppose $k > j$, and let $s = k - j$ and $r = \gamma^k(i)$. Again by Proposition 12, $T_2 - h_{ik}^r = T_1 + h_{ij}^r - h_{is}^r$, which is also adjacent to $T_1$.

Finally, if $F$ is the $i$th big face of $T_1$, then $T_1$ is adjacent to $T_2$ at the $i$th notched face, so the previous paragraph (with the roles of $T_1$ and $T_2$ interchanged) yields that $T_2$ locally resembles $\mathcal{F}^Y$. □

We can now complete the proof of the Theorem. Let $\mathcal{F}$ be a tiling. (Recall that $K$ is a tile of $\mathcal{F}$.) Let $y \in \text{Cyc}(n)$ be such that $K$ locally resembles $\mathcal{F}^Y$. Now let $\mathcal{F}_0 \subset \mathcal{F}$ consist of all those tiles $T$ of $\mathcal{F}$ such that $T \in \mathcal{F}^Y$ and $T$ locally resembles $\mathcal{F}^Y$. Clearly $K \in \mathcal{F}_0 \subseteq \mathcal{F}^Y$. If $T \in \mathcal{F}_0$, then all tiles in $\mathcal{F}$ adjacent to $T$ are also in $\mathcal{F}^Y$, and by Lemma 13 each of them locally resembles $\mathcal{F}^Y$. Since $\mathcal{F}^Y$ is connected (Lemma 6) it follows that $\mathcal{F}_0 = \mathcal{F}^Y$, so that $\mathcal{F}_0 = \mathcal{F}$. Therefore $\mathcal{F} = \mathcal{F}^Y$.

This completes the proof of the Theorem.

**Reference**