Schröder equation and commuting functions on the circle

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Abstract

We show that if \( F : S^1 \to S^1 \) is a homeomorphism of the unit circle \( S^1 \) and the rotation number \( \alpha(F) \) of \( F \) is irrational, then the Schröder equation

\[
\Phi(F(z)) = e^{2\pi i \alpha(F)} \Phi(z), \quad z \in S^1,
\]

has a unique (up to a multiplicative constant) continuous at a point of the limit set of \( F \) solution. We apply this result to prove that if \( F \) is a non-trivial continuous and disjoint iteration group or semigroup on \( S^1 \) and a continuous at least at one point function \( G : S^1 \to S^1 \) commutes with a suitable element of \( F \), then \( G \in F \).

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1. Introduction

Let \( X \) be either an open real interval or the unit circle \( S^1 \). Recall that a family of functions \( \{f^t : X \to X, \ t \in \mathbb{R}\} \) (respectively \( \{f^t : X \to X, \ t > 0\} \) such that for all \( s, t \in \mathbb{R} \) (respectively \( s, t > 0 \)),

\[
f^s \circ f^t = f^{s+t}
\]

is called an iteration group (respectively iteration semigroup). An iteration group/semigroup is said to be continuous (respectively measurable) if for every \( x \in X \) the mapping \( t \mapsto f^t(x) \) is continuous (respectively measurable).

Matkowski [8] proved that if \( X \) is an open interval and a continuous at least at one point function \( g : X \to X \) commutes with two mappings \( f^a, \ f^b \) belonging to a measurable iteration group \( F = \{f^t : X \to X, \ t \in \mathbb{R}\} \) such that \( f^1 \) is a strictly increasing surjection without fixed points, each \( f^t \) is continuous (every such an iteration group is in fact continuous) and \( \frac{b}{a} \) is irrational, then \( g \in F \). Ciepliński [6] showed that the same assertion holds under another assumptions on the iteration group \( F \).

Recall also that an iteration group/semigroup of homeomorphisms of the circle is called disjoint if every its element either is the identity mapping or has no fixed point (see Zdun [11] and Ciepliński [2–4] and [5]). A complete description of disjoint iteration groups on the circle was given in [4] and [5], whereas disjoint and continuous iteration semigroups on \( S^1 \) were studied in [11].

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The aim of this note is to discuss the case when $X = \mathbb{S}^1$ and $g$ commutes with only one suitable element of a non-trivial continuous and disjoint iteration group or semigroup. In order to do this we prove a result concerning solutions of the Schröder equation on the circle.

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

It is well known (see for instance Cornfeld et al. [7]) that for every homeomorphism $F: \mathbb{S}^1 \to \mathbb{S}^1$ there exists a homeomorphism $f: \mathbb{R} \to \mathbb{R}$, which is unique up to translation by an integer, and a unique $k \in \{-1, 1\}$ such that

$$F(e^{2\pi i x}) = e^{2\pi i f(x)} \quad \text{and} \quad f(x + 1) = f(x) + k, \quad x \in \mathbb{R}. $$

We say that $F: \mathbb{S}^1 \to \mathbb{S}^1$ preserves orientation if $f$ is increasing. For every such homeomorphism $F$ the number $\alpha(F) \in [0, 1)$ defined by

$$\alpha(F) := \lim_{n \to \infty} \frac{f^n(x)}{n} \pmod 1, \quad x \in \mathbb{R},$$

always exists and does not depend on $x$ and $f$. Furthermore, $\alpha(F) \in \mathbb{Q}$ if and only if $F$ has a periodic point. If $\alpha(F)$ is irrational, then the non-empty set $L_F$ (the limit set of $F$) of all cluster points of the set $\{F^n(z), \ n \in \mathbb{Z}\}$ does not depend on $z \in \mathbb{S}^1$ and either $L_F = \mathbb{S}^1$ or $L_F$ is a perfect nowhere dense subset of $\mathbb{S}^1$.

3. Main results

We start with a theorem which is an extension of the well-known result of Poincaré (see for instance Cornfeld et al. [7]) and Lemma 7 in Ciepliński [2].

**Theorem 1.** If $F: \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation-preserving homeomorphism without periodic points, then the Schröder equation

$$\Phi(F(z)) = e^{2\pi i \alpha(F)} \Phi(z), \quad z \in \mathbb{S}^1, \quad (1)$$

has a unique up to a multiplicative constant continuous at a point of $L_F$ solution $\Phi: \mathbb{S}^1 \to \mathbb{S}^1$. Moreover, this solution is continuous.

**Proof.** It is well known (see for instance Cornfeld et al. [7]) that there exists a continuous mapping $\Phi: \mathbb{S}^1 \to \mathbb{S}^1$ fulfilling (1). Now, fix $z_0 \in L_F$ and assume that $\Phi_1: \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous at $z_0$ solution of (1). Put $\Psi := \frac{\Phi}{\Phi_1}$ and observe that $\Psi: \mathbb{S}^1 \to \mathbb{S}^1$ is also continuous at $z_0$ and $\Psi(F(z)) = \Psi(z)$ for $z \in \mathbb{S}^1$. Consequently,

$$\Psi(F^n(z)) = \Psi(z), \quad n \in \mathbb{Z}, \quad z \in \mathbb{S}^1. \quad (2)$$

Fix a $z \in \mathbb{S}^1$ and let $(n_k)_{k \in \mathbb{N}}$ be a sequence of integers with $\lim_{k \to \infty} F^{n_k}(z) = z_0$. Then, using (2) and the continuity of $\Psi$ at $z_0$, we get

$$\Psi(z_0) = \lim_{k \to \infty} \Psi(F^{n_k}(z)) = \Psi(z).$$

Thus $\Psi$ is constant and $\Phi_1 = c \cdot \Phi$ for a $c \in \mathbb{S}^1$. \hfill \Box

**Remark 2.** One can reduce Eq. (1) to the reals, but we then obtain a system of two functional equations instead of one equation.

A verification of the proof of Theorem 1 shows that

**Proposition 3.** If $F: \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation-preserving homeomorphism without periodic points, then Eq. (1) has a unique up to a multiplicative constant continuous at a point of $L_F$ solution $\Phi: \mathbb{S}^1 \to \mathbb{C}$. Moreover, this solution is continuous.
An immediate consequence of Theorem 1 is the following corollary generalizing Theorem 1 in Bae et al. [1].

**Corollary 4.** The only continuous at least at one point mappings commuting with a rotation without periodic points are the rotations.

From Remark and Corollary 1 in Zdun [10] one can conclude

**Proposition 5.** Let \( \mathcal{F} = \{ F^t : \mathbb{S}^1 \to \mathbb{S}^1, \ t \in \mathbb{R} \} \) be a continuous iteration group. Then \( \mathcal{F} \) is disjoint if and only if each \( F^t \) is continuous and \( F^1 : \mathbb{S}^1 \to \mathbb{S}^1 \) is a homeomorphism such that either \( F^1 = \text{id}_{\mathbb{S}^1} \) or \( F^1 \) has no fixed point.

According to Theorem 3.3 in Zdun [11], Theorem 2 in Zdun [10] and Proposition 5 we have

**Corollary 6.** Let \( I \) be either \( \mathbb{R} \) or \( (0, +\infty) \). If \( \{ F^t : \mathbb{S}^1 \to \mathbb{S}^1, \ t \in I \} \) is a non-trivial continuous and disjoint iteration group or semigroup, then there exist a homeomorphism \( \Phi : \mathbb{S}^1 \to \mathbb{S}^1 \) and a \( c \in \mathbb{R} \setminus \{0\} \) such that

\[
F^t(z) = \Phi(e^{2\pi i c t} \Phi^{-1}(z)), \quad z \in \mathbb{S}^1, \ t \in I. \tag{3}
\]

A continuous iteration group \( \{ F^t : \mathbb{S}^1 \to \mathbb{S}^1,\ t \in \mathbb{R} \} \) of homeomorphisms is said to be *positively equicontinuous* if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for any \( z, w \in \mathbb{S}^1 \) and \( t \geq 0 \), \( d(z, w) < \delta \) implies \( d(F^t(z), F^t(w)) < \epsilon \), where \( d(z, w) := |x - y| \) for some \( x, y \in \mathbb{R} \) with \( |x - y| \leq \frac{1}{2} \) such that \( e^{2\pi i x} = z \) and \( e^{2\pi i y} = w \). Such iteration groups have been studied in Bae et al. [1].

Since (see Corollary 1 in Ciepliński [3]) every positively equicontinuous iteration group is disjoint, an immediate consequence of Corollary 6 is the following extension of Theorem 9 in Bae et al. [1].

**Corollary 7.** If \( \{ F^t : \mathbb{S}^1 \to \mathbb{S}^1, \ t \in \mathbb{R} \} \) is a non-trivial positively equicontinuous iteration group, then there exist a homeomorphism \( \Phi : \mathbb{S}^1 \to \mathbb{S}^1 \) and a \( c \in \mathbb{R} \setminus \{0\} \) satisfying (3) (with \( I = \mathbb{R} \)).

**Remark 8.** From Theorem 2 in Piekarska [9] it follows that every continuous iteration group such that

\[
0 < \inf\{ t > 0 : F^t(z_0) = z_0 \} < +\infty =: \inf \emptyset
\]

for a \( z_0 \in \mathbb{S}^1 \) is also disjoint.

We can now formulate the following:

**Theorem 9.** Let \( I \) be either \( \mathbb{R} \) or \( (0, +\infty) \). Assume also that \( \mathcal{F} = \{ F^t : \mathbb{S}^1 \to \mathbb{S}^1, \ t \in \mathbb{R} \} \) is a non-trivial continuous and disjoint iteration group or semigroup and a homeomorphism \( \Phi : \mathbb{S}^1 \to \mathbb{S}^1 \) and a \( c \in \mathbb{R} \setminus \{0\} \) are such that (3) holds true. If a continuous at least at one point function \( G : \mathbb{S}^1 \to \mathbb{S}^1 \) commutes with a mapping \( F^a \in \mathcal{F} \) for which \( ca \) is irrational, then \( G \in \mathcal{F} \).

**Proof.** As \( G \) commutes with \( F^a \), from (3) it follows that

\[
\Phi(e^{2\pi i c a} \Phi^{-1}(G(z))) = G(\Phi(e^{2\pi i c a} \Phi^{-1}(z))), \quad z \in \mathbb{S}^1.
\]

and consequently

\[
\Phi(e^{2\pi i c a} \Phi^{-1}(G(\Phi(z)))) = G(\Phi(e^{2\pi i c a} z)), \quad z \in \mathbb{S}^1.
\]

Hence, putting

\[
\psi := \Phi^{-1} \circ G \circ \Phi,
\]

we get

\[
\psi(e^{2\pi i c a} z) = e^{2\pi i c a} \psi(z), \quad z \in \mathbb{S}^1.
\]
Corollary 4 shows that there is an $\alpha_0 \in \mathbb{R}$ such that $\Psi(z) = e^{2\pi i \alpha_0 z}$ for $z \in \mathbb{S}^1$. (4) together with (3) now yields
\[ G(z) = \Phi(e^{2\pi i \alpha_0 \Phi^{-1}(z)}) = F^{\alpha_0}(z), \quad z \in \mathbb{S}^1, \]
which means that $G \in \mathcal{F}$. $\square$

We shall finally apply Theorem 9 to prove the following result corresponding to Theorem 3 in Matkowski [8] and Theorem 2 in Ciepliński [6].

**Theorem 10.** Let $I$ be either $\mathbb{R}$ or $(0, +\infty)$. Assume also that $\{F^t : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \ t \in I\}$ is a non-trivial continuous and disjoint iteration group or semigroup and a homeomorphism $\Phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and a $c \in \mathbb{R} \setminus \{0\}$ are such that (3) holds true. If $\mathcal{G} = \{G^t : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \ t > 0\}$ is a continuous iteration semigroup such that each $G^t$ is continuous at least at one point and
\[ F^t \circ G^t = G^t \circ F^t, \quad t > 0, \tag{5} \]
then there exists a function $C : (0, +\infty) \rightarrow I$ for which the Cauchy difference takes values in $\mathbb{Z}/c$, that is
\[ C(s + t) - C(s) - C(t) \in \mathbb{Z}/c, \quad s, t > 0, \tag{6} \]
and such that $G^t = F^{C(t)}$ for $t > 0$.

**Proof.** By (5) we have
\[ F^s \circ G^{w_0 s} = G^{w_0 s} \circ F^s, \quad s, w > 0, \ w \in \mathbb{Q}. \]
Fix $s, t > 0$ and let $(w_n)_{n \in \mathbb{N}}$ be a sequence of positive rationals such that $\lim_{n \rightarrow +\infty} w_n = \frac{t}{s}$. Then,
\[ F^s \circ G^{w_n s} = G^{w_n s} \circ F^s, \quad n \in \mathbb{N}, \]
so the continuity of $F^s$ and the iteration semigroup $\mathcal{G}$ gives $F^s \circ G^t = G^t \circ F^s$. Theorem 9 now shows that there is a $C(t) \in I$ for which $G^t = F^{C(t)}$. Therefore,
\[ F^{C(s+t)} = G^{s+t} = G^s \circ G^t = F^{C(s)+C(t)} \]
and (3) yields (6). $\square$

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**References**