Spectral analysis of family of singular non-self-adjoint differential operators of even order

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Abstract

This work examines the spectrum of a family of certain non-self-adjoint singular differential operators of even order on a whole axis. The coefficients of such operators depend on a complex spectral parameter in a polynomial manner. The scope of our work is also engaged in the construction of the resolvent and a multiple spectral expansion which is corresponding to such operators. This process is performed under the hypothesis that the coefficients of the differential expression are not infinitely small. The similar problems on a semi-axis and a whole axis were investigated in earlier papers [F.G. Maksudov, E.E. Pashayeva, About multiple expansion in terms of eigenfunctions for one-dimensional non-self-adjoint differential operator of even order on a semi-axis, in: Spectral Theory of Operators and Its Applications, vol. 3, Elm Press, Baku, 1980, pp. 34–101 (in Russian)] and [E.E. Pashayeva, About one multiple expansion in terms of solutions of differential equation on the whole axis, in: Spectral Theory of Operators and Its Applications, vol. 5, Elm Press, Baku, 1984, pp. 145–151 (in Russian)], respectively. However, in those papers, the coefficients of the differential expression were decreasing rapidly enough as $x$ was approaching to infinity.

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1. Introduction

This work performs the spectral analysis of the differential operator constructed by the following differential expression

$$l_\lambda(y) = L_0 y + L_1(\lambda)y$$

where

$$L_0 = \sum_{j=0}^{2n} q_j \frac{(-id)^{2n-j}}{dx^{2n-j}}$$

$q_0 \equiv 1$, $q_1 = 0$, coefficients $q_j$, $j = 2, \ldots, 2n$, are complex-valued. Moreover

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Let $D_m$ be the regions that are simply connected, bounded away from branch points and contain none of the \( \gamma_{jk} \) curves in their interiors, where

\[
\gamma_{jk} = \{ \lambda \mid \text{Im} \mu_j = \text{Im} \mu_k \}.
\]

Let $D_{L(\lambda),m}$ denote the set of all functions $y(x, \lambda)$ square integrable with respect to $x \in (-\infty, \infty)$, $\lambda \in D_m$, $m = 1, \ldots, 2n$, and

\[
L_1(\lambda) = \sum_{j=2}^{2n} \int \frac{p_j(x, \lambda)}{dx^{2n-j}} (-id)^{2n-j},
\]

\[
P_j(\lambda) = \lambda^{j-1} p_j(1) + \cdots + p_{jj}(x),
\]

where $p_{kj}(x)$, $j \leq k$, are complex-valued functions.

We suppose that

\[
(\lambda^2 + 1)^r p_{jk}(x) \in L_2(-\infty, \infty)
\]

for chosen $r$, where $r$ is the highest multiplicity of the root of $p'(\mu) = 0$ and

\[
p(\mu) = \mu^{2n} + q_1 \mu^{2n-1} + q_2 \mu^{2n-2} + \cdots + q_{2n-1} \mu + q_{2n}.
\]

With the differential expression (1.1) we associate an operator $L(\lambda)$ operating on the Hilbert space $L_2(\infty, \infty)$. We define the operator $L(\lambda)$ by the formula $L(\lambda)y = l(\lambda)$ on functions $y \in L_2(-\infty, \infty)$ which have derivatives $y^{(v)}$, $v = 1, \ldots, 2n - 1$, absolutely continuous on every interval $[b, b]$, $b > 0$ and which are such that $l(\lambda) = L_2(-\infty, \infty)$. Since we do not assume that $\text{Imm} k(x) = 0$, the differential expression (1.1) and the operator $L(\lambda)$ are not self-adjoint.

This work examines the solutions of the equation $l(\lambda) + \lambda^{2n} = 0$ and their asymptotic behaviors as $x \rightarrow \pm \infty$ and $|\lambda| \rightarrow \infty$. We also investigate the spectrum, then construct the resolvent and a multiple spectral expansion which is corresponding to the operator $L(\lambda)$. This process is performed under the hypothesis that the coefficients of the differential expression are not infinitely small. As a result of our analysis the multiple expansion of arbitrary test functions was obtained. We got the expansion in terms of the sum of eigenfunctions and the integral that involves some solutions of the corresponding differential equation.

This work generalizes some previously established results. The similar problems on a semi-axis and a whole axis were investigated in earlier papers [4] and [6], respectively. However, in those papers, the coefficients of the differential expression were decreasing rapidly enough as $x$ was approaching to infinity.

R.R.D. Kemp [2] has examined a problem for similar operator where the coefficients were not infinitely small, but they did not depend on a complex parameter. He analyzed the spectrum of a corresponding operator and obtained an expansion in characteristic functions for suitably restricted class of functions. We examine a more general spectral problem. In our case the coefficients of the differential operator are not infinitely small, but depend on complex spectral parameter. We also obtain a suitable multiple expansion of arbitrary test functions.

2. Solutions of the equation $l(\lambda) + \lambda^{2n} = f$

Consider the case when the equation

\[
p(\mu) = \lambda^{2n}
\]

has no the real values.

Note that for real $t$ the equation $p(t) = \lambda^{2n}$ defines a curve in the complex $\lambda$-plane. This equation will in general split the complex $\lambda$-plane up into the regions $D_m$, $m = 1, \ldots, 2n$.

For large $|\mu|$, $\mu_1, i = 1, \ldots, 2n$, can be renumbered so that

\[
\mu_j = \alpha_j \lambda \left(1 + O(\frac{|\lambda|^{-1}})\right), \quad 0 \leq \arg \lambda \leq \frac{\pi}{n}, \quad m = 1, 2, \ldots, 2n,
\]

where $\alpha_j = \exp \frac{2ij}{n}$ are the roots of unity, and

\[
\text{Im} \mu_1 \geq \text{Im} \mu_2 \geq \cdots \geq \text{Im} \mu_{2n}.
\]

Let $D_m$ be the regions that are simply connected, bounded away from branch points and contain none of the $\gamma_{jk}$ curves in their interiors, where

\[
\gamma_{jk} = \{ \lambda \mid \text{Im} \mu_j = \text{Im} \mu_k \}.
\]

Let $D_{L(\lambda),m}$ denote the set of all functions $y(x, \lambda)$ square integrable with respect to $x \in (-\infty, \infty)$, $\lambda \in D_m$, $m = 1, \ldots, 2n$, and
We define the operator $L(\lambda)$ as follows:

for each $\lambda \in D_m$ its domain is $D_{L(\lambda),m}$ and $L(\lambda)y = l_x(y)$.

Consider the following differential equation

$$l_x(y) + \lambda^{2n}y = f.$$  \hfill (2.2)

It is equivalent to the system of $n$ equations of the first order

$$Y' = [Q + P]Y + F,$$  \hfill (2.3)

where

$$Q = (q_{jk}) = \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nn} \end{pmatrix}, \quad \text{here } q_1 = 0,$$

$$p_{jk} = -\delta_{jk} - \delta_{jk}i^{2n-k+1}q_{jn-k+1} - \delta_{jk}i^{2n-k+1}q_{nj-k+1}.$$  \hfill (2.4)

In order to obtain the Green’s function for (2.2) we (see [2]) first construct certain solutions of (2.3) for $F = 0$, and use them to construct the Green’s matrix for (2.3).

If $\lambda$ is such that $p(\mu) = \lambda^{2n}$ has $2n$ distinct solutions $\mu_1, \mu_2, \ldots, \mu_{2n}$ then $Z' = QZ$ has a fundamental matrix $M \exp[i\theta x]$ where $i\theta = [i\mu_j \delta_{jk}]$ and $M = [i(\mu_j)^{k-1}]$.

Using this we obtain an integral equation equivalent to (2.3) with $F = 0$ in the form

$$Y(x, \lambda) = M \exp[i\theta x]C_0 + \int_0^x M \exp[i\theta(x - \xi)]M^{-1}P(\xi, \lambda)Y(\xi, \lambda)\,d\xi,$$  \hfill (2.5)

where $C_0$ is a constant $2n \times 1$ matrix and the lower limits on the integrals (each element in the column matrix) are arbitrary.

If $\lambda_1$ is such that $p(\mu) = \lambda_1^{2n}$ has $2n$ distinct solutions then the same is true for $\lambda$ sufficiently close to $\lambda_1$ and the $2n$ solutions $\mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_{2n}(\lambda)$ of $p(\mu) = \lambda^{2n}$ are analytic functions of $\lambda$ in this neighborhood of $\lambda_1$. This functions have branch points at $\lambda_{j0}^{2n} = p(\mu_{j0})$, $j = 1, 2, \ldots, 2n - 1$, where $\mu_{j0}, j = 1, 2, \ldots, 2n - 1$, are the solutions of $p'(\mu) = 0$.

Thus in any simply connected region containing no $\lambda_{j0}$ the functions $\mu_k(\lambda)$ are analytic.

**Theorem 2.1.** Let (1.4) holds. There are linearly independent solutions $y_1, y_2, \ldots, y_{2n}$ and $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2n}$ of (2.2) which exist for all bounded

$\lambda \in D_m, \quad m = 1, \ldots, 2n, \ x \in (-\infty, \infty).$
The matrices

\[ Y(x, \lambda) = [y_1, y_2, \ldots, y_{2n}] \quad \text{and} \quad \tilde{Y}(x, \lambda) = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2n}] \]

are continuous in both \((x, \lambda), x \in (-\infty, \infty), \lambda \in \tilde{D}_m, m = 1, \ldots, 2n,\) are analytic in \(\lambda \in \bar{D}_m, m = 1, \ldots, 2n,\) for fixed \(x \in (-\infty, \infty)\) and have the following asymptotic behavior

\[ Y(x, \lambda) = M \exp[i\theta x](I + o(1)), \quad x \to \infty, \]

\[ \tilde{Y}(x, \lambda) = M \exp[i\theta x](I + o(1)), \quad x \to -\infty \]

uniformly in \(\lambda \in \tilde{D}_m, m = 1, \ldots, 2n,\) where \(I = [\delta_{kj}]\) is the identity matrix.

Theorem 2.1 is a modification of Theorem 8.1 in [1, p. 92], and its proof will be omitted.

Now we will find the asymptotic behavior of the solutions \(Y(x, \lambda), \tilde{Y}(x, \lambda)\) as \(|\lambda| \to \infty\).

Consider Eq. (2.2) for \(f = 0,\)

\[ l\lambda(y) + \lambda^{2n}y = 0. \quad (2.5) \]

It is equivalent to the system

\[ \frac{dU}{dx} = \lambda AU + A_0(x)U + \frac{1}{\lambda} R(x, \lambda)U, \quad -\infty < x < \infty, \quad |\lambda| \geq r > 0, \quad (2.6) \]

where \(A_0(x), A'_0(x)\) and \(R(x, \lambda)\) are absolutely integrable.

Obviously, the \(A\) matrix does not have multiple eigenvalues and each pair of eigenvalues \(w_i\) and \(w_j (\text{Im } w_i - \text{Im } w_j)\) does not change the sign.

So, the system (2.6) possesses the same form as in [3, p. 48].

Arguing similarly as in [4, pp. 45–48], we obtain the asymptotic behavior of our solutions provided \(|\lambda| \to \infty\) while \(\lambda \in D_m, m = 1, \ldots, 2n,\)

\[ Y(x, \lambda) = M \exp[i\theta x] \exp \left[ -\frac{1}{\Delta} \int_{-\lambda}^{\lambda} \beta(t) \, dt \right] \left[ 1 + O(|\lambda|^{-1}) \right] \quad (2.7) \]

and

\[ \tilde{Y}(x, \lambda) = M \exp[i\theta x] \exp \left[ -\frac{1}{\Delta} \int_{-\lambda}^{\lambda} \beta(t) \, dt \right] \left[ 1 + O(|\lambda|^{-1}) \right] \]

uniformly in \(x\) for \(x \in (-\infty, \infty)\) and \(|\lambda| \to \infty.\) Here

\[ \beta = [\beta_k \delta_{ki}], \quad \beta_k(x) = \frac{1}{\Delta} \left( -i^{2n} p_{2n,1}(x) \mu_k - i^{2n-1} p_{2n-1,1}(x) \mu_k^2 - \cdots - i^2 p_{2,1}(x) \mu_k^{2n-1} \right), \quad (2.8) \]

and \(\Delta\) is a determinant of matrix \(M.\)

It is not hard to see from (2.1) that such regions \(D_m, m = 1, \ldots, 2n,\) do exist, as well as \(|\lambda| \to \infty, \Delta \) cannot be identically equal to zero.
Finally we shall also need solutions of (2.3) when \( \lambda \) is a branch point of the functions \( \mu_j(\lambda) \). We see [2, p. 643] that at such points \( \mu_j \)'s coincide in groups and the solutions of \( p(\mu) = \lambda^{2n} \) will be denoted by \( \mu_1, \mu_2, \ldots, \mu_r \) with multiplicities \( m_1, m_2, \ldots, m_r \).

\[
\left( \sum_{j=1}^r \mu_j = 2n \right).
\]

A fundamental matrix \( Z' = QZ \) will have the more complicated form \( M_1 A \exp[iQ_1x] \), where \( Q_1 \) is a diagonal matrix with \( m_1 \)'s, then \( m_2 \)'s etc. down the diagonal. Thus the equation corresponding to (2.4) is

\[
Y(x, \lambda) = M_1 A(x) \exp[i\theta_1 x] C_0 + \int M_1 A(x - \xi) \exp[i\theta_1 (x - \xi)] M_1^{-1} P(\xi, \lambda) Y(\xi, \lambda) d\xi.
\]

(2.9)

**Theorem 2.2.** There exist solutions \( y_{kj}(x, \lambda) \) and \( \tilde{y}_{kj}(x, \lambda) \) of (2.9) for \( j = 1, \ldots, r, \ k = 1, \ldots, m_j \) and uniformly in \( \lambda \in \bar{D}_m, m = 1, \ldots, 2n \), such that

\[
y_{kj}(x, \lambda) = \frac{x^{k-1}}{(k-1)!} \exp(i \mu_j x)(z_{1j} + o(1)), \quad \text{as } x \to \infty.
\]

(2.10)

The solution \( \tilde{y}_{kj}(x, \lambda) \) has the same asymptotic behavior as \( x \to -\infty \) uniformly in \( \lambda \in \bar{D}_m, m = 1, \ldots, 2n \).

Also as \( |\lambda| \to \infty \)

\[
y_{kj}(x, \lambda) = \frac{x^{k-1}}{(k-1)!} \exp[i \mu_i x] \exp[-\frac{1}{\Delta} \int_{-x}^{x} \beta_i(t) dt] [z_{1j} + O(|\lambda|^{-1})]
\]

uniformly in \( x \in (-\infty, \infty), \lambda \in \bar{D}_m, m = 1, \ldots, 2n \).

The proof of this theorem is a modification of that of Theorem 2.1. and of the Problem 35 [1, p. 106], and will be omitted.

### 3. Construction of the Green’s function

We shall now discuss the solution of (2.3) for \( F = 0 \) when the equation \( p(\mu) = \lambda^{2n} \) has no real solutions.

The results of this section are based on [2, pp. 644–648] and [4, pp. 66–79].

As \( p(\mu) = \lambda^{2n} \) has no real solutions we can number the solutions \( \mu_1, \mu_2, \ldots, \mu_{2n} \) so that

\[
\text{Im} \mu_1 \geq \text{Im} \mu_2 \geq \cdots \geq \text{Im} \mu_m > 0 > \text{Im} \mu_{m+1} \geq \cdots \geq \text{Im} \mu_{2n}.
\]

Thus \( y_1, y_2, \ldots, y_m \) are exponentially small at \( \infty \) and \( \tilde{y}_{m+1}, \ldots, \tilde{y}_{2n} \) are exponentially small at \( -\infty \). We shall partition our matrices as follows:

\[
Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{21} & \tilde{Y}_{22} \end{pmatrix},
\]

where \( Y_{11} \) and \( \tilde{Y}_{11} \) are \( m \times m \) matrices.

We now define

\[
Z = \begin{pmatrix} Y_{11} & \tilde{Y}_{12} \\ Y_{21} & \tilde{Y}_{22} \end{pmatrix}
\]

and provided that \( Z^{-1} \) exists

\[
K(x, \xi, \lambda) = \begin{cases} 
\begin{pmatrix} Y_{11}(x, \lambda) & 0 \\ Y_{21}(x, \lambda) & 0 \end{pmatrix} Z^{-1}(\xi, \lambda), & \xi \leq x, \\
\begin{pmatrix} 0 & \tilde{Y}_{12}(x, \lambda) \\ 0 & \tilde{Y}_{22}(x, \lambda) \end{pmatrix} Z^{-1}(\xi, \lambda), & \xi > x.
\end{cases}
\]

(3.1)
Theorem 3.1. If \( Z \) is non-singular for a particular value of \( \lambda, \lambda \in \bar{D}_m, m = 1, \ldots, 2n \), then \( K(x, \xi, \lambda) \) is the Green’s matrix for (2.3).

Proof. By this we mean, that if \( F \) is a vector function with
\[
\|F\|_p = \left\{ \int_{-\infty}^{\infty} |F(x)|^p \, dx \right\}^{1/p} < \infty,
\]
we can prove that the vector-function
\[
y(x, \lambda) = \int_{-\infty}^{\infty} K(x, \xi, \lambda) F(\xi) \, d\xi
\]
(3.2)
is the unique solution of (2.3) which belongs to \( L^p(-\infty, \infty) \). Also we can show that if \( x \leq 0 \leq \xi \) as well as if \( x \geq 0 \geq \xi \), then
\[
|K(x, \xi, \lambda)| \leq K \exp[-\delta|x-\xi'|].
\]
where \( K \) is a positive constant and \( \delta < \min[|\text{Im}\mu_m|, |\text{Im}\mu_{m+1}|] \).

Thus, from (3.2) by Hölder’s inequality for each \( \lambda \in \bar{D}_m, m = 1, \ldots, 2n \),
\[
|y(x, \lambda)| \leq K \int_{-\infty}^{\infty} e^{-\delta|x-\xi'|} |F(\xi)| \, d\xi
\]
\[
\leq K \left\{ \int_{-\infty}^{\infty} e^{-p\delta|x-\xi'|/2} |F(\xi)|^p \, d\xi \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} e^{-q|x-\xi'|/2} \, d\xi \right\}^{1/q}.
\]
Thus, using a change of variables and the Fubini theorem, we obtain
\[
\int_{-\infty}^{\infty} |y(x, \lambda)|^p \, dx \leq K^p (4/q\delta)^{p/q} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-p\delta|x-\xi'|/2} |F(x-\xi)|^p \, d\xi \, dx
\]
\[
\leq K^p (4/q\delta)^{p/q} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x)|^p \, dx \int_{-\infty}^{\infty} e^{-p\delta|x-\xi'|/2} \, d\xi \leq K_1 \|F\|_p^p
\]
for each \( \lambda \in \bar{D}_m, m = 1, \ldots, 2n \).

Thus \( y(x, \lambda) \) exists and belongs to \( L^p(-\infty, \infty) \) for all \( p \geq 1 \), and for each \( \lambda \in \bar{D}_m, m = 1, \ldots, 2n \). \( \square \)

Corollary 3.1. The Green’s function for the differential operator \( L(\lambda) \) is \( G(x, \xi, \lambda) = i^{2n} K_{1,2n}(x, \xi, \lambda) \) where \( K_{1,2n}(x, \xi, \lambda) \) is the element in the first row and 2nth column of \( K(x, \xi, \lambda) \), provided that \( K(x, \xi, \lambda) \) exists.

Suppose \( \lambda_0 \) is in one of the regions \( D_m \) and does not lie on any \( \gamma_{jk} \), then in a neighborhood of \( \lambda_0 \) no \( \text{Im}\mu_i \) changes sign. Thus in this neighborhood \( m \) is fixed and \( K(x, \xi, \lambda) \) is analytic in \( \lambda \). In crossing a \( \gamma_{jk} \) while remaining in \( D_j \) the \( y_j \)’s and \( \tilde{y}_j \)’s may change, but in the first part we saw if on one side \( \text{Im}\mu_1 \geq \text{Im}\mu_2 \geq \cdots \geq \text{Im}\mu_{2n} \) then on \( \gamma_{jk} \) the \( Y \) matrix from that side, \( Y_1 \) will have the asymptotic form
\[
Y_1 = M \exp[i\theta x] A_1 [I + o(1)] \quad \text{as} \quad x \to \infty,
\]
where \( A_1 \) has units along the main diagonal and zeros below it. From the other side the only difference will be that the order of the \( \mu_j \)’s has been altered. Thus
\[
Y_2 = M \exp[i\theta x] Q A_2 [I + o(1)] \quad \text{as} \quad x \to \infty,
\]
where \( A_2 \) is of the same form as \( A_1 \) and \( Q \) rearranging the columns of \( M \exp[i\theta x] \) appropriately. Hence the \( Y_1 = Y_2 A_2^{-1} Q^{-1} A_1 \). Similarly \( \tilde{Y}_1 = \tilde{Y}_2 B_2^{-1} Q^{-1} B_1 \) where \( B_1 \) and \( B_2 \) have units along the main diagonal and zeros above.
Note that $A_{2}^{-1}$ and $B_{2}^{-1}$ have the same form as $A_{2}$ and $B_{2}$ respectively and that it is impossible to have $\gamma_{jk}$ here where $j < m < k$ or $k < m < j$. This implies that $Q = [Q_{ij}]$, partitioned as before, we have $Q_{12} = Q_{21} = 0$. Using this and the definition of $Z$ we see that

$$Z_1 = Z_2 \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix},$$

where $P_{11}$ and $P_{22}$ have determinant $\pm 1$ (as $A_j$, $B_j$ and $Q$ have this property).

These remarks can be summarized in the following theorem.

**Theorem 3.2.** If the branch points are removed from the regions $D_j$, $j = 1, \ldots, 2n$, the matrix $K(x, \xi, \lambda)$ is analytic in the remaining portion except the points where $Z$ is singular.

With the help of the auxiliary boundary problem

$$l_{\xi}(y) + \lambda^{2n}y = 0, \quad i = 1, \ldots, n; \quad 0 < b < +\infty,$$

$$\begin{align*}
U_i(y) &= y(i-1)(-b) = 0, \\
U_{n+i}(y) &= y(i-1)(+b) = 0,
\end{align*}$$

(3.4) taking (2.1) into account, it can be shown [4, pp. 70–79], that

$$|K(x, \xi, \lambda)| \leq \frac{C}{|\lambda|^{2n-1}}, \quad C = \text{const},$$

(3.6) uniformly in each bounded square $-a \leq x, \xi \leq a$, $a > 0$ as $\lambda$ belongs to the resolvent set of $L(\lambda)$ and when $|\lambda|$ is sufficiently large.

Now we define a function

$$W_j(\lambda) = \pm (\det Z)$$

(3.7) by fixing the sign at some point and then choosing the coherent signs on the two sides of each $\gamma_{jk}$ we obtain a function locally analytic in $D_m$ except at branch points, which may be double-valued.

In a neighborhood of some arc extending to $\infty$ we may use (2.7) to conclude that in this neighborhood

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(3.7) by fixing the sign at some point and then choosing the coherent signs on the two sides of each $\gamma_{jk}$ we obtain a function locally analytic in $D_m$ except at branch points, which may be double-valued.
We rewrite the operator $L(\lambda)$ on $L_2(-\infty, \infty)$ which we associate with the following differential expression

$$
I_\lambda^*(z) = (i)^{2n}z(2n) + (i)^{2n-2}(\bar{P}_2(x, \lambda) + \bar{q}_2)z(2n-2) + (i)^{2n-3}(\bar{P}_3(x, \lambda) + \bar{q}_3)z(2n-3) + \cdots + (\bar{P}_{2n}(x, \lambda) + \bar{q}_{2n} + \lambda^{2n})z).
$$

We define the operator $L^*(\lambda)$ by the formula $L^*(\lambda)f = I_\lambda^*(f)$ on functions $f \in L_2(-\infty, \infty)$ whose derivatives $f^{(v)}, v = 1, \ldots, 2n - 1,$ exist and are absolutely continuous in $x$ on every interval $[-b, b], 0 < b < \infty$ for $\lambda \in D_m, m = 1, \ldots, 2n,$ and are such that $I_\lambda^*(z) \in L_2(-\infty, \infty)$.

We shall assume that $ho(L) = \sum_{\mu \in \rho(L)} |p(\mu)|^2$ solutions neither has $p(\mu)$ by

$$
\rho(L) = \sum_{\mu \in \rho(L)} |p(\mu)|^2
$$

by

$$
\rho(L) = \sum_{\mu \in \rho(L)} |p(\mu)|^2
$$

between two $D_j$’s in which the $p(\mu)$.

Conversely we see that if $p(\mu) = \lambda^{2n}$ then $\lambda \in \rho(L)$ then $\lambda \in \rho(L^*)$ then $\lambda \in \rho(L)$ and $\lambda \in \rho(L^*)$ then $\lambda \in \rho(L).$ In the case as $p(\mu) = \lambda^{2n}$ has no real solutions neither has $p(\mu) = \lambda^{2n}$ so $\lambda \in \rho(L)$ implies that $\lambda \in \rho(L^*)$.

Conversely we see that if $p(\mu) = \lambda^{2n}$ has no real solutions and $\lambda \in \rho(L)$ then $\lambda \in \rho(L^*)$. Thus, from Theorem 4.2 applied to the $L^*(\lambda)$ we can find that the curve $\lambda^{2n} = p(t)$ splits up into $\rho(L^*)$, $\sigma(L^*)$, and $C\sigma(L^*)$ and is contained in $\sigma(L^*)$.

This proves

**Corollary 4.1.** If $\lambda$ does not lie on $\Omega_m^*(\lambda)$ then $\lambda \in \rho(L^*)$ or $\rho(L^*)$ according as $\lambda \in \rho(L^*)$ or $\rho(L)$. The curves $\Omega_m^*(\lambda)$ split up into $\rho(L^*)$, $\sigma(L^*)$, and $C\sigma(L^*)$ and are contained in $\sigma(L^*)$.

5. The spectral expansion corresponding to the operator $L(\lambda)$

We shall obtain the $(2n)$-fold expansion of arbitrary test functions in terms of the sum of eigenfunctions and the integral involving some solutions of the corresponding differential equation in the space $L_2(-\infty, \infty)$.

We shall assume that $\rho(L)$ and $\rho(L^*)$ are finite and do not intersect $\Omega_m, m = 1, \ldots, 2n,$ and $\Omega_m^*, m = 1, \ldots, 2n,$ respectively.

We shall also assume that on $\Omega_m, m = 1, \ldots, 2n,$ and $\Omega_m^*, m = 1, \ldots, 2n,$ the functions $W_j(\lambda)$ and $W_j^*(\lambda)$ which are defined, are not zero.

It is possible that the method may be generalized to deal with the less special case where the only additional assumption is that $\rho(L)$ and $\rho(L^*)$ are finite (see [5]).

Let the functions $p_{k,j}(x)$ satisfy the condition

$$
|p_{k,j}(x)| \leq Ce^{-\epsilon|x|}, \quad j \leq k, \quad C = \text{const.}
$$

We rewrite the operator $L(\lambda)$ in the following form:

$$
L(\lambda) = \lambda^{2n} + \sum_{i=1}^{2n} A_i \lambda^{2n-i},
$$

where

$$
A_j = p_{2n,j}(x) + (-i)p_{2n-1,j-1}(x) \frac{d}{dx} + \cdots + (-i)^{j-1} p_{2n-j+1,1}(x) \frac{d^{j-1}}{dx^{j-1}},
$$

(5.3)
\[ A_{2n} = (p_{2n,2n}(x) + q_{2n}) + (-i) \left[ p_{2n-1,2n-1}(x) + q_{2n-1} \right] \frac{d}{dx} + \cdots \]
\[ + (-i)^{2n-2} \left[ p_{2,2}(x) + q_2 \right] \frac{d^{2n-2}}{dx^{2n-2}} + (-i)^{2n} \frac{d^{2n}}{dx^{2n}}. \]

We define the resolvent \( R_\lambda \) of the operator \( L(\lambda) \) to be
\[ L(\lambda) R_\lambda = E, \tag{5.4} \]
here \( E \) is the identity operator in \( L_2(-\infty, \infty) \).

Or, substituting the expression of (5.2) in (5.4) several times, we arrive at the following expression for the resolvent \( R_\lambda \):
\[ R_\lambda = \sum_{j=1}^{k} (-1)^{j-1} \frac{E}{\lambda^{2nj}} \left( \sum_{i=1}^{2n} A_i \lambda^{2n-i} \right)^{j-1} + (-1)^{k} \frac{1}{\lambda^{2nk}} \sum_{i=1}^{2n} (A_i \lambda^{2n-i})^{k} R_\lambda \tag{5.5} \]
for \( k = 1, \ldots, 2n \).

Let \( f_k, k = 0, 2n - 1 \), are arbitrary differentiable test functions. We shall consider the following auxiliary Cauchy problem
\[ L \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) u(x, t) = 0, \]
\[ u^{(j)}(x, 0) = f_j, \quad j = 0, \ldots, 2n - 1. \tag{5.6} \]

Here
\[ L \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) u(x, t) = (-i)^{2n} u^{(2n)}(x, t) + (-i)^{2n-2} \left[ \left( p_{2,1}(x) \frac{\partial}{\partial t} + p_{2,2}(x) + q_2 \right) u^{(2n-2)}(x, t) \right] + \cdots \]
\[ + (-i)^{2n-3} \left[ \left( p_{3,1}(x) \frac{\partial^2}{\partial t^2} + p_{3,2}(x) \frac{\partial}{\partial t} + p_{3,3}(x) \right) u^{(2n-3)}(x, t) \right] + \cdots \]
\[ + \left[ p_{2n,1}(x) \frac{\partial^{2n-1}}{\partial t^{2n-1}} + p_{2n,2}(x) \frac{\partial^{2n-2}}{\partial t^{2n-2}} + \cdots + p_{2n,2n-1}(x) \frac{\partial}{\partial t} + p_{2n,2n}(x) + q_{2n} \right] u(x, t) \]
\[ + \frac{\partial^{2n}}{\partial x^{2n}} u(x, t). \]

Denoting by
\[ y(x, \lambda) = \int_{0}^{\infty} e^{-\lambda t} u(x, t) \, dt \]
and performing the Laplace transformation in (5.6), we obtain
\[ (-i)^{2n} y^{(2n)} + (-i)^{2n-2} \left[ P_2(x, \lambda) + q_2 \right] y^{(2n-2)} + \cdots + (-i)^{2n} \left[ P_{2n-1}(x, \lambda) + q_{2n-1} \right] y' \]
\[ + \left[ P_{2n}(x, \lambda) + q_{2n} + \lambda^{2n} \right] y = \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i, \tag{5.7} \]
here
\[ \phi_i = \sum_{j=1}^{i-1} A_{i-j} f_{j-1} + f_{i-1} \tag{5.8} \]
and \( A_i \) are defined by (5.3).

By the definition of \( R_\lambda \) the solution of (5.7) can be written in the form
\[ y(x, \lambda) = R_\lambda \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right). \tag{5.9} \]
Using the familiar relationship
\[ \int_{\Gamma_R} \lambda^n \, d\lambda = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1, \end{cases} \]
where \( \Gamma_R \) is a circumference of \( R \) radius centered at the origin, it is easily seen that
\[ \left| \int_{\Gamma_{R, \varepsilon}} \lambda^n \, d\lambda \right| \leq \begin{cases} \varepsilon R^{n+1}, & n \neq -1, \\ 2\pi - \varepsilon, & n = -1, \end{cases} \quad (5.10) \]
where \( \Gamma_{R, \varepsilon} = \Gamma_R \setminus \{\text{arcs of a total length } \varepsilon\} \).

We shall consider the contour integral
\[ I(N, \varepsilon) = \frac{1}{2\pi i} \int_{\Gamma_{N, \varepsilon}} y(x, \lambda) \, d\lambda, \quad (5.11) \]
where \( \Gamma_{N, \varepsilon} \) is a compound contour which we shall now describe.

Let \( H_{\varepsilon} = \{ \lambda \mid \text{which are within a distance not more than } \varepsilon \text{ of } \Omega_m, \ m = 1, \ldots, 2n \} \)
and let \( \tilde{\Gamma}_{N, \varepsilon} \) be the contour of
\[ H_{N, \varepsilon} = \{ \lambda \mid |\lambda| \leq N, \ 	ext{and } \lambda \notin H_{\varepsilon} \} \]
positively oriented.

Let \( \lambda_1, \lambda_2, \ldots, \lambda_p \) be the simple eigenvalues of the operator \( L(\lambda) \), \( y_1(x), y_2(x), \ldots, y_p(x) \) the corresponding eigenfunctions, and \( z_1(x), z_2(x), \ldots, z_p(x) \) be the corresponding eigenfunctions of the related adjoint problem which is defined in the usual manner. We assume \( \varepsilon \) is so small and \( N \) is so large that \( P \sigma(L) \subset H_{N, \varepsilon} \) and so that for each \( D_j, j = 1, \ldots, 2n, D_j \cap H_{N, \varepsilon} \) is non-empty.

Let \( \tilde{\Gamma}_{N, \varepsilon} \) be the portion of \( \tilde{\Gamma}_{N, \varepsilon} \) belonging to \( \Gamma_N \).

The evaluation of \( I(N, \varepsilon) \) will be performed in two ways: by residues, and by direct integration for the case \( \varepsilon \to 0, N \to \infty \).

We shall first proceed with the direct technique as \( \varepsilon \to 0 \) and \( N \to \infty \). Taking (5.5), (5.9), and (5.10) into account we are led to the tentative result
\[ I(N, \varepsilon) = \frac{1}{2\pi i} \int_{\Gamma_{N, \varepsilon}} y(x, \lambda) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_{N, \varepsilon}} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \left( \sum_{j=1}^{i-1} A_{i-j} f_{j-1} + f_{i-1} \right) \right) R_{\lambda} \]
\[
\times \left\{ \int_0^\infty K(x, \xi, \lambda) \left[ \sum_{i=1}^{2n} \lambda^{2n-i-i} \left( \sum_{j=1}^{i-1} A_{i-j} f_{j-1} + f_{i-1} \right) \right] d\xi \right\} d\lambda
\]
\[= f_0 + o(\varepsilon) + \frac{1}{2\pi i} \int_{I_{N,\varepsilon}} M(x, \lambda) \left( \sum_{i=1}^{2n} A_i \lambda^{2n-i} \right)^2 d\lambda,
\]
where
\[
M(x, \lambda) = \int_{-\infty}^\infty K(x, \xi, \lambda) \left[ \sum_{i=1}^{2n} \lambda^{2n-i} \left( \sum_{j=1}^{i-1} A_{i-j} f_{j-1} + f_{i-1} \right) \right] d\xi.
\]
Taking the estimation (3.6) into account, we obtain that
\[
\int_{I_{N,\varepsilon}} M(x, \lambda) \lambda^4 \left( \sum_{i=1}^{2n} A_i \lambda^{2n-i} \right)^2 d\lambda = O\left( \frac{1}{|\lambda|^2} \right).
\]
So, for \(\lambda \in D_j, j = 1, \ldots, 2n\), and for \(|\lambda|\) sufficiently large the formula takes the form
\[
\frac{1}{2\pi i} \int_{I_{N,\varepsilon}} y(x, \lambda) d\lambda = f_0 + o(\varepsilon) + O\left( \frac{1}{|\lambda|^2} \right). \tag{5.12}
\]
Obviously, as \(|\lambda| \to \infty\) the right side approaches \(f_0\).
Analogously it is easily seen that
\[
\frac{1}{2\pi i} \int_{I_{N,\varepsilon}} \lambda^k y(x, \lambda) d\lambda = f_k + o(\varepsilon) + O\left( \frac{1}{|\lambda|^2} \right) \tag{5.13}
\]
for \(\lambda \in D_j, j = 1, \ldots, 2n\), and for \(|\lambda|\) sufficiently large.
Now we shall proceed to the evaluation of \(I(N, \varepsilon)\) by residue technique. It is clear that crossing \(D_j\) we will get the leap of a resolvent \(R_{\lambda}\) of \(L(\lambda)\). The direct computation with the integrals yields
\[
\frac{1}{2\pi i} \lim_{N \to \infty} \int_{I_{N,\varepsilon}} \lambda^k y(x, \lambda) d\lambda
\]
\[= \frac{1}{2\pi i} \lim_{N \to \infty} \int_{I_{N,\varepsilon}} \lambda^k R_{\lambda} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) d\lambda
\]
\[= \frac{1}{2\pi i} \lim_{N \to \infty} \int_{I_{N,\varepsilon}} R_{\lambda} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) d\lambda + \frac{1}{2\pi i} \lim_{N \to \infty} \sum_{m=1}^{2n} \int_{\Omega_{m,-\varepsilon}} R_{\lambda} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) d\lambda
\]
\[- \int_{\Omega_{m,+\varepsilon}} R_{\lambda} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) d\lambda
\]
\[= \sum_{s=1}^p \text{res} \lambda^k R_{\lambda} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) \big|_{\lambda = \lambda_s} + \frac{1}{2\pi i} \sum_{m=1}^{2n} \int_{\Omega_m} \lambda^k \left[ R_{\lambda-i0} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) - R_{\lambda+i0} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) \right] d\lambda. \tag{5.14}
\]
Here $\Omega_{m,+\varepsilon}$ is the curve from one side and $\Omega_{m,-\varepsilon}$ from another side parallel to the curves $\Omega_m$, $m = 1, \ldots, 2n$, and $R_{\lambda+0}$ is the limit of $R_\lambda$ from one side and $R_{\lambda-0}$ is the limit from another side of the curves $\Omega_m$, $m = 1, \ldots, 2n$. Our assumption that $W_j(\lambda) \neq 0$ in $\Omega_m$, $m = 1, \ldots, 2n$, implies that these limits exist and are continuous.

Now we shall find the leap of the resolvent $R_\lambda$

$$R_{\lambda+0} - R_{\lambda-0}.$$ 

We know that:

$$\sum_{i=1}^{2n} \lambda^{2n-i} \phi_i = \int_{-\infty}^{\infty} [K(x, \xi, \lambda + i0) - K(x, \xi, \lambda - i0)] \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) d\xi.$$ 

Proceeding as in [4, p. 95], we obtain

$$\frac{1}{2\pi i} \int_{\Omega_m} \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) d\lambda = \frac{1}{2\pi i} \sum_{m=1}^{2} \int \lambda^k C(x, \lambda) F(\lambda) d\lambda. \quad (5.15)$$

Here $C(x, \lambda)$ are some differences of solutions $y_j(x, \lambda)$ and $F(\lambda)$ are the functions depending on the solutions of the corresponding adjoint problem. The integrals in (5.15) converge uniformly and absolutely for all $x \in (-\infty, \infty)$.

Taking (5.15) and (5.13) into account of (5.14), we obtain

$$f_k = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\Gamma_{N,\varepsilon}} \lambda^k y(x, \lambda) d\lambda = \sum_{s=1}^{p} \text{Res} \lambda^k R_\lambda \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) \bigg|_{\lambda=\lambda_s} - \frac{1}{2\pi i} \sum_{m=1}^{2n} \int \lambda^k C(x, \lambda) F(\lambda) d\lambda. \quad (5.16)$$

for all $\lambda \in \bar{D}_m$, $m = 1, \ldots, 2n$.

Applying the familiar formulae for the calculation of residues [4, p. 97], it is easily seen that

$$\sum_{s=1}^{p} \text{Res} \lambda^k R_\lambda \left( \sum_{i=1}^{2n} \lambda^{2n-i} \phi_i \right) \bigg|_{\lambda=\lambda_s} = \sum_{s=1}^{p} \lambda_s^k a_s y_s(x), \quad (5.17)$$

where

$$a_s = \frac{\int_{-\infty}^{\infty} |y_s(\xi)|^2 d\xi \cdot \int_{-\infty}^{\infty} \sum_{i=1}^{2n} \lambda_s^{2n-i} \phi_i(\xi) \bar{y}_s(\xi) d\xi}{\int_{-\infty}^{\infty} y_s(\xi) \bar{y}_s(\xi) d\xi \cdot \int_{-\infty}^{\infty} [L'(\lambda_s) y_s(\xi)] \bar{y}_s(\xi) d\xi}, \quad s = \overline{1, p}.$$ 

Thus, using (5.17), we arrive to the following expansion formula for arbitrary differentiable test functions $f_k$, $k = 0, \ldots, 2n - 1$,

$$f_k = \sum_{s=1}^{p} \lambda_s^k a_s y_s(x) - \frac{1}{2\pi i} \sum_{m=1}^{2n} \int \lambda^k C(x, \lambda) F(\lambda) d\lambda. \quad (5.18)$$

Thus we have proved the following theorem.

**Theorem 5.1.** Let (5.1) holds, and let the set of eigenvalues of the operator $L(\lambda)$ be finite and eigenvalues be simple and not lie in the continuous spectrum. Then for arbitrary differentiable test functions $f_k$, $k = 0, \ldots, 2n - 1$, we have the $(2n)$-fold expansion of the form (5.18), where integrals converge uniformly and absolutely along $\Omega_m$, $m = 1, \ldots, 2n$.

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