DISCRETE MATHEMATICS

α-Domination

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Abstract

Let $G=(V,E)$ be any graph with n vertices, m edges and no isolated vertices. For some α with $0 < \alpha \leqslant 1$ and a set $S \subseteq V$, we say that S is α-*dominating* if for all $v \in V-S$, $|N(v) \cap S| \geqslant \alpha |N(v)|$. The size of a smallest such S is called the α-*domination number* and is denoted by $\gamma_\alpha(G)$. In this paper, we introduce α-domination, discuss bounds for $\gamma_{1/2}(G)$ for the King's graph, and give bounds for $\gamma_\alpha(G)$ for a general α, $0 < \alpha \leqslant 1$. Furthermore, we show that the problem of deciding whether $\gamma_\alpha(G) \leqslant k$ is NP-complete.

1. Introduction

Graph terminology not presented here can be found in Chartrand and Lesniak [1]. Let $G=(V,E)$ be a graph with n vertices, m edges and no isolated vertices. Consider an $s \times t$ chessboard. The *King's graph* $K[s \times t]$ is formed by letting each vertex represent one cell of the chessboard and joining two vertices in the graph if and only if the cells represented by these vertices are a King's move apart on the chessboard. (A King can move one cell either horizontally, vertically or diagonally.)

A puzzle mentioned by David Woolbright [15] involves occupying each cell of a 6×6 array with a guard or a prisoner, subject only to the constraint that every prisoner is adjacent to at least as many guards as prisoners (where adjacency is vertical, horizontal or diagonal). It was shown by Mark Liatti [10] that the size of a smallest set of guards is, in this case, 14 as shown in Fig. 1.

This puzzle generalizes to a graph invariant in the following way. We may say that the *Woolbright number* of a graph G is the size of a smallest set of vertices with the

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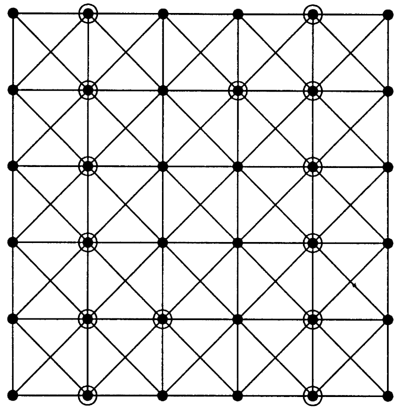

Fig. 1. A half-dominating set for $K[6 \times 6]$.

property that every vertex not in S has at least as many neighbors in S as neighbors not in S. Solving the puzzle above then amounts to finding the Woolbright number of the King's graph $K[6 \times 6]$.

The (*open*) *neighborhood* $N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to v. The *closed neighborhood* $N[v]$ of v is $N(v) \cup \{v\}$. For any set $S \subseteq V$, the *neighborhood* $N(S)$ of S is defined as $\cup_{v \in S} N(v)$, and the *closed neighborhood* $N[S]$ of S is $N(S) \cup S$. Generalizing the Woolbright number, we introduce the concept of α-domination: For any α with $0 < \alpha \leqslant 1$ and a set $S \subseteq V$, we say that S is α*-dominating* if for all $v \in V - S$, $|N(v) \cap S| \geqslant \alpha |N(v)|$. The size of a smallest such S is called the α*-domination number* and is denoted by $\gamma_\alpha(G)$. Thus the Woolbright number of a graph G is $\gamma_{1/2}(G)$. The size of a largest minimal such set S is called the *upper* α*-domination number* and is denoted by $\Gamma_\alpha(G)$.

Recall that a set $S \subseteq V$ is said to *dominate* a graph if every vertex in the graph is either in S or is adjacent to a vertex in S. Stated in the current context, we might say that for every vertex $v \in V - S$, $|N(v) \cap S| \geqslant 1$. The size of a smallest such set is called the *domination number* and is denoted by $\gamma(G)$. The size of a largest minimal dominating set is called the *upper domination number* and is denoted by $\Gamma(G)$. Since a smallest α-dominating set is a dominating set, it is immediate to see that $\gamma(G) \leqslant \gamma_\alpha(G)$ for all G and for every α.

The following results for specific graphs are straightforward and are given without proof.

Proposition 1. *If P_n is a path with n vertices, then*

$$\gamma_\alpha(P_n) = \lceil n/3 \rceil \quad \textit{if } 0 < \alpha \leqslant \tfrac{1}{2},$$
$$\gamma_\alpha(P_n) = \lfloor n/2 \rfloor \quad \textit{if } 1/2 < \alpha \leqslant 1.$$

Proposition 2. *If C_n is a cycle with n vertices, then*

$$\gamma_\alpha(C_n) = \lceil n/3 \rceil \quad \text{if } 0 < \alpha \leqslant \tfrac{1}{2},$$
$$\gamma_\alpha(C_n) = \lceil n/2 \rceil \quad \text{if } 1/2 < \alpha \leqslant 1.$$

Proposition 3. *If K_n is a complete graph with n vertices, then*

$$\gamma_\alpha(K_n) = \lceil \alpha(n-1) \rceil.$$

Proposition 4. *If $K_{m,n}$ is a complete bipartite graph with $1 \leqslant m \leqslant n$, then*

$$\gamma_\alpha(K_{m,n}) = \min\{\lceil \alpha m \rceil + \lceil \alpha n \rceil, m\}.$$

2. Preliminary results

We will be interested in studying the relationship between the α-domination parameter and other domination-related parameters. One can easily find that more than 80 types of domination and domination-related parameters have appeared in the literature (see [9]). Although this new parameter adds to the list in a unique way, it is related to other types of domination.

Fink and Jacobson [5] introduced the concept of r-domination. A set $S \subseteq V$ is *r-dominating* if for every vertex $v \in V - S$, $|N(v) \cap S| \geqslant r$. That is, every vertex not in the r-dominating set S has at least r of its neighbors in S. For k-regular graphs, r-domination and α-domination are the same when $\alpha = r/k$, but for general graphs, these two concepts are different. Dunbar et al. [4] introduced the concept of signed domination. A function $f : V \to \{-1, 1\}$ is called a *signed dominating function* if for every vertex $v \in V$, $f(N[v]) \geqslant 1$. The signed domination number $\gamma_s(G)$ equals the minimum weight of a signed dominating function f of G. Note that if f is a signed dominating function, for every u with $f(u) = -1$, more than half of u's neighbors have weight 1. Thus, signed domination is similar to, but not the same as, $\frac{1}{2}$-domination.

We have already mentioned that for any graph G the standard domination parameter $\gamma(G)$ is a lower bound for $\gamma_\alpha(G)$. An upper bound is found by examining the vertex cover number. The *vertex cover number* $\alpha_0(G)$ is the size of a smallest set of vertices S such that every edge has at least one end vertex in S. Clearly, $\gamma_1(G) = \alpha_0(G)$.

For any graph G and for any α, with $0 < \alpha \leqslant 1$, if S is any set of minimum size which α-dominates G, we will call S a *γ_α-set*. Similarly, if S is a set of minimum size which dominates G, we call S a *γ-set*. An immediate result is the following: If $\alpha_1 < \alpha_2$, then $\gamma_{\alpha_1}(G) \leqslant \gamma_{\alpha_2}(G)$. Since $\gamma_1(G) = \alpha_0(G)$, we obtain the following proposition.

Proposition 5. *For any graph G, $\gamma(G) \leqslant \gamma_\alpha(G) \leqslant \alpha_0(G)$ for all α with $0 < \alpha \leqslant 1$.*

Next sufficient conditions are examined to guarantee that the parameter $\gamma_\alpha(G)$ equals its upper or lower bound. In any graph G, we will denote the maximum (minimum) degree of a vertex by $\Delta(G)$ (respectively, $\delta(G)$).

Proposition 6. *If G has maximum degree $\Delta(G)$, then the following holds: if $0 < \alpha \leqslant 1/\Delta(G)$, then $\gamma_\alpha(G) = \gamma(G)$.*

Proof. Since every γ_α-set is a dominating set, $\gamma(G) \leqslant \gamma_\alpha(G)$ for all α. Let S be a γ-set. If $v \notin S$, since S is a dominating set we know $|N(v) \cap S| \geqslant 1$ and by definition of $\Delta(G)$, $|N(v)| \leqslant \Delta(G)$. So $|N(v) \cap S| \geqslant 1 \geqslant \alpha\Delta(G) \geqslant \alpha|N(v)|$. Thus S is an α-dominating set and hence $\gamma_\alpha(G) \leqslant |S| = \gamma(G)$. □

The converse to Proposition 6 does not hold. The *double star* is the graph obtained by joining the vertices of maximum degree in the two stars $K_{1,r}$ and $K_{1,s}$. This graph has $\gamma_\alpha(G) = 2 = \gamma(G)$, even when $\alpha = \frac{1}{2} > 1/\Delta(G)$.

Proposition 7. *Let G be a graph. If $1 \geqslant \alpha > (\Delta(G) - 1)/\Delta(G)$, then $\gamma_\alpha(G) = \alpha_0(G)$.*

Proof. Suppose G is a graph and S is a γ_α-set, with $\alpha > (\Delta(G) - 1)/\Delta(G)$. Let v be a vertex with $v \notin S$. Then $|N(v) \cap S| > (\Delta(G) - 1)|N(v)|/\Delta(G)$. Thus $\Delta(G)|N(v) \cap S| > \Delta(G)|N(v)| - |N(v)|$. Let $k \geqslant 0$ be an integer satisfying $k = |N(v) - S|$. Then $|N(v)| - k = |N(v) \cap S|$, and so $\Delta(G)[|N(v)| - k] > \Delta(G)|N(v)| - |N(v)|$. Thus $\Delta(G)k < |N(v)|$. If $k \geqslant 1$, this contradicts the definition of maximum degree. Hence $k = 0$. Since v was an arbitrary vertex outside S, it must be true that every vertex in $V - S$ has its open neighborhood contained in S. So there is no edge in G with both end vertices in $V - S$. Consequently, S is a vertex cover for G and $\alpha_0(G) \leqslant |S|$. Since the reverse inequality always holds, we must have equality. □

Thus we have established sufficient conditions for the α-domination number to achieve two parameters which bound it. Clearly, equality is obtained for any graph with equal domination and vertex covering number. These are examined in [8,13]. One such class is trees whose non-stem vertices form an independent set. (A stem is a vertex adjacent to a vertex of degree 1.)

It seems natural to search for other sufficient conditions which imply that $\gamma_\alpha(G) = \gamma(G)$ or $\gamma_\alpha(G) = \alpha_0(G)$. However, consider the family of complete graphs $G = K_n$ for $n \geqslant 4$. If $1/\Delta(G) < \alpha \leqslant (\Delta(G) - 1)/\Delta(G)$, then $\gamma(G) < \gamma_\alpha(G) < \alpha_0(G)$. So any condition that guarantees $\gamma_\alpha(G) = \gamma(G)$ for *all* graphs G must also imply that $\alpha/\Delta(G)$, and any condition that guarantees $\gamma_\alpha(G) = \alpha_0(G)$ for all graphs G must imply that $\alpha > (\Delta(G) - 1)/\Delta(G)$.

It is also natural at this point to consider the question of intermediate values; i.e., if $\gamma(G) = r$, $\alpha_0(G) = t$ and $r < s < t$, is there always a value of α for which $\gamma_\alpha(G) = s$? This question is answered in the negative by the following example: Consider a cycle C_n, for $n \geqslant 11$. This graph has $\gamma(C_n) = \lceil n/3 \rceil$ and $\alpha_0(C_n) = \lceil n/2 \rceil$. However, by Proposition 2, $\gamma_\alpha(C_n)$ is equal to $\gamma(C_n)$ or $\alpha_0(C_n)$ for all α.

3. Nordhaus–Gaddum-type results

For any graph G with n vertices, the *complement of* G, denoted $\bar{G}$, is defined in the following way: $V(\bar{G}) = V(G)$, and two vertices u and v are adjacent in $\bar{G}$ if and only if u and v are not adjacent in G. Nordhaus and Gaddum [12] provided best possible bounds on the sum of the chromatic number of a graph and its complement. We seek bounds for the sum of the α-domination number of a graph and its complement.

First, we state the following result which follows from an unpublished theorem by Cowen and Emerson (1985). Its proof appears in a survey article by Woodall [14]. For any graph $G = (V, E)$ and a set $C = \{1, 2, \ldots, k\}$, we define a *coloring* of G to be a function $f : V \to C$. We denote by S_i the set of all vertices with label i.

Theorem 8. *Let $G = (V, E)$ be a graph, let $C = \{1, 2, \ldots, k\}$ and let $p_1, p_2, \ldots, p_k$ be non-negative real numbers such that $\Sigma_i p_i \geqslant 1$. Then there exists a coloring $f : V \to C$ such that $|N(v) \cap S_i| \leqslant p_{f(v)} |N(v)|$ for each vertex v.*

The following result on the α-domination number follows from Theorem 8.

Theorem 9. *If $0 < \alpha < 1$, then for any graph G, $\gamma_\alpha(G) + \gamma_{1-\alpha}(G) \leqslant n$.*

Proof. We let $k = 2$, let $p_1 = \alpha$, and $p_2 = 1 - \alpha$. Then Theorem 8 provides a coloring of the vertices of G such that $|N(v) \cap S_i| \leqslant p_{f(v)} |N(v)|$ for each vertex v.

Suppose now that $v \notin S_1$, i.e., $f(v) = 2$. By Theorem 8 $|N(v) \cap S_2| \leqslant p_2 |N(v)| = (1 - \alpha)|N(v)|$. Hence

$$|N(v) \cap S_1| = |N(v)| - |N(v) \cap S_2| \geqslant |N(v)| - (1 - \alpha)|N(v)| = \alpha|N(v)|.$$

Since this is true for any vertex not in S_1, we see that S_1 forms an α-dominating set for G. In the same way it is easy to see that S_2 is a $(1 - \alpha)$-dominating set for G. Therefore, $\gamma_\alpha(G) + \gamma_{1-\alpha}(G) \leqslant |S_1| + |S_2| = n$. $\square$

The upper bound in Theorem 9 is achieved for a complete graph K_n whenever $\alpha(n - 1)$ is not an integer. However, a good lower bound for this expression is not possible. If G is a star $(K_{1,n-1})$, then $\gamma_\alpha(G) + \gamma_{1-\alpha}(G) = 2$ for all values of α. Further, if G is a complete graph on n vertices, we can use Proposition 3 to see that $n - 1 \leqslant \gamma_\alpha(G) + \gamma_{1-\alpha}(G) \leqslant n$ for all values of α.

The following corollaries are immediate.

Corollary 10. *For any graph G, $\gamma_{1/2}(G) \leqslant \lfloor n/2 \rfloor$.*

This bound is sharp for all n, since $\gamma_{1/2}(K_n) = \lceil (n-1)/2 \rceil$. The following corollary is of the Nordhaus-Gaddum type for $\alpha = \frac{1}{2}$.

Corollary 11. *If G and $\bar{G}$ have no isolates, then*

$$\gamma_{1/2}(G) + \gamma_{1/2}(\bar{G}) \leqslant n.$$

This bound is sharp as shown by examining C_4. We do not have a Nordhaus–Gaddum-type result for general α. However, it is possible to obtain a generalization for Theorem 9 in a natural way.

Corollary 12. *Let G be a graph. If for some $k \geqslant 2$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k$, where $0 < \alpha_i < 1$ for all i, then $\gamma_{\alpha_1}(G) + \gamma_{\alpha_2}(G) + \cdots + \gamma_{\alpha_k}(G) \leqslant kn/2$.*

Proof. The proof is by induction on k. If $k=2$, we have $\alpha_1 + \alpha_2 \leqslant 1$. By Theorem 9 we have $\gamma_{\alpha_1}(G) + \gamma_{\alpha_2}(G) \leqslant \gamma_{\alpha_1}(G) + \gamma_{1-\alpha_1}(G) \leqslant n$. Assume that $k > 2$ and the result holds for integers less than k. At least one of the values α_i must satisfy $\alpha_i \leqslant \frac{1}{2}$. Without loss of generality assume $\alpha_k \leqslant \frac{1}{2}$. Then from Corollary 10 we know $\gamma_{\alpha_k}(G) \leqslant n/2$. Finally, using the inductive hypothesis, we see that $[\gamma_{\alpha_1}(G) + \gamma_{\alpha_2}(G) + \cdots + \gamma_{\alpha_{k-1}}(G)] + \gamma_{\alpha_k}(G) \leqslant (k-1)n/2 + n/2 = kn/2$. $\square$

A *universal vertex* is a vertex of degree $n-1$. A slightly more general result than Corollary 11 is obtained.

Corollary 13. *If G and $\bar{G}$ have no isolates, and if $0 < \alpha < 1$, then*

$$\gamma_{\alpha}(G) + \gamma_{1-\alpha}(\bar{G}) \leqslant \lfloor (3n/2) \rfloor - 2.$$

Proof. Without loss of generality, we may assume that $\alpha \leqslant \frac{1}{2}$. Then $\gamma_{\alpha}(G) \leqslant \gamma_{1/2}(G) \leqslant \lfloor n/2 \rfloor$. Since G has no isolates, $\bar{G}$ has no universal vertex. So there are two vertices, x and y, such that xy is not an edge in $\bar{G}$. The set $V - \{x, y\}$ is thus a vertex cover for $\bar{G}$ of cardinality $n-2$. Therefore, $\gamma_{1-\alpha}(\bar{G}) \leqslant \alpha_0(\bar{G}) \leqslant n-2$. Combining these we see that $\gamma_{\alpha}(G) + \gamma_{1-\alpha}(\bar{G}) \leqslant \lfloor (3n/2) \rfloor - 2$. $\square$

It is straightforward to see that this upper bound is achieved if $G = mK_2$ and $\alpha \leqslant 1/(n-2)$.

4. Bounds on $\gamma_{\alpha}(G)$

Let S be an α-dominating set with $|S| = \gamma_{\alpha}(G)$. Let M be the set of edges between S and $V - S$. Counting the edges from S to $V - S$, we see that $|M| \leqslant \Sigma_{v \in S} \deg(v)$. Further, counting the number of edges from $V - S$ to S, we see that $|M| \geqslant \Sigma_{v \in V-S} \alpha \deg(v)$.

Combining these it is clear that

$$\Delta(G)|S| \geqslant \Sigma_{v\in S}\deg(v) \geqslant \Sigma_{v\in V-S}\alpha\deg(v) \geqslant \alpha\delta(G)|V-S|. \tag{1}$$

Using (1) we obtain the following proposition.

Proposition 14. *For any graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$,*

$$\gamma_\alpha(G) \geqslant \frac{\alpha\delta(G)n}{\Delta(G)+\alpha\delta(G)}.$$

Using Theorem 9 we can obtain an upper bound on $\gamma_\alpha(G)$.

Proposition 15. *For any graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$,*

$$\gamma_\alpha(G) \leqslant \frac{\Delta(G)n}{\Delta(G)+(1-\alpha)\delta(G)}.$$

Proof. By Theorem 9, we know that $\gamma_\alpha(G) \leqslant n-\gamma_{1-\alpha}(G)$. Then using Proposition 14, we see that this quantity is no larger than

$$n - \frac{(1-\alpha)\delta(G)n}{\Delta(G)+(1-\alpha)\delta(G)}$$

and the result follows. □

The following corollary is immediate.

Corollary 16. *For any tree T and for any α with $0<\alpha\leqslant 1$,*

$$\frac{\alpha n}{\Delta(T)+\alpha} \leqslant \gamma_\alpha(T) \leqslant \frac{\Delta(T)n}{\Delta(T)+1-\alpha}.$$

A different lower bound is obtained by again using (1).

Proposition 17. *For any graph G with m edges,*

$$\gamma_\alpha(G) \geqslant \frac{2\alpha m}{(\alpha+1)\Delta(G)}.$$

Proof. First, we may observe that since

$$\Sigma_{v\in S}\deg(v) \geqslant \Sigma_{v\in V-S}\alpha\deg(v)$$

we also obtain

$$\alpha\Sigma_{v\in S}\deg(v) + \Sigma_{v\in S}\deg(v) \geqslant \alpha\Sigma_{v\in S}\deg(v) + \Sigma_{v\in V-S}\alpha\deg(v)$$

$$(\alpha+1)|S|\Delta(G) \geqslant (\alpha+1)\Sigma_{v\in S}\deg(v) \geqslant \alpha\Sigma_{v\in V}\deg(v) = \alpha 2m$$

and the result follows. □

We may also use Proposition 17 to find another upper bound. The proof is similar to that of Proposition 15 and is omitted here.

Proposition 18. *For any graph G with maximum degree $\Delta(G)$ and m edges,*

$$\gamma_\alpha(G) \leqslant \frac{(2-\alpha)\Delta(G)n - (2-2\alpha)m}{(2-\alpha)\Delta(G)}.$$

Next, we will consider bounds for regular graphs. If every vertex of a graph G has degree k, we say that G is *k-regular*. Clearly, if G is k-regular, then $\gamma_{1/k}(G) = \gamma(G)$. In a k-regular graph, the number of edges $m = kn/2$ and $\Delta(G) = \delta(G) = k$. The next corollary follows from Propositions 17 and 18.

Corollary 19. *For a k-regular graph G, and for any α with $0 < \alpha \leqslant 1$,*

$$\frac{\alpha n}{1+\alpha} \leqslant \gamma_\alpha(G) \leqslant \frac{n}{2-\alpha}.$$

Letting $\alpha = i/k$ where i is an integer with $1 \leqslant i \leqslant k$, we obtain the following lower bound.

Corollary 20. *If G is a k-regular graph and i is an integer with $1 \leqslant i \leqslant k$, then $\gamma_{i/k}(G) \geqslant \lceil [i/(i+k)]n \rceil$.*

To see that this result is sharp, let H be any $(k-i)$-regular graph with $2k$ vertices, and let S be a set of $2i$ independent vertices. Construct G by forming the join of i of the vertices of S with k of the vertices of H, and then joining the rest of the S vertices with the rest of the vertices of H. It is immediate to see that G is k-regular, that $n = 2k + 2i$, and that the set S is a $\gamma_{i/k}$-dominating set for G of cardinality $2i = [i/(i+k)]n$.

Corollary 19 provides an upper bound as well.

Corollary 21. *If G is k-regular, then*

$$\gamma_{i/k}(G) \leqslant \left\lceil \frac{kn}{2k-i} \right\rceil.$$

5. The King's graph

We return to the original problem of finding $\gamma_{1/2}(G)$, where $G = K[s \times t]$. We may think of the King's graph $K[s \times t]$ as having s rows of vertices and t columns of vertices. For any $K[s \times t]$ graph, if S is a $\gamma_{1/2}$-set for the graph, then we say a column is *empty* if the column has no vertices in S. Similarly, a *full* column is one which has all its vertices in S. First, we look at small King's graphs. It is immediate to see that $\gamma_{1/2}(K[2 \times 1]) = 1$. Moreover, $\gamma_{1/2}(K[2 \times 2]) = \gamma_{1/2}(K[2 \times 3]) = 2$. Finally, $\gamma_{1/2}(K[2 \times 4]) = 4$. For larger values of t we have the following result.

Proposition 22. *For a* $2 \times t$ *King's graph,* $\gamma_{1/2}(K[2 \times t]) = t - 1$, *for all* $t \geqslant 5$.

Proof. We will use induction on t to show that $t-1$ is an upper bound for $\gamma_{1/2}(K[2\times t])$. Let G be a $K[2 \times 5]$ graph. Then the vertices in the second and fourth columns $\frac{1}{2}$-dominate the graph, and so $\gamma_{1/2}(K[2 \times 5]) \leqslant 4$. The vertices in the second and fifth columns together with a vertex from the third column show that $\gamma_{1/2}(K[2 \times 6]) \leqslant 5$. Next assume that $\gamma_{1/2}(K[2 \times k]) \leqslant k - 1$ for all k with $5 \leqslant k \leqslant t - 1$, and consider the graph $K[2 \times t]$ for some $t \geqslant 7$. If the first two columns are omitted, the resulting graph has a $\gamma_{1/2}$-set S' with cardinality at most $t-3$. Now, adding in the second column to S' yields a set S which $\frac{1}{2}$-dominates the graph $K[2 \times t]$. Thus $\gamma_{1/2}(K[2 \times t]) \leqslant |S| = t - 1$.

To show that $t-1$ is a lower bound for $\gamma_{1/2}(K[2 \times t])$, assume S is a $\gamma_{1/2}$-set. Let x be the number of columns in the graph with both vertices in S. Let y be the number of columns with exactly one vertex in S, and let z be the number of columns in the graph with no vertices in S. It is immediate to see that an empty column (except column t) is followed by a full column within two steps to the right. Thus $z \leqslant x + 1$. Clearly, $|S| = 2x + y$. Therefore, we have $t = x + y + z \leqslant 2x + y + 1$. So $t - 1 \leqslant |S|$, and the result follows. □

The following lemma will be useful in determining the Woolbright number of a $3 \times t$ King's graph.

Lemma 23. *If* G *is a* $3 \times t$ *King's graph, then there exists a* $\gamma_{1/2}$*-set* S *with the property that following every empty column* (*except column* t) *there is a full column within two steps to the right.*

Proof. Suppose G is a $3 \times t$ King's graph with a $\gamma_{1/2}$-set S and assume column j is empty. If column $j+1$ is full, the result holds. If column $j+1$ has one vertex in S, it must be the vertex in the middle row, in which case column $j+2$ is full and we are done.

Suppose column $j+1$ has two vertices in S. Then column $j+2$ must have at least two vertices in S to cover the vertex in column $j+1$ which is not in S. We may create a $\gamma_{1/2}$-set S' in the following way. Shift one vertex of S from column $j+1$ to the empty spot in column $j+2$, and move the remaining vertex (if necessary) to the middle position in column $j+1$. The set S' is then a $\frac{1}{2}$-dominating set with the same number of vertices as S.

When this process is repeated for every empty column except for column t, the result follows. □

Proposition 24. *If* G *is a* $3 \times t$ *King's graph, then* $\gamma_{1/2}(G) = t$.

Proof. Let G be a $3 \times t$ King's graph. Then a $\frac{1}{2}$-dominating set is found by choosing all vertices on the second row of G. Thus if S is a $\gamma_{1/2}$-set for G, we know $|S| \leqslant t$.

Next, we show that t is also a lower bound for $\gamma_{1/2}(G)$. Let S be any $\gamma_{1/2}$-set for G. Let w be the number of columns of G which have all three vertices in S. Similarly, let x (respectively, y and z) be the number of columns of G with two (respectively, one and zero) vertices in S. Then $|S| = 3w + 2x + y$ and $t = x + y + z + w$. By Lemma 23 we know that $z \leqslant w + 1$. Thus $t \leqslant 2w + x + y + 1$ and consequently $|S| - t \geqslant w + x - 1$.

If $w = x = 0$, then we must also have $z = 0$, and so every column has one vertex in S. Thus $|S| = t$ and we are done. If at least one of w or x is positive, then we have $|S| \geqslant t + w + x - 1 \geqslant t$. In either case t is a lower bound for the cardinality of S, and the result follows. □

For larger $K[s \times t]$ graphs, we seek bounds for the $\frac{1}{2}$-domination number. The first is obvious from the previous proof.

Proposition 25. *Let k and t be positive integers. Then*

$$\gamma_{1/2}(K[(2k+1) \times t]) \leqslant kt.$$

Proof. We may find a $\frac{1}{2}$-dominating set for such a graph by choosing all t vertices in each of the k even-numbered rows of the graph. □

Next, we examine graphs of the form $K[s \times t]$, with s an even integer, and t sufficiently large. First, we note that, $\gamma_{1/2}(K[4 \times 4]) \leqslant 7$, by taking as our $\frac{1}{2}$-dominating set all the vertices in the third column, together with the vertices v_{11}, v_{31}, v_{32} where v_{ij} denotes the vertex in row i and column j. It is straightforward to see that six vertices will not suffice as a $\gamma_{1/2}$-dominating set for $K[4 \times 4]$, and hence $\gamma_{1/2}(K[4 \times 4]) = 7$. □

Proposition 26. *If $G = K[4 \times t]$, then $\gamma_{1/2}(G) \leqslant 2t - 2$ for $t \geqslant 5$.*

Proof. By selecting the second and fourth columns of $K[4 \times 5]$, a $\frac{1}{2}$-dominating set is found with eight vertices, so $\gamma_{1/2}(K[4 \times 5]) \leqslant 8$. Moreover, for $t = 6$, by selecting the second and fifth columns, together with two vertices from different rows in columns three or four, we have $\gamma_{1/2}(K[4 \times 6]) \leqslant 10$. Thus the result holds for $t = 5$ and 6. Suppose G is a $K[4 \times t]$ graph for some $t \geqslant 7$ and assume that the result holds for all $K[4 \times j]$ graphs with $5 \leqslant j < t$. Let G' be the graph obtained from G by omitting the last two columns. Then $G' = K[4 \times (t-2)]$. Let S' be a $\gamma_{1/2}$-set for G'. Then $|S'| \leqslant 2(t-2) - 2$ by our inductive hypothesis. Create a $\frac{1}{2}$-dominating set S for G by adding all the vertices of column $t-1$ to the vertices in S'. Then $|S| \leqslant 2t - 6 + 4 = 2t - 2$ and the result follows. □

That the bound in Proposition 26 is not always sharp is seen by noting that there exists a $\frac{1}{2}$-dominating set of size 13 for the graph $K[4 \times 8]$, as shown in Fig. 2. Thus for the graph $G = K[4 \times 8]$, $\gamma_{1/2}(G) < 14$.

It should be recalled here that the Woolbright number for the graph $K[6 \times 6]$ has been shown to be 14 [15]. Further, by noting that the graph $K[6 \times 5]$ has the same

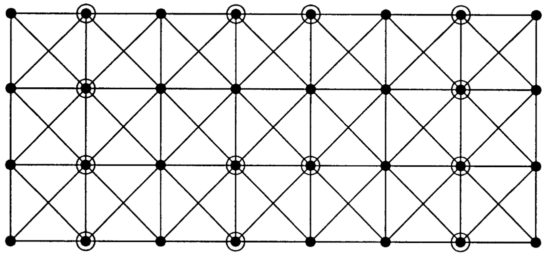

Fig. 2. A half-dominating set for $K[4 \times 8]$.

$\frac{1}{2}$-domination number as the graph $K[5 \times 6]$, Proposition 25 yields the value 12 as an upper bound for $\gamma_{1/2}(K[6 \times 5])$. Using these two facts as the basis statements, an inductive argument yields the following upper bound.

Proposition 27. *For* $t \geqslant 5$, $\gamma_{1/2}(K[6 \times t]) \leqslant 3t - 3$.

Finally, we may use these propositions to find an upper bound for $\gamma_{1/2}(K[s \times t])$ for any even integer $s \geqslant 6$. Suppose $s = 2k$ for some $k \geqslant 3$. Then using Proposition 25 we again find that $\gamma_{1/2}(K[2k \times 5]) = \gamma_{1/2}(K[5 \times 2k]) \leqslant 2(2k) \leqslant (k)(5) - 3$. And using Proposition 27 we obtain $\gamma_{1/2}(K[2k \times 6]) \leqslant 3(2k) - 3 = 6k - 3$. With these facts as a basis step, the proof of the following proposition follows by induction as before.

Proposition 28. *For* $k \geqslant 3$ *and* $t \geqslant 5$, $\gamma_{1/2}(K[2k \times t]) \leqslant kt - 3$.

6. α-Irredundance and α-independence

We begin with two definitions. Let $S \subseteq V$. We say S is an *irredundant set* if for all $x \in S$ there is a vertex $y \in N[x]$ which is not in $S - \{x\}$ such that $|N(y) \cap (S - \{x\})| = 0$. We let $\mathrm{IR}(G)$ ($\mathrm{ir}(G)$) denote the size of a largest (respectively, smallest maximal) irredundant set of vertices in G. Any minimal dominating set is a maximal irredundant set. We say S is an *independent set* if no two vertices in S are adjacent, or equivalently, if $v \in S$, $|N(v) \cap S| = 0$. Let $\beta(G)$ ($i(G)$) denote the size of a largest (respectively, smallest maximal) independent set of vertices. Any maximal independent set is a minimal dominating set. This gives the inequality for all graphs G [2]

$$\mathrm{ir}(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \beta(G) \leqslant \Gamma(G) \leqslant \mathrm{IR}(G).$$

We say S is an α-*irredundant* set if the following condition holds. For all $x \in S$ there is a vertex $y \in N[x]$ which is not in $S - \{x\}$ such that $|N(y) \cap (S - \{x\})| < \alpha|N(y)|$.

It may be noted here that every minimal α-dominating set is a maximal α-irredundant set. We will let $\mathrm{IR}_\alpha(G)$ ($\mathrm{ir}_\alpha(G)$) denote the size of a largest (respectively, smallest maximal) α-irredundant set of vertices in G.

In light of the relationship between α-domination and α-irredundance, the following inequality chain emerges.

$$\mathrm{ir}_\alpha(G) \leqslant \gamma_\alpha(G) \leqslant \Gamma_\alpha(G) \leqslant \mathrm{IR}_\alpha(G).$$

Lemma 29. *Let $\alpha_1 < \alpha_2$ be positive real numbers both less than 1. Then for any graph G,*

$$\mathrm{IR}_{\alpha_1}(G) \leqslant \mathrm{IR}_{\alpha_2}(G).$$

Proof. Let S be an α_1-irredundant set with $|S| = \mathrm{IR}_{\alpha_1}(G)$. We claim that S is an α_2-irredundant set. Let $x \in S$. Then there exists a $y \notin S - \{x\}$, with $y \in N[x]$, such that $|N(y) \cap (S - \{x\})| < \alpha_1 |N(y)| < \alpha_2 |N(y)|$ and the result follows. □

Proposition 30. *For any graph G and any $0 < \alpha \leqslant 1$, $\mathrm{IR}(G) \leqslant \mathrm{IR}_\alpha(G)$.*

Proof. Let S be an IR-set. Then for all $x \in S$, there is a vertex $y \notin S - \{x\}$, with $y \in N[x]$ and with $|N(y) \cap (S - \{x\})| = 0 < \alpha |N(y)|$. So $\mathrm{IR}_\alpha(G) \geqslant |S| = \mathrm{IR}(G)$. □

Using Proposition 30 we have the following inequality:

$$\Gamma(G) \leqslant \mathrm{IR}(G) \leqslant \mathrm{IR}_\alpha(G).$$

On the other hand, $\Gamma_\alpha(G)$ is not comparable with the standard upper domination parameters. Consider an $s \times t$ chessboard. The *Rook's graph* $R[s \times t]$ is formed by letting each vertex represent one cell of the chessboard and joining two vertices in the graph if and only if the cells represented by these vertices are a Rook's move apart on the chessboard. (A Rook can move any number of squares either horizontally or vertically.) In the graph $R[5 \times 5]$ (the Rook's graph on a $5X5$ board), we see that $\Gamma_{1/8}(G) = \Gamma(G) = 5$, while $\mathrm{IR}(G) = 6$. The *prism* is the graph constructed by joining two copies of C_3 by a perfect matching. In the prism, we have $\Gamma_{2/3}(G) = 4$ while $\mathrm{IR}(G) = 3$. Furthermore for this graph, $\mathrm{IR}_{3/4}(G) = 4$, so the inequality of Proposition 30 can be strict. To see that $\Gamma_\alpha(G)$ is not comparable with $\Gamma(G)$, again the prism provides an example of a graph with $\Gamma_{2/3}(G) = 4$ and $\Gamma(G) = 3$. Finally, construct a graph G by adding two leaves to each vertex of a path on three vertices. This graph has $\Gamma_{2/3}(G) = 5$ while $\Gamma(G) = 6$.

For any graph G, we know that $\mathrm{IR}(G) \leqslant n - \delta(G)$ [3]. Although this inequality does not hold for α-irredundance, (again using the prism) we do know that $\mathrm{IR}_\alpha(G) \leqslant n - \gamma(G)$ as shown below.

Proposition 31. *For any graph G, $\mathrm{IR}_\alpha(G) \leqslant n - \gamma(G)$.*

Proof. Let S be an α-irredundant set with $|S| = \mathrm{IR}_\alpha(G)$. We claim that $V - S$ is a dominating set. For if not we may assume that there is a vertex $x \in S$ which is not dominated by $V - S$. Then $N[x] \subseteq S$. But since S is an α-irredundant set, there is a vertex y in $N[x]$ such that y is not in $S-\{x\}$, for which $|N(y)\cap(S-\{x\})| < \alpha|N(y)|$ which is a contradiction. $\square$

We say S is an *α-independent set* if S is an independent set that is also an α-dominating set. We let $\beta_\alpha(G)$ ($i_\alpha(G)$) denote the size of a largest (respectively, smallest maximal) α-independent set. Note that not every graph has an α-independent set for all α. For example, the graph K_n for $n\geqslant 3$, does not have an α-independent set for any $\alpha > 1/(n-1)$. On the other hand, every graph has an α-independent set for $\alpha/\Delta(G)$ and here $i_\alpha(G)=i(G)$. Now, if S is an α-independent set for some α, $0<\alpha$, then S is a minimal α-dominating set, since the set $S-\{x\}$ is no longer dominating for all $x\in S$. Thus we get the following inequalities for those graphs G which have an α-independent set.

$$\gamma_\alpha(G)\leqslant i_\alpha(G)\leqslant \beta_\alpha(G)\leqslant \Gamma_\alpha(G),$$
$$i(G)\leqslant i_\alpha(G)\leqslant \beta_\alpha(G)\leqslant \beta(G).$$

A classical theorem in graph theory is due to Gallai [6].

Proposition 32 (Gallai [6]). *For any graph G, $\beta_0(G)+\alpha_0(G)=n$.*

We use Gallai's result to obtain another bound on the α-domination number.

Proposition 33. *Suppose that G has an α-independent set for some α, $0<\alpha\leqslant 1$. Then $\gamma_\alpha(G)\leqslant \lfloor n/2\rfloor$.*

Proof. By the above inequalities and by Gallai's theorem, $\gamma_\alpha(G)\leqslant i_\alpha(G)\leqslant \beta(G)=n-\alpha_0(G)$. Furthermore, $\gamma_\alpha(G)\leqslant \alpha_0(G)$. Thus, $\gamma_\alpha(G)\leqslant n/2$. $\square$

7. Complexity results

In this section, we will consider the complexity of computing $\gamma_\alpha(G)$. We will state the corresponding decision problem and show that it is NP-complete. In [7], the following two decision problems are shown to be NP-complete:

VERTEX-COVER
Instance: Graph $G=(V,E)$, positive integer k.
Question: Does G have a vertex cover S with $|S|\leqslant k$?
3-REGULAR DOMINATION
Instance: 3-regular graph $G=(V,E)$, positive integer k.
Question: Does G have a dominating set S with $|S|\leqslant k$?

For each α, $0<\alpha$, we will show that the following decision problem is NP-complete. The proof is based on a result by McRae [11].

α-DOMINATION

Instance: Graph $G=(V,E)$, positive integer k.

Question: Does G have an α-dominating set S with $|S|\leqslant k$?

Theorem 34. *For every α with $0<\alpha\leqslant 1$, the problem α-DOMINATION is NP-complete.*

Proof. We prove the theorem by considering the following cases separately: (1) $\alpha=1$; (2) $0<\alpha\leqslant\frac{1}{3}$; (3) $\frac{1}{3}<\alpha<1$.

Case 1: If $\alpha=1$, then $\gamma_\alpha(G)=\alpha_0(G)$, the vertex covering number of G and since VERTEX COVER is known to be NP-complete [7], the result follows.

Case 2: $0<\alpha\leqslant 1/3$. Suppose G is a 3-regular graph, then by Proposition 6, $\gamma_\alpha(G)=\gamma(G)$. Since 3-REGULAR DOMINATION is known to be NP-complete [7], the result follows.

Case 3: $1/3<\alpha<1$. For this case, we will use a transformation from 3-REGULAR DOMINATION. Suppose α is a fixed number such that $\frac{1}{3}<\alpha<1$, and let x be the smallest integer such that $(x+1)/(x+3)\geqslant\alpha$. Let y be the largest integer with $y>x$ and $x/y\geqslant\alpha$. Let Y be a complete graph on $y+1$ vertices, and let $X=\{v_1,v_2,\ldots,v_x\}$ be any x vertices of Y.

Now, let G be a 3-regular graph. Join each vertex of X to every vertex of G, and denote the resulting graph as G^*. Since α, x and y are fixed, this transformation can be done in polynomial time. We will show that G has a dominating set of size at most k if and only if G^* has an α-dominating set of size at most $x+k$.

Suppose first that G has a dominating set S with $|S|\leqslant k$. Form the set $D=S\cup X$. It is straightforward to verify that this is an α-dominating set for G^* with $|D|\leqslant x+k$.

Now, suppose G^* has an α-dominating set with size at most $x+k$. Out of the α-dominating sets of G^* of size at most $x+k$, choose one which uses as many vertices from X as possible. Call this set D. Then there must be a vertex in $Y-X$ that is not in D. We will show that the set $D\cap V(G)$ forms a dominating set for G of size at most k.

Claim 1. *$D\cap V(G)$ is a dominating set for G.*

Proof of Claim 1. Let $v\in V(G)-D$. By the choice of x, $(x+1)/(x+3)\geqslant\alpha>x/(x+2)$, so that $\alpha(x+3)>x(x+3)/(x+2)>x$. Now, $\deg(v)=x+3$, and since D is an α-dominating set for G^*, $|N(v)\cap D|\geqslant\alpha(x+3)>x$, so that $|N(v)\cap D|\geqslant x+1$ and v must have a neighbor in $D\cap V(G)$. Thus $D\cap V(G)$ is a dominating set for G.

Claim 2. $|D\cap V(G)|\leqslant k$.

Proof of Claim 2. Let $u\in Y-(X\cup D)$. By the choice of y, $x/y\geqslant\alpha>x/(y+1)$ so that $\alpha y>x-\alpha$. Now, $|N(u)\cap D|\geqslant\alpha y>x-\alpha>x-1$. Thus $|D\cap Y|\geqslant x$ so that $|D\cap V(G)|\leqslant k$.

Since the 3-REGULAR DOMINATION problem is known to be NP-complete [7], the α-DOMINATION problem is also NP-complete for $1/3 < \alpha < 1$.

Combining all the cases, the theorem is proved. □

8. Open problems

We close with the following open problems:

1. For n odd and $\alpha > \frac{1}{2}$, we note that $\gamma_\alpha(C_n) > \gamma_\alpha(P_n)$. Thus it is possible for $\gamma_\alpha(G)$ to increase when an edge is added to G. We have not tried to ascertain which graphs have this property or the effect on $\gamma_\alpha(G)$ caused by deleting an edge or a vertex from G.
2. If the value of α ranges between 0 and 1, we have minimal results concerning the behavior of the sum $\gamma_\alpha(G) + \gamma_{1-\alpha}(G)$. We know that for some classes of graphs (paths, cycles, stars) this value is constant and does not change with α. Further, we know that for complete graphs, the value alternates between n and $n-1$, depending on whether $\alpha(n-1)$ is an integer. But for graphs outside these classes and for general α, we have not addressed the problem.
3. Although we have a Nordhaus–Gaddum-type bound for $\alpha = \frac{1}{2}$, a bound for general α has not been established. Also characterizations of graphs achieving the bounds in Section 3 have not been found.
4. There exist examples of graphs for which $\mathrm{ir}_\alpha(G) > \mathrm{ir}(G)$ for some α. For example, the graph K_n for $n \geqslant 4$ has $\mathrm{ir}(K_n) = 1$, and $\mathrm{ir}_{1/2}(K_n) = \lfloor n/2 \rfloor$. We do not know whether $\mathrm{ir}_\alpha(G)$ is comparable to $\mathrm{ir}(G)$ for all G.

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