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# Bifurcation from Zero or Infinity in Sturm–Liouville Problems Which Are Not Linearizable

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We consider the nonlinear Sturm–Liouville problem

$$Lu := -(pu')' + qu = \lambda au + h(\cdot, u, u', \lambda), \qquad \text{in } (0, \pi),$$
$$a_0 u(0) + b_0 u'(0) = 0, \qquad a_1 u(\pi) + b_1 u'(\pi) = 0,$$

where $a_i, b_i$ are real numbers with $|a_i| + |b_i| > 0$, $i = 0, 1$, $\lambda$ is a real parameter, and the functions $p$ and $a$ are strictly positive on $[0, \pi]$. Suppose that the nonlinearity $h$ satisfies a condition of the form

$$|h(x, \xi, \eta, \lambda)| \le M_0|\xi| + M_1|\eta|, \qquad (x, \xi, \eta, \lambda) \in [0, \pi] \times \mathbb{R}^3,$$

as either $|(\xi, \eta)| \to 0$ or $|(\xi, \eta)| \to \infty$, for some constants $M_0, M_1$. Then we show that there exist global continua of nontrivial solutions $(\lambda, u)$ bifurcating from $u = 0$ or "$u = \infty$," respectively. These global continua have properties similar to those of the continua found in Rabonowitz' well-known global bifurcation theorem.


## 1. INTRODUCTION

We consider the nonlinear Sturm–Liouville problem

$$\begin{aligned} Lu := -(pu')' + qu = \lambda au + h(\cdot, u, u', \lambda), \qquad &\text{in } (0, \pi), \\ a_0 u(0) + b_0 u'(0) = 0, \qquad a_1 u(\pi) + b_1 u'(\pi) = 0, & \end{aligned} \tag{1.1}$$

where $a_i, b_i$ are real numbers with $|a_i| + |b_i| > 0$, $i = 0, 1$, $\lambda$ is a real parameter, and the functions $p, a \in C^1[0, \pi]$, $q \in C^0[0, \pi]$. We also as-

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sume that $p$ and $a$ are strictly positive on $[0, \pi]$. The function $h: [0, \pi] \times \mathbb{R}^3 \to \mathbb{R}$ is assumed to be continuous and to have the form $h = f + g$, where $f, g$ are continuous functions on $[0, \pi] \times \mathbb{R}^3$ and there are constants $M_0, M_1$ such that

$$|f(x, \xi, \eta, \lambda)| \leq M_0|\xi| + M_1|\eta|, \qquad (x, \xi, \eta, \lambda) \in [0, \pi] \times \mathbb{R}^3. \quad (1.2)$$

These assumptions will be taken to hold throughout. In addition, at various points in the paper we will impose one or the other (or both) of the following conditions on $g$: for any bounded interval $\Lambda \subset \mathbb{R}$,

$$g(x, \xi, \eta, \lambda) = o(|(\xi, \eta)|), \qquad \text{as } |(\xi, \eta)| \to 0, \qquad (1.3)$$

or

$$g(x, \xi, \eta, \lambda) = o(|(\xi, \eta)|), \qquad \text{as } |(\xi, \eta)| \to \infty, \qquad (1.4)$$

uniformly for $(x, \lambda) \in [0, \pi] \times \Lambda$ (here $|\cdot|$ denotes the Euclidean norm).

If condition (1.3) holds then $(\lambda, 0)$ is a solution of Eq. (1.1) for any $\lambda \in \mathbb{R}$ and we can consider bifurcation from $u = 0$, i.e., bifurcation of nontrivial solutions from the set of trivial solutions $\mathbb{R} \times \{0\}$ (a solution $(\lambda, u)$ of (1.1), or of any other equation in this paper, is said to be *nontrivial* if $u \neq 0$). Similar problems have been considered before in, for example, [3], [5], [9], and [15]. These papers prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1[0, \pi]$ emanating from "bifurcation intervals" (in $\mathbb{R} \times \{0\}$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the linear problem obtained from (1.1) by setting $h \equiv 0$. However, these papers suppose that $M_1 = 0$ (except in a remark in [15] where a special case of (1.1), with $M_1 < 1/\pi$, is considered). By extending the approximation technique used in [3] we obtain similar results for the above more general problem (with larger bifurcation intervals). Bailey also considered a similar problem in [1], although with a much stronger Lipschitz-continuity condition on the nonlinearity, and he obtained less information about the behaviour of the bifurcating sets of solutions.

If condition (1.4) holds then we can consider bifurcation from "$u = \infty$," i.e., the existence of solutions of (1.1) having arbitrarily large $u$. If, in addition to (1.4), $f \equiv 0$, then the problem is said to be asymptotically linear and the existence of solutions $(\lambda, u)$ of (1.1) with large $u$ "bifurcating from infinity" is discussed in the papers [12] and [16]. The approach used in [12] and [16] is to transform the bifurcation from infinity problem to a problem involving bifurcation from zero at eigenvalues of the linearization of (1.1), and then apply the standard global bifurcation theory from [11]. However, if $f \not\equiv 0$ and satisfies (1.2), then problem (1.1) need not be asymptotically linear and the transformed problem may not have a linearization at $u = 0$.

Thus the standard global bifurcation results are not immediately applicable and the proofs in [12] and [16] are not valid in this case. However, by extending the approximation technique from [3] and combining it with the global results in [8] and [11] we prove the existence, in this case, of global sets of solutions bifurcating from infinity which are similar to those obtained in [12] and [16].

In [10] Przybycin uses the same transformation together with the approximation technique from [3] to obtain a theorem on bifurcation from "intervals at infinity." He also assumes that $M_1 = 0$.

The general results on bifurcation from infinity just mentioned do not require any assumptions on the behaviour of the function $h$ at $u = 0$. However, if (1.3) holds, in addition to (1.4), then these general results can be improved. Also, continua of solutions bifurcate from both $u = 0$ and $u = \infty$ in this case. Furthermore, if $g \equiv 0$ then we can obtain additional information on the location of the bifurcating sets of solutions.

Finally, it can be shown (see the concluding remarks below) that condition (1.2) can be weakened slightly to allow the $M_i$ to depend on $\lambda$ to some extent—however, this is at the cost of even larger bifurcation intervals.

## 2. PRELIMINARY RESULTS

For any integer $k \geq 0$, let $C^k[0, \pi]$ denote the Banach space of real-valued, continuous functions on $[0, \pi]$, having continuous derivatives up to order $k$ on $(0, \pi)$, which extend continuously to $[0, \pi]$, and let $|\cdot|_k$ denote the standard sup-norm on $C^k[0, \pi]$. Let $E$ be the subspace of $C^1[0, \pi]$ consisting of those functions which satisfy the boundary conditions in (1.1). A *solution* of (1.1) is a pair $(\lambda, u) \in \mathbb{R} \times C^2[0, \pi]$ satisfying (1.1) (similarly for other equations below). Thus we may consider the structure of the solution set in $\mathbb{R} \times E$. For any positive integer $k$, let $S_k$ denote the set of functions $u \in E$ which have only simple zeros in $[0, \pi]$ and exactly $k - 1$ such zeros in $(0, \pi)$. Let $S_k^+$ be the set of functions $u \in S_k$ which are positive in a deleted neighbourhood of $x = 0$, and let $S_k^- = -S_k^+$. The sets $S_k^+$, $S_k^-$, and $S_k$ are open in $E$. From now on $\nu$ will denote either $+$ or $-$; $-\nu$ will denote the opposite sign to $\nu$.

If $h \equiv 0$ then (1.1) is a standard linear Sturm–Liouville problem. Thus there exists a strictly increasing sequence of simple eigenvalues $\mu_k$, $k = 1, 2, \ldots,$ with corresponding eigenfunctions $\phi_k \in S_k$. We also define the numbers $\mu_k^0 = \pi^2 k^2 / l^2$, $k \geq 1$, where

$$l = \int_0^\pi \left( \frac{a(z)}{p(z)} \right)^{1/2} dz.$$

Now consider the problem

$$\begin{aligned} Lu &= \lambda a u + q_0 u + q_1 u', \qquad \text{in } (0, \pi), \\ a_0 u(0) + b_0 u'(0) &= 0, \qquad a_1 u(\pi) + b_1 u'(\pi) = 0, \end{aligned} \tag{2.1}$$

with general functions $q_0, q_1 \in C^0[0, \pi]$.

LEMMA 2.1. *There exist positive constants $c_0, c_1$, independent of $q_0, q_1$, such that if $(\lambda, u) \in \mathbb{R} \times S_k$, $k \geq 1$, is a solution of* (2.1) *then*

$$|\lambda - \mu_k^0| \leq c_0\left(1 + |q_0|_0 + |q_1|_0^2\right) + c_1(1 + |q_1|_0)k. \tag{2.2}$$

*Remark.* The numbers $c_0, c_1$ (and the constants $c_i$ in the following proof) depend on $|p|_1, |q|_0, |a|_1, |p^{-1}|_0, |a^{-1}|_0$, and on the boundary conditions in (2.1). However, we are regarding the functions $p$, $q$, and $a$, and the boundary conditions as fixed, so we suppress the dependence of $c_0$ and $c_1$ on them.

*Proof of Lemma* 2.1. Let

$$y = \int_0^x \left( \frac{a(z)}{p(z)} \right)^{1/2} dz.$$

By transforming the independent variable $x$ on the interval $[0, \pi]$ to the variable $y$ on the interval $[0, l]$, and defining the function $v$ on $[0, l]$ by $v(y(x)) = u(x)$, we obtain an equation for $v$ having the form

$$v'' + r_1 v' + (\lambda + r_0)v = 0, \qquad \text{in } (0, l), \tag{2.3}$$

where $r_i \in C^0[0, \pi]$, $i = 0, 1$ (this transformation is similar to the well-known Liouville transformation, which is described in Section 10.9 of [4]; the assumption that $a, p \in C^1[0, \pi]$ is needed here). Clearly $u \in S_k \Leftrightarrow v \in S_k$. Let $R_i = |r_i|_0$, $i = 0, 1$. From the transformation it can be verified that $R_0 \leq c_2(1 + |q_0|_0)$, $R_1 \leq c_3(1 + |q_1|_0)$.

We now construct a number $\lambda_0$ such that any nontrivial solution $(\lambda, v)$ of (2.3) (together with the transformed boundary conditions—these depend only on the original boundary conditions and on $p$ and $a$) has $\lambda \geq \lambda_0$. We sketch the construction. By multiplying (2.3) by $v$ and integrating over the interval $(0, l)$ we obtain

$$\lambda \|v\|^2 = \|v'\|^2 + \text{b.c.} - \langle r_1 v', v \rangle - \langle r_0 v, v \rangle, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the $L^2(0, l)$ inner product and norm, respectively, and b.c. denotes the boundary terms obtained from integrating

$\langle v'', v\rangle$ by parts. Now, it can be shown that for any $\epsilon_1 > 0$, $e \in [0, l]$, and any $w \in C^2[0, \pi]$,

$$|w(e)|^2 \le 2\epsilon_1 \|w'\|^2 + 2\epsilon_1^{-1}\|w\|^2.$$

Hence, we can show that

$$|\text{b.c.}| \le c_4\left(2\epsilon_1\|v'\|^2 + 2\epsilon_1^{-1}\|v\|^2\right),$$

where $c_4$ depends only on the coefficients $a_i, b_i$, $i = 0, 1$, in the boundary conditions in (2.1) and on $p, a$. Also, for any $\epsilon_2 > 0$,

$$|\langle r_1 v', v\rangle| \le \|v'\|\,\|r_1 v\| \le 2\epsilon_2\|v'\|^2 + 2\epsilon_2^{-1}R_1^2\|v\|^2.$$

Thus, choosing $\epsilon_1, \epsilon_2$, such that $c_4\epsilon_1 + \epsilon_2 < 1/4$, we obtain from (2.4),

$$\lambda \ge \lambda_0 := -2c_4\epsilon_1^{-1} - 2\epsilon_2^{-1}R_1^2 - R_0.$$

We now apply the results in [2] to Eq. (2.3). For any real numbers $K, L$, such that $\zeta := K - L^2/4 > 0$, the functions $\alpha(L, K), \beta(L, K)$ defined in [2] satisfy

$$\left|\alpha(L, K) - \zeta^{-1/2}\pi/2\right| \le L\zeta^{-1/2}, \qquad \left|\beta(L, K) - \zeta^{-1/2}\pi/2\right| \le L\zeta^{-1/2},$$

when $L\zeta^{-1/2} \le c_5$, where $c_5$ is a constant, independent of $K$ and $L$. Also, in the present situation, the constants $L_i, K_i$, $i = 1, 2$, used in [2] satisfy $|L_i| \le R_1$, $\lambda - R_0 \le K_i \le \lambda + R_0$. Let $h_{\max}$ and $h_{\min}$ denote, respectively, the maximum and minimum distance between any two consecutive zeros of any nontrivial solution $v$ of (2.3). Theorem 3.1 of [2] shows that $h_{\max}$ and $h_{\min}$ satisfy the following estimates:

$$\begin{aligned} h_{\max} &\le \alpha(L_1, K_1) + \beta(L_2, K_1) \\ &\le \left(\lambda - R_0 - R_1^2\right)^{-1/2}\left[\pi + 2R_1\left(\lambda - R_0 - R_1^2\right)^{-1/2}\right], \\ h_{\min} &\ge \alpha(L_2, K_2) + \beta(L_1, K_2) \\ &\ge \left(\lambda + R_0 + R_1^2\right)^{-1/2}\left[\pi - 2R_1\left(\lambda - R_0 - R_1^2\right)^{-1/2}\right], \end{aligned}$$

when $\lambda \ge R_1^2/c_5^2 + R_0 + R_1^2/4$. Now, the number of zeros, $k$, of $v$ in $(0, l)$ satisfies

$$l/h_{\max} - 1 \le k \le l/h_{\min} + 1,$$

and hence (2.2) follows from the above estimates for $\lambda \ge \lambda_1 := c_6(1 + R_0 + R_1^2)$ (for sufficiently large $c_6$). Thus the lemma holds in the region $\lambda \ge \lambda_1$.

Now consider a nontrivial solution $(\lambda, v)$ of (2.3) with $\lambda_0 \le \lambda \le \lambda_1$. Then the simple estimate (1.7) in [2] shows that $Kh_{\min}^2 + 2Lh_{\min} \ge \pi^2$, where $K = |\lambda + r_0|_0 \le (|\lambda_0| + |\lambda_1| + R_0)$, $L = R_1$. Hence $h_{\min} \ge ((|\lambda_0| + |\lambda_1| + R_0)^{1/2} + R_1)^{-1}$, and so $k$ must satisfy

$$k^2 \le c_7\left(1 + R_0 + R_1^2\right).$$

Therefore, if $c_0, c_1$, are chosen appropriately the lemma also holds in the region $\lambda_0 \le \lambda \le \lambda_1$. This completes the proof of the lemma.

*Remark.* By using the results of [2] rather more carefully in the above proof, we could obtain estimates for the "asymptotic" values of the constants $c_0, c_1$, i.e., values of the constants which would ensure that (2.2) holds for all sufficiently large $k$ and $\lambda$. We note also that it is in the proof of Lemma 2.1 that we use the assumption that $a \in C^1[0, \pi]$. It is known that finding asymptotic estimates for the eigenvalues of (2.1) is more difficult when $a \notin C^1[0, \pi]$, and the estimate (2.2) may not be true—to avoid these problems we assume that $a \in C^1[0, \pi]$, even though $a \in C^0[0, \pi]$ would seem more natural in (1.1).

Let $K_0 = c_0(1 + \sqrt{2} M_0 + 2M_1^2)$, $K_1 = c_1(1 + \sqrt{2} M_1)$ (the reason for the $\sqrt{2}$ factors will be seen in the proof of Lemma 2.2 below) and for any $k \ge 1$, $\nu$, and any $\delta, \zeta > 0$, let

$$I_k = \left\{\lambda \in \mathbb{R}: |\lambda - \mu_k^0| \le K_0 + K_1 k\right\},$$

$$I_k(\delta) = \left\{\lambda \in \mathbb{R}: |\lambda - \mu_k^0| < K_0 + K_1 k + \delta\right\},$$

$$U_k^\nu(\delta, \zeta) = \left\{(\lambda, u) \in \mathbb{R} \times E: \lambda \in I_k(\delta), u \in S_k^\nu, |u|_1 > \zeta^{-1}\right\}.$$

Clearly $U_k^\nu(\delta, \zeta)$ is open in $\mathbb{R} \times E$. Also, it follows from Lemma 2.2, with $q_0 = q_1 = 0$, that $\mu_k \in I_k$ for all $k \ge 1$.

In the proof of our main result on bifurcation from infinity we will also need to consider the modified nonlinear problem

$$\begin{gathered} Lu = \lambda au + |u|_1^{-2\epsilon} f(\cdot, |u|_1^\epsilon u, |u|_1^\epsilon u', \lambda) + g(\cdot, u, u', \lambda), \\ a_0 u(0) + b_0 u'(0) = 0, \qquad a_1 u(\pi) + b_1 u'(\pi) = 0, \end{gathered} \tag{2.5}$$

with $\epsilon \in [0, 1/2]$. In a sense, this problem approximates problem (1.1) when $\epsilon$ is small; this form of approximation is similar to that used in [3].

LEMMA 2.2. *Suppose that condition* (1.4) *holds,* $\Lambda \subset \mathbb{R}$ *is a compact interval, and* $\delta > 0$. *Then there exists* $\zeta_U = \zeta_U(\delta, \Lambda) > 0$ *such that if* $(\lambda, u, \epsilon) \in \Lambda \times E \times [0, 1/2]$ *is a solution of* (2.5) *with* $|u|_1 > \zeta_U^{-1}$, *then*

$$(\lambda, u) \in \bigcup_{(k, \nu)} U_k^\nu(\delta, \zeta_U).$$

*Proof.* Suppose that the result is not true. Then there exists a sequence $(\lambda_n, u_n, \epsilon_n) \in \Lambda \times E \times [0, 1/2]$, $n = 1, 2, \ldots,$ of solutions of (2.5), with $\lambda_n \to \lambda_\infty \in \Lambda$, $|u_n|_1 > n$, and

$$(\lambda_n, u_n) \notin \bigcup_{(k,\nu)} U_k^\nu(\delta, n^{-1}). \tag{2.6}$$

Define the functions $f_n, g_n \in C^0[0, \pi]$ by

$$f_n(x) = |u_n|_1^{-1-2\epsilon_n} f\big(x, |u_n|_1^{\epsilon_n} u_n(x), |u_n|_1^{\epsilon_n} u_n'(x), \lambda_n\big),$$
$$g_n(x) = |u_n|_1^{-1} g\big(x, u_n(x), u_n'(x), \lambda_n\big).$$

For each $n \geq 1$ let $w_n = u_n/|u_n|_1 \in E$. Dividing (2.5) by $|u_n|_1$ shows that $w_n$ satisfies the equation

$$Lw_n = \lambda_n a w_n + f_n + g_n. \tag{2.7}$$

Also, it follows from the properties of $f$ and $g$ that

$$\begin{aligned} |f_n(x)| &\leq M_0|w_n(x)| + M_1|w_n'(x)|, \\ \rho_n &:= |g_n|_0 \to 0. \end{aligned} \tag{2.8}$$

Thus from (2.7) we have

$$|w_n''(x)| \leq K\big(|w_n(x)| + |w_n'(x)| + \rho_n\big), \qquad x \in [0, \pi], \tag{2.9}$$

for some constant $K > 0$.

We will now show that there exist $\delta_1 > 0$, $N_1 > 0$ such that, for $n \geq N_1$,

$$\big|(w_n(x), w_n'(x))\big| > \delta_1, \qquad x \in [0, \pi]. \tag{2.10}$$

Suppose that this is not true. Then there exists a subsequence of the sequence $\{w_n\}$ (which we will relabel as $\{w_n\}$), and $\tau \in [0, \pi]$ such that $|(w_n(\tau), w_n'(\tau))| \to 0$. By following the argument on p. 379 of [3] (using Gronwall's inequality) we find that this, together with (2.9), implies that $w_n \to 0$ in $E$. However, this contradicts the fact that $|w_n|_1 = 1$ for all $n$. Thus (2.10) must be true.

It follows from (2.10) that for $n \geq N_1$ we can write $f_n(x)$, $x \in [0, \pi]$, in the form

$$\begin{aligned} f_n(x) &= \frac{f_n(x) M_0^2 w_n(x)}{F_n(x)^2} w_n(x) + \frac{f_n(x) M_1^2 w_n'(x)}{F_n(x)^2} w_n'(x) \\ &= f_n^0(x) w_n(x) + f_n^1(x) w_n'(x), \end{aligned}$$

where $F_n(x)^2 = M_0^2 w_n(x)^2 + M_1^2 w_n'(x)^2 \geq \delta_2 > 0$, and for each $i = 0, 1$, $f_n^i \in C^0[0, \pi]$ and

$$|f_n^i(x)| \leq \sqrt{2} \frac{|f_n(x)|}{M_0|w_n(x)| + M_1|w_n'(x)|} M_i \leq \sqrt{2} M_i,$$

by (2.8). In a similar manner we have

$$g_n(x) = g_n^0(x) w_n(x) + g_n^1(x) w_n'(x), \qquad x \in [0, \pi],$$

with $g_n^i \in C^0[0, \pi]$ and $|g_n^i|_0 \to 0$ as $n \to \infty$. Thus Eq. (2.7) can be rewritten as

$$Lw_n = \lambda_n a w_n + \left(f_n^0 + g_n^0\right) w_n + \left(f_n^1 + g_n^1\right) w_n', \tag{2.11}$$

with

$$|f_n^i + g_n^i|_0 \leq \sqrt{2} M_i + |g_n^i|_0. \tag{2.12}$$

It also follows from (2.10) that, for each $n \geq N_1$, $(\lambda_n, u_n) \in \Lambda \times S_k$ for some $k$ (possibly depending on $n$). Hence it follows from (2.11), (2.12), Lemma 2.1, and the definition of $I_k(\delta)$ that $\lambda_n \in I_k(\delta)$ if $n$ is sufficiently large. Thus $(\lambda_n, u_n) \in U_k^+(\delta, n^{-1}) \cup U_k^-(\delta, n^{-1})$, which contradicts (2.6) and so completes the proof of Lemma 2.2.

When condition (1.3) holds a similar result (with a similar proof) is valid for small $|u|_1$. For any $k \geq 1$, $\nu$, and any $\delta, \zeta > 0$, let

$$V_k^\nu(\delta, \zeta) = \{(\lambda, u) \in \mathbb{R} \times E : \lambda \in I_k(\delta), u \in S_k^\nu, |u|_1 < \zeta\}.$$

LEMMA 2.3. *Suppose that condition* (1.3) *holds*, $\Lambda \subset \mathbb{R}$ *is a compact interval, and* $\delta > 0$. *Then there exists* $\zeta_V = \zeta_V(\delta, \Lambda) > 0$ *such that if* $(\lambda, u, \epsilon) \in \Lambda \times E \times [0, 1/2]$ *is a nontrivial solution of* (2.5), *with* $|u|_1 < \zeta_V$, *then*

$$(\lambda, u) \in \bigcup_{(k,\nu)} V_k^\nu(\delta, \zeta_V).$$

## 3. MAIN RESULTS

Let $\mathcal{S} \subset \mathbb{R} \times E$ be the set of nontrivial solutions of (1.1). For any $\lambda \in \mathbb{R}$, we say that a subset $\mathcal{C} \subset \mathcal{S}$ *meets* $(\lambda, \infty)$ (respectively, $(\lambda, 0)$) if there exists a sequence $(\lambda_n, u_n) \in \mathcal{C}$, $n = 1, 2, \ldots$, such that $\lambda_n \to \lambda$, $|u_n|_1 \to \infty$ (respectively, $|u_n|_1 \to 0$) as $n \to \infty$. Furthermore, we will say that $\mathcal{C} \subset \mathcal{S}$ meets $(\lambda, \infty)$ (respectively, $(\lambda, 0)$) *through* $\mathbb{R} \times S_k^\nu$ if the sequence

$(\lambda_n, u_n) \in \mathscr{C}$, $n = 1, 2, \ldots$, can be chosen so that $u_n \in S_k^\nu$ for all $n$. If $I \subset \mathbb{R}$ is a bounded interval we say that $\mathscr{C} \subset \mathscr{S}$ *meets* $I \times \{\infty\}$ (respectively, $I \times \{0\}$) if $\mathscr{C}$ meets $(\lambda, \infty)$ (respectively, $(\lambda, 0)$) for some $\lambda \in I$; we define $\mathscr{C}$ meets $I \times \{\infty\}$ or $I \times \{0\}$ *through* $\mathbb{R} \times S_k^\nu$ similarly.

We consider first the case of bifurcation from $u = \infty$ and suppose, for now, that condition (1.4) holds (bifurcation from $u = 0$, when (1.3) holds, will be considered below—we deal with this case second because it is simpler than bifurcation from $u = \infty$, and so the proofs may be omitted). Putting $\epsilon = 0$ in Lemma 2.2, it follows that any set $\mathscr{C} \subset \mathscr{S}$ which meets $(\lambda, \infty)$, for some $\lambda$, must do so through $\mathbb{R} \times S_k^\nu$, for at least one pair $(k, \nu)$ ($\mathscr{C}$ may meet more than one "point" $(\lambda, \infty)$ through more than one set $\mathbb{R} \times S_k^\nu$). Furthermore, if $\mathscr{C}$ meets $(\lambda, \infty)$ through $\mathbb{R} \times S_k^\nu$ then $\lambda \in I_k$. For each $k \geq 1$ and $\nu$ we define the set $\mathscr{D}_k^\nu \subset \mathscr{S}$ to be the union of all the components of $\mathscr{S}$ which meet $I_k \times \{\infty\}$ through $\mathbb{R} \times S_k^\nu$ (we will show below that this set is nonempty). The set $\mathscr{D}_k^\nu$ may not be connected in $\mathbb{R} \times E$ although, by adding the "points at infinity" $(\lambda, \infty)$, $\lambda \in I_k$, to $\mathbb{R} \times E$ and defining an appropriate topology on the resulting set, the set $\mathscr{D}_k^\nu \cup (I_k \times \{\infty\})$ is connected—we will not pursue this further.

For any set $A \subset \mathbb{R} \times E$ we let $P_R(A)$ denote the natural projection of $A$ onto $\mathbb{R} \times \{0\}$.

THEOREM 3.1. *If* (1.4) *holds then, for every integer $k \geq 1$ and each $\nu$, the set $\mathscr{D}_k^\nu$ is nonempty and at least one of the following holds*:

(i) $\mathscr{D}_k^\nu$ *meets* $I_{k'} \times \{\infty\}$ *through* $\mathbb{R} \times S_{k'}^{\nu'}$ *for some* $(k', \nu') \neq (k, \nu)$;

(ii) $\mathscr{D}_k^\nu$ *meets* $(\lambda, 0)$ *for some* $\lambda \in \mathbb{R}$;

(iii) $P_R(\mathscr{D}_k^\nu)$ *is unbounded.*

*In addition, if the union* $\mathscr{D}_k = \mathscr{D}_k^+ \cup \mathscr{D}_k^-$ *does not satisfy* (ii) *or* (iii) *then it must satisfy* (i) *with* $k' \neq k$.

*Proof.* For any nontrivial $(\lambda, v) \in \mathbb{R} \times E$ we define the function $\tilde{f}(\lambda, v) \in C^0[0, \pi]$ by

$$\tilde{f}(\lambda, v)(x) = |v|_1^2 f\big(x, v(x)/|v|_1^2, v'(x)/|v|_1^2, \lambda\big), \qquad x \in [0, \pi],$$

and let $\tilde{f}(\lambda, 0) = 0$; we define $\tilde{g}$ similarly. By our basic assumptions and (1.4), the functions $\tilde{f}, \tilde{g}$: $\mathbb{R} \times E \to C^0[0, \pi]$ are continuous and satisfy

$$\begin{aligned} \big|\tilde{f}(\lambda v)\big|_0 &\leq (M_0 + M_1)|v|_1, \\ |\tilde{g}(\lambda, v)|_0 &= o(|v|_1), \qquad \text{as } |v|_1 \to 0, \end{aligned} \tag{3.1}$$

where the $o$ estimate holds uniformly for $\lambda$ in any bounded interval $\Lambda \subset \mathbb{R}$. Now let $(\lambda, u)$ be a nontrivial solution of (1.1). Setting $v = u/|u|_1^2$, we have $|v|_1 = 1/|u|_1$ and $u = v/|v|_1^2$. Dividing (1.1) by $|u|_1^2$ yields the equation

$$Lv = \lambda av + \tilde{f}(\lambda, v) + \tilde{g}(\lambda, v) \tag{3.2}$$

(the boundary conditions in (1.1) also hold here and below; for brevity we will not explicitly display them again). The transformation $(\lambda, u) \to T(\lambda, u) := (\lambda, v)$ was used in the papers [12] and [16] and turns a "bifurcation at infinity" problem into a "bifurcation at zero" problem. However, here Eq. (3.2) does not have a linearization at $v = 0$ so we cannot immediately apply standard global bifurcation theory to obtain the desired result, as was done in [12] and [16]. To deal with this problem we will also consider the equation

$$Lv = \lambda av + \tilde{f}(\lambda, |v|_1^{\epsilon} v) + \tilde{g}(\lambda, v), \tag{3.3}$$

for $\epsilon \in (0, 1/2]$ (it can be seen from these definitions that (3.3) is equivalent to (2.5)). For fixed $\epsilon \in (0, 1/2]$ it follows from (3.1) that $|\tilde{f}(\lambda, |v|_1^{\epsilon} v)|_0 = o(|v|_1)$, so the global bifurcation results in [8] and [11] are applicable to this equation. Also, we will see that as $\epsilon \to 0$, (3.3) approximates (3.2) in a suitable sense. We now choose some fixed (arbitrary) $k_0 \geq 1$ and $\nu_0$, and we will prove the theorem for $k = k_0$ and $\nu = \nu_0$.

Let $\tilde{\mathscr{S}} \subset \mathbb{R} \times E$ be the set of nontrivial solutions of (3.2). By construction, the transformation $(\lambda, u) \to T(\lambda, u)$ maps $\mathscr{S}$ into $\tilde{\mathscr{S}}$ and, heuristically, interchanges points at $u = 0$ (respectively, $u = \infty$) with points at $v = \infty$ (respectively, $v = 0$). Let $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ be the union of all the components of $\tilde{\mathscr{S}}$ which meet $I_{k_0} \times \{0\}$ through $\mathbb{R} \times S_{k_0}^{\nu_0}$. Then $\mathscr{D}_{k_0}^{\nu_0}$ is the inverse image $T^{-1}(\tilde{\mathscr{D}}_{k_0}^{\nu_0})$ of $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ under the transformation $T$. Thus to prove the theorem it suffices to show that the set $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ is nonempty and either meets some interval $I_k \times \{0\}$ through $\mathbb{R} \times S_k^{\nu}$, with $(k, \nu) \neq (k_0, \nu_0)$, or is unbounded in $\mathbb{R} \times E$ (the alternatives (ii) and (iii) stated in the theorem for $\mathscr{D}_{k_0}^{\nu_0}$ correspond, via $T$, to the various ways in which $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ can be unbounded).

Suppose that $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ is nonempty, but neither of the above possibilities holds (the possibility that $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ is empty will be dealt with below). Then $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ is bounded and we can choose a compact interval $\Lambda \subset \mathbb{R}$ such that $P_R(\tilde{\mathscr{D}}_{k_0}^{\nu_0}) \cup I_{k_0}$ is in the interior of $\Lambda$. Let $\tilde{\mathscr{S}}_{\Lambda} = \tilde{\mathscr{S}} \cap (\Lambda \times E)$. We note that for any $k$, $\nu$, and $\delta$ the transformation $T$ maps $U_k^{\nu}(\delta)$ into $V_k^{\nu}(\delta)$. Also, the statement of Lemma 2.2 holds in the transformed situation if $u$ is changed to $v$, $U_k^{\nu}$ to $V_k^{\nu}$, and $|u|_1 > \zeta_U^{-1}$ to $|v|_1 < \zeta_U$. Now, for any $\delta, \zeta > 0$, let

$$W_0(\delta, \zeta) = \bigcup_{(k, \nu) \neq (k_0, \nu_0)} V_k^{\nu}(\delta, \zeta),$$

$$\partial' W_0(\delta, \zeta) = \{(\lambda, v) \in \partial W_0(\delta, \zeta) : |v|_1 = \zeta\}.$$

The set $W_0(\delta, \zeta)$ is open in $\mathbb{R} \times E$, and we let $\overline{W}_0(\delta, \zeta)$ denote its closure. Also, the location of the intervals $I_k$ ensures that any bounded subset of $\mathbb{R} \times E$ intersects only finitely many sets $V_k^\nu(\delta, \zeta)$. It can now be seen that we can choose $\delta_0, \zeta_0 > 0$, with $\zeta_0 \leq \zeta_U(\delta_0, \Lambda)$, such that:

(a) $I_{k_0}(\delta_0) \subset \Lambda$;

(b) $\tilde{\mathscr{S}}_\Lambda \cap \partial W_0(\delta_0, \zeta_0) \subset \partial' W_0(\delta_0, \zeta_0)$;

(c) $\tilde{\mathscr{D}}_{k_0}^{\nu_0} \cap \overline{W}_0(\delta_0, \zeta_0) = \varnothing$

((b) follows from Lemma 2.2 with $\epsilon = 0$; if (c) did not hold then by Lemma 2.2 with $\epsilon = 0$, $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ would meet $I_k \times \{0\}$ through $\mathbb{R} \times S_k^\nu$ for some $(k, \nu) \neq (k_0, \nu_0)$, contrary to our assumption on $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$).

Now let $\tilde{C}_0 = \tilde{\mathscr{D}}_{k_0}^{\nu_0} \cup (I_{k_0} \times \{0\}) \subset \mathbb{R} \times E$. It follows from Lemma 2.2 and (c) that $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ cannot meet any point $(\lambda, 0)$ with $\lambda \notin I_{k_0}$, so $\tilde{C}_0$ is closed; it is also connected. Furthermore, Eq. (3.2) shows that bounded subsets of $\tilde{\mathscr{S}}$ in $\mathbb{R} \times E$ are also bounded in $\mathbb{R} \times C^2[0, \pi]$, so $\tilde{C}_0$ is compact. Thus we can choose a bounded open set $\mathscr{O}_0 \subset \Lambda \times E$ such that

$$\tilde{C}_0 \subset \mathscr{O}_0, \qquad \partial' W_0(\delta_0, \zeta_0) \cap \overline{\mathscr{O}}_0 = \varnothing, \qquad \partial \mathscr{O}_0 \cap \left(\bar{\tilde{\mathscr{S}}} \backslash W_0(\delta_0, \zeta_0)\right) = \varnothing \tag{3.4}$$

(since $\operatorname{dist}(\tilde{C}_0, \partial' W_0(\delta_0, \zeta_0)) > 0$ it is clear that an open set with the first two properties exists; we obtain the final property by following the arguments in Lemmas 1.1 and 1.2 of [11] or in Step 1 on p. 677 of [18]). On the other hand, a set $\mathscr{O}_0$ with these properties can also be constructed if $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$ is empty, so the remainder of the proof covers all the above possibilities for $\tilde{\mathscr{D}}_{k_0}^{\nu_0}$. Let

$$\mathscr{O} = \mathscr{O}_0 \backslash \overline{W}_0(\delta_0, \zeta_0). \tag{3.5}$$

The set $\mathscr{O}$ is open and, for sufficiently small $\delta_1, \zeta_1 > 0$, has the properties

$$\tilde{\mathscr{D}}_{k_0}^{\nu_0} \subset \mathscr{O}, \qquad V_{k_0}^{\nu_0}(\delta_1, \zeta_1) \subset \mathscr{O}, \qquad \partial \mathscr{O} \cap \tilde{\mathscr{S}} = \varnothing \tag{3.6}$$

(the final property in (3.6) follows from (b) above and (3.4)).

By Theorem 2 in [8], for each fixed $\epsilon \in (0, 1/2]$ there exists a component $\tilde{\mathscr{D}}_{k_0}^{\nu_0}(\epsilon) \subset \mathbb{R} \times E$ of nontrivial solutions of (3.3) with the following properties: $\tilde{\mathscr{D}}_{k_0}^{\nu_0}(\epsilon)$ meets $(\mu_{k_0}, 0)$ through $\mathbb{R} \times S_{k_0}^{\nu_0}$ (and so intersects $V_{k_0}^{\nu_0}(\delta_1, \zeta_1)$); $\tilde{\mathscr{D}}_{k_0}^{\nu_0}(\epsilon)$ is either unbounded or there is some $(k, \nu) \neq (k_0, \nu_0)$ such that $\tilde{\mathscr{D}}_{k_0}^{\nu_0}(\epsilon)$ meets $(\mu_k, 0)$ through $\mathbb{R} \times S_k^\nu$ (and so intersects $W_0(\delta_0, \zeta_0)$). These nodal properties follow from the nodal properties of the linear problem and Lemma 1.24 of [11]; see [11, p. 502]. Now, these properties together with the construction of the set $\mathscr{O}$ imply that the component $\tilde{\mathscr{D}}_{k_0}^{\nu_0}(\epsilon)$ intersects both $\mathscr{O}$ and the complement of $\mathscr{O}$, and so $\tilde{\mathscr{D}}_{k_0}^{\nu_0}(\epsilon) \cap \partial\mathscr{O} \neq \emptyset$. Thus, for each $\epsilon$ there exists a nontrivial solution $(\lambda_\epsilon, v_\epsilon) \in \partial\mathscr{O}$ of (3.3). Since $\mathscr{O}$ is bounded in $\mathbb{R} \times E$, Eq. (3.3) shows that the set of points $(\lambda_\epsilon, v_\epsilon)$ is bounded in $\mathbb{R} \times C^2[0, \pi]$, independently of $\epsilon$. Therefore there is a sequence $\epsilon_n$, $n = 1, 2, \dots,$ such that $\epsilon_n \to 0$ and $(\lambda_{\epsilon_n}, v_{\epsilon_n})$ converges in $\mathbb{R} \times E$ to a solution $(\lambda_\infty, v_\infty)$ of (3.2). We will show that $(\lambda_\infty, v_\infty)$ is nontrivial, which implies that $(\lambda_\infty, v_\infty) \in \partial\mathscr{O} \cap \tilde{\mathscr{S}}$. Since this contradicts (3.6), this will complete the proof of the theorem.

Suppose that $|v_{\epsilon_n}|_1 \to 0$. Then it follows from Lemma 2.2 that, for all sufficiently large $n$,

$$\left(\lambda_{\epsilon_n}, v_{\epsilon_n}\right) \in \partial\mathscr{O} \cap \left(V_{k_0}^{\nu_0}(\delta_2, \zeta_2) \cup W_0(\delta_2, \zeta_2)\right),$$

where $\delta_2 = \min\{\delta_0, \delta_1\}$, $\zeta_2 < \min\{\zeta_0, \zeta_1\}$. However, this contradicts (3.5) and (3.6). Thus we can conclude that $(\lambda_\infty, v_\infty)$ is nontrivial, which completes the proof.

*Remarks.* (i) If, for each $\lambda \in \mathbb{R}$, there is an $x$ such that $h(x, 0, 0, \lambda) \neq 0$ then alternative (ii) in Theorem 3.1 cannot hold.

(ii) Unlike in the case of bifurcation from zero in [11], it need not be the case that $\mathscr{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$ in Theorem 3.1 (see [12, Remark 2.12]). This makes the approximation argument in the above proof more complicated than it would be if we had $\mathscr{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$, as in [3]; this point appears to have been overlooked in [10].

(iii) Under further differentiability hypotheses on the functions $p$, $q$, $a$, and $h$, the genericity results in [13] can be adapted to show that "generically" the sets $\mathscr{D}_k^\nu$ are collections of smooth, one-dimensional curves in $\mathbb{R} \times E$ (see [13, Theorem 2.5]). We will not give precise statements of what "generically" means here; as an example, this result holds whenever the coefficient function $p$ in (1.1) belongs to a certain residual subset of the subspace of $C^3[0, \pi]$ consisting of positive functions.

We will now suppose that condition (1.3) holds and consider the case of bifurcation from $u = 0$. The above methods enable us to extend some of the results in [3], [5], [9], and [15] for this situation. As before, putting $\epsilon_n = 0$, for all $n \geq 1$, in Lemma 2.3 shows that any set $\mathscr{C} \subset \mathscr{S}$ which meets $(\lambda, 0)$, for some $\lambda$, must do so through $\mathbb{R} \times S_k^\nu$ for at least one pair $(k, \nu)$. Furthermore, if $\mathscr{C}$ meets $(\lambda, 0)$ through $\mathbb{R} \times S_k^\nu$ then $\lambda \in I_k$. Thus for each $k$ and $\nu$ we can define the set $\mathscr{C}_k^\nu \subset \mathscr{S}$ to be the union of all the components of $\mathscr{S}$ which meet $I_k \times \{0\}$ through $\mathbb{R} \times S_k^\nu$ (the following theorem will show that this set is nonempty for all $k$ and $\nu$). The set $\mathscr{C}_k^\nu \cup (I_k \times \{0\})$ is connected in $\mathbb{R} \times E$, but $\mathscr{C}_k^\nu$ may not be connected.

A further consequence of condition (1.3), together with (1.2), is that if $(\lambda, u)$ is a solution of (1.1) with $u \in \partial S_k^\nu$, then $u = 0$ (see the first paragraph of the proof of Theorem 1 in [3]); hence the nodal structure of the solutions is preserved along any component of $\mathscr{S}$.

THEOREM 3.2. *If* (1.3) *holds then, for every integer* $k \geq 1$ *and each* $\nu$, *the set* $\mathscr{C}_k^\nu$ *is nonempty, is unbounded in* $\mathbb{R} \times E$, *and* $\mathscr{C}_k^\nu \subset \mathbb{R} \times S_k^\nu$.

*Proof.* The proof is similar to the proof of Theorem 1 in [3], using Lemma 2.3 rather than Lemma 1 of [3]. We note that the preservation of nodal properties along components of $\mathscr{S}$ makes the proof of this theorem rather simpler than the proof of Theorem 3.1.

*Remarks.* (i) Theorem 3.2 is very similar to Berestycki's Theorem 1 in [3], except that $M_1 = 0$ in [3]. The extension to $M_1 > 0$ here is gained at the expense of having larger bifurcation intervals $I_k$; in fact, the bifurcation intervals in [3] have constant length, whereas here they grow with $k$.

(ii) Schmitt and Smith also proved a similar result (see [15, Theorems 3.4 and 3.5]). However, their theorems also require $M_1 = 0$, and they only consider $k \geq k_0$ for some $k_0$ sufficiently large that their bifurcation intervals $I_k$ (which are the same as Berestycki's) do not overlap for $k \geq k_0$. Also, they do not show that the individual sets $\mathscr{C}_k^+, \mathscr{C}_k^-$ are nonempty and unbounded, only that the union $\mathscr{C}_k^+ \cup \mathscr{C}_k^-$ is. In addition, in Lemma 3.3 of [15] they consider the case $0 < M_1 < 1/\pi$, for a particular equation, and obtain larger, nonoverlapping bifurcation intervals, for $k$ sufficiently large, and the remark following Theorem 3.4 notes that a corresponding bifurcation theorem holds for this case. Theorem 3.2 above extends this result to a general equation, for all $M_1 > 0$ and all $k$ and $\nu$ (the bifurcation intervals $I_k$ used here may overlap, but the use of nodal properties ensures that this does not invalidate the bifurcation results).

(iii) In a sequence of papers Makhmudov and Aliev also give results on bifurcation intervals when the nonlinearity again satisfies a similar condition (see, for instance, [9] and the references therein). Their applications are usually to higher order Sturm−Liouville equations, but in essence their results also require $M_1 = 0$ and $k$ sufficiently large to obtain bifurcation.

Next, if conditions (1.3) and (1.4) both hold then we can improve Theorems 3.1 and 3.2 as follows.

THEOREM 3.3. *If* (1.3) *and* (1.4) *hold then, for every integer* $k \geq 1$ *and each* $\nu$, $\mathscr{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$, *and alternative* (i) *of Theorem* 3.1 *cannot hold. Furthermore, if* $\mathscr{D}_k^\nu$ *meets* $(\lambda, 0)$, *for some* $\lambda \in \mathbb{R}$, *then* $\lambda \in I_k$. *Similarly, if* $\mathscr{C}_k^\nu$ *meets* $(\lambda, \infty)$, *then* $\lambda \in I_k$.

*Proof.* The remark preceding Theorem 3.2 shows that if (1.3) holds then $\mathscr{S} \cap (\mathbb{R} \times \partial S_k^\nu) = \emptyset$. Hence the sets $\mathscr{S} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathscr{S} \backslash (\mathbb{R} \times S_k^\nu)$ are mutually separated in $\mathbb{R} \times E$ (see [17, Definition 26.4]). Thus by Corollary 26.6 of [17] any component of $\mathscr{S}$ must be a subset of one or another of these sets. Since $\mathscr{D}_k^\nu$ is the union of components of $\mathscr{S}$ which intersect $\mathbb{R} \times S_k^\nu$, each of these components must be a subset of $\mathbb{R} \times S_k^\nu$, and hence $\mathscr{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$. This ensures that alternative (i) of Theorem 3.1 cannot hold. It now follows from Lemma 2.3 that $\mathscr{D}_k^\nu$ can only meet $(\lambda, 0)$ if $\lambda \in I_k$. Similarly, by Lemma 2.2, $\mathscr{C}_k^\nu$ can only meet $(\lambda, \infty)$ if $\lambda \in I_k$.

Imposing a further restriction on the nonlinearity will enable us to say rather more about the location of the sets $\mathscr{C}_k^\nu$ and $\mathscr{D}_k^\nu$, and which of the alternatives in Theorem 3.3 holds. We suppose that

$$g \equiv 0 \tag{3.7}$$

(in effect, we suppose that the nonlinearity $h$ itself satisfies (1.2)).

THEOREM 3.4. *Suppose that* (3.7) *holds. Then for every integer $k \geq 1$ and each $\nu$,*

$$\mathscr{C}_k^\nu \cup \mathscr{D}_k^\nu \subset I_k \times E. \tag{3.8}$$

*Hence, $P_R(\mathscr{C}_k^\nu)$ and $P_R(\mathscr{D}_k^\nu)$ are bounded and $\mathscr{C}_k^\nu$ meets $I_k \times \{\infty\}$ while $\mathscr{D}_k^\nu$ meets $I_k \times \{0\}$ (each through $\mathbb{R} \times S_k^\nu$).*

*Proof.* Relation (3.8) follows immediately from Lemma 2.1 and Theorems 3.2 and 3.3. The other results then follow from Theorem 3.3 and the definitions of the sets $\mathscr{C}_k^\nu, \mathscr{D}_k^\nu$.

*Remarks.* (i) Theorem 3.4 does not appear to imply that $\mathscr{C}_k^\nu \cap \mathscr{D}_k^\nu \neq \emptyset$. Conceivably, for instance, the set $\mathscr{C}_k^\nu$ could consist of a countable collection of components of nontrivial solutions, each of which is bounded but the union of which meets $I_k \times \{\infty\}$, and similarly for $\mathscr{D}_k^\nu$. If this were so we would have $\mathscr{C}_k^\nu \cap \mathscr{D}_k^\nu = \emptyset$. This situation seems rather unlikely, and will certainly not happen if there is a simple bifurcation at either $u = 0$ or $u = \infty$, since then there will be a unique curve of solutions near $u = 0$ or $u = \infty$, i.e., there will be a single bifurcating component (see [6] or [18] for simple bifurcation from $u = 0$ and [7] or [14] for simple bifurcation from $u = \infty$). This will ensure that $\mathscr{C}_k^\nu \cap \mathscr{D}_k^\nu \neq \emptyset$, but not necessarily $\mathscr{C}_k^\nu = \mathscr{D}_k^\nu$.

(ii) Theorem 3.4 is similar to Proposition 2 in [5], except that there $M_1 = 0$, $k$ is required to be sufficiently large, and only the sets $\mathscr{C}_k^+ \cup \mathscr{C}_k^-$ are considered.

(iii) For large $k$ we could obtain more precise estimates for the sizes of the bifurcation intervals $I_k$ (see the remark following the proof of

Lemma 2.1), and we could even have different intervals for bifurcation from zero or infinity if we had different constants $M_i$ in the estimates for the behaviour of $f$ at zero and infinity. Furthermore, if more precise information about the behaviour of the nonlinearity at either zero or infinity is available then Theorems 3.3 and 3.4 can be improved further. For instance, if $|h(x, \xi, \eta, \lambda)| = o(|(\xi, \eta)|)$, as either $|(\xi, \eta)| \to 0$ or $|(\xi, \eta)| \to \infty$, and if a continuum of solutions meets $(\lambda, 0)$ or $(\lambda, \infty)$, respectively, then $\lambda$ must be an eigenvalue $\mu_k$ of the linear problem.

(iv) We can weaken condition (1.2) to a certain extent and still obtain all the above results (albeit with larger bifurcation intervals $I_k$). Suppose, instead of (1.2), that for any number $L > 0$, condition (1.2) holds for $(x, \xi, \eta, \lambda) \in [0, \pi] \times \mathbb{R}^2 \times [-L, L]$, with constants $M_0(L), M_1(L)$ and suppose that $M_0(L) + M_1(L)^2 = o(L)$ as $L \to \infty$. Then if (3.7) holds, Lemma 2.1 says that any nontrivial solution $(\lambda, u) \in \mathbb{R} \times S_k$ of (1.1), with $|\lambda| \le L$, satisfies

$$|\lambda - \mu_k^0| \le c_0\big(1 + M_0(L) + M_1(L)^2\big) + c_1\big(1 + M_1(L)\big)k.$$

Hence, some analysis shows that there is a decreasing function $\alpha: \mathbb{N} \to \mathbb{R}$ such that $\alpha(k) \to 0$ as $k \to \infty$ and any nontrivial solution $(\lambda, u) \in \mathbb{R} \times S_k$ satisfies

$$|\lambda - \mu_k^0| \le \alpha(k)k^2.$$

If instead of (3.7), one or the other of (1.3) or (1.4) is satisfied then this estimate holds for the limiting values of $\lambda$ obtained from sequences of solutions $(\lambda_n, u_n)$ with $|u_n|_1 \to 0$ or $|u_n|_1 \to \infty$. In any case, replacing the intervals $I_k$ used previously with the intervals $I'_k = [\mu_k^0 - \alpha(k)k^2, \mu_k^0 + \alpha(k)k^2]$, all of the above results still hold (any compact interval $\Lambda$ intersects only finitely many of the new intervals $I'_k$, so the proofs are identical). The lengths of the new intervals $I'_k$ are $2\alpha(k)k^2$, and the size of this, for large $k$, depends on the rate of convergence of $(M_0(L) + M_1(L)^2)/L$ to zero as $L \to \infty$; they could be substantially larger than the intervals $I_k$.

(v) The following (linear) problems have no nontrivial solutions and show that we cannot weaken the condition in the previous remark further to $M_0(L) + M_1(L)^2 = O(L)$:

(a) $-u'' = \lambda u + h$, with $u(0) = u(\pi) = 0$, and $h = -\lambda u$;

(b) $-u'' = \lambda u + h$, with $u(0) = u(\pi) = 0$, and $h = 4\lambda^{1/2}u'$.

(vi) It follows from remark (iv) that if $M_0(L) + M_1(L)^2 = o(L)$ and if (3.7) holds, then the projections $P_R(\mathscr{C}_k^\nu)$ and $P_R(\mathscr{D}_k^\nu)$ are bounded and lie in $I'_k$, and the sets $\mathscr{C}_k^\nu$ and $\mathscr{D}_k^\nu$ may intersect (this is the extended version of Theorem 3.4). Theorem 3.3 in [16] gives conditions on the nonlinearity $h$ which ensure that there are constants $R_1, R_2$, with $R_1 < R_2$, such that if $(\lambda, u) \in \mathscr{C}_k^\nu$ then $|u|_0 \le R_1$, while if $(\lambda, u) \in \mathscr{D}_k^\nu$ then $|u|_0 \ge R_2$; thus the sets $\mathscr{C}_k^\nu$ and $\mathscr{D}_k^\nu$ cannot meet and each of $P_R(\mathscr{C}_k^\nu)$ and $P_R(\mathscr{D}_k^\nu)$ must be unbounded. In a sense this is the reverse of the situation just described. In Toland's conditions, $M_0(L) = O(L)$, $M_1(L) \equiv 0$.

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