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Mehler Integral Transforms Associated with Jacobi Functions with Respect to the Dual Variable

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We prove a Mehler representation for Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the dual variable λ. We exploit this representation to define a pair of dual integral transforms $\chi_{\alpha,\beta}$ and its transposed ${}^t\chi_{\alpha,\beta}$. We define two second order difference operators $P_{\alpha,\beta}$ and Q such that $\varphi_\lambda^{(\alpha,\beta)}(t)$ is an eigenfunction of $P_{\alpha,\beta}$ with respect to the dual variable λ, and $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are permutation operators between $P_{\alpha,\beta}$ and Q. Next we give some spaces of functions on which $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are isomorphisms and we establish inversion formulas for these transforms.

INTRODUCTION

A Jacobi function $t \to \varphi_\lambda^{(\alpha,\beta)}(t)$ of order (α,β) with $\alpha \geq -1/2$, $\beta \in \mathbb{R}$, and $\lambda \in \mathbb{C}$, is defined as the even C^∞-function on $\mathbb{R}$ which satisfies the differential equation,

$$\begin{cases} \dfrac{d^2u}{dt^2} + \dfrac{A'_{\alpha,\beta}(t)}{A_{\alpha,\beta}(t)}\dfrac{du}{dt} + \left(\lambda^2 + (\alpha+\beta+1)^2\right)u = 0 \\ u(0) = 1, \qquad \dfrac{d}{dt}u(0) = 0, \end{cases}$$

where

$$A_{\alpha,\beta}(t) = 2^{2(\alpha+\beta+1)}(\operatorname{sh} t)^{2\alpha+1}(\operatorname{ch} t)^{2\beta+1}.$$

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For special values of α and β, the Jacobi functions are interpreted as spherical functions on non-compact riemannian symmetric spaces of rank one; for more details see the very interesting survey of T. H. Koornwinder [13].

It is well known that the function $\varphi_\lambda^{(\alpha,\beta)}(t)$ possesses a Mehler representation given by

$$\forall t > 0, \forall \lambda \in \mathbb{C}, \qquad \varphi_\lambda^{(\alpha,\beta)}(t) = \int_0^\infty K_{\alpha,\beta}(t,s) \cos \lambda s \, ds, \tag{0.1}$$

where $K_{\alpha,\beta}(t,.)$, $t > 0$, is a nonnegative function continuous on $]-t,t[$ and supported in $[-t,t]$. This Mehler representation was exploited by T. H. Koornwinder to define an integral transform (called the Abel transform) $F_{\alpha,\beta}$ (see [11]), which relates the Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}$ and the Fourier-cosine transform $\mathcal{F}_0$, that is,

$$\mathcal{F}_{\alpha,\beta} = \mathcal{F}_0 \circ F_{\alpha,\beta}. \tag{0.2}$$

Here $\mathcal{F}_{\alpha,\beta}$ is given for an even C^∞-function on $\mathbb{R}$, with compact support, by

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = (2\pi)^{-1/2} \int_0^\infty f(t) \varphi_\lambda^{(\alpha,\beta)}(t) A_{\alpha,\beta}(t) \, dt. \tag{0.3}$$

We assume that $\alpha \geq \beta \geq -1/2$; then the inverse of the Fourier–Jacobi transform is given by

$$\mathcal{F}_{\alpha,\beta}^{-1}(f)(t) = (2\pi)^{-1/2} \int_0^\infty \mathcal{F}_{\alpha,\beta}(f)(\lambda) \varphi_\lambda^{(\alpha,\beta)}(t) \frac{d\lambda}{|c_{\alpha,\beta}(\lambda)|^2}, \tag{0.4}$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma((\alpha+\beta+1+i\lambda)/2)\Gamma((\alpha-\beta+1+i\lambda)/2)}.$$

In this work, we are interested with a dual Mehler representation of the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the dual argument λ. More precisely, we shall prove that the function $\varphi_\lambda^{(\alpha,\beta)}(t)$ possesses the integral representation

$$\forall t \geq 0, \forall \lambda_0 \in \mathbb{R}, \qquad \varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \frac{2}{\pi} \int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) \cos \lambda t \, d\lambda. \tag{0.5}$$

This result was pointed out by M. Flensted-Jensen and T. H. Koornwinder in a remark made at the end of the paper [8]; also they indicated that the nonnegativity of the kernel $b_{\alpha,\beta}$ should be obtained by using an addition formula for the function $\varphi_{\lambda_0}^{(\alpha,\beta)}$ given in the same paper. Here we work out in details this remark, we establish the nonnegativity of the kernel $b_{\alpha,\beta}$, we state some properties of the kernel $b_{\alpha,\beta}$, and we mark out the contrast between this kernel and the kernel $K_{\alpha,\beta}$ defined in (0.1).

We observe that we can write formula (0.5) as

$$\forall t \geq 0, \forall \lambda_0 \in \mathbb{R}, \qquad \varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \chi_{\alpha,\beta}(\cos(.\,t))(\lambda_0), \tag{0.6}$$

where

$$\chi_{\alpha,\beta}(f)(\lambda_0) = \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0,\lambda) f(\lambda)\, d\lambda, \tag{0.7}$$

with f an even continuous and bounded function on $\mathbb{R}$.

Hence this allows us to define a pair of dual integral transforms which we call the dual Mehler transform denoted $\chi_{\alpha,\beta}$ and given by the formula (0.7) and its transposed ${}^t\chi_{\alpha,\beta}$ defined for an even continuous function f on $\mathbb{R}$, with compact support, by

$${}^t\chi_{\alpha,\beta}(f)(\lambda) = \int_0^\infty b_{\alpha,\beta}(\lambda_0,\lambda) f(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0), \tag{0.8}$$

where $\nu_{\alpha,\beta}$ is the measure given by $d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-1/2}(d\lambda/|c_{\alpha,\beta}(\lambda)|^2)$. The transform ${}^t\chi_{\alpha,\beta}$ is related to the inverse Fourier–Jacobi transform $F_{\alpha,\beta}^{-1}$ defined by (0.4), by the relation analogous to (0.2),

$$F_{\alpha,\beta}^{-1} = F_0^{-1} \circ {}^t\chi_{\alpha,\beta}, \tag{0.9}$$

here F_0 denotes the Fourier-cosine transform.

In this paper, we construct a second order difference operator $P_{\alpha,\beta}$ such that $\varphi_\lambda^{(\alpha,\beta)}(t)$ is an eigenfunction of the operator $P_{\alpha,\beta}$ with respect to the dual variable λ; precisely we state that for all λ in $\mathbb{C}$ and $t \geq 0$, we have

$$P_{\alpha,\beta}\big(\varphi_\lambda^{(\alpha,\beta)}(t)\big) = -\operatorname{ch} 2t\, \varphi_\lambda^{(\alpha,\beta)}(t).$$

Next we study the transforms $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ and we establish some results analogous to the ones given by K. Trimèche for the J. L. Lions transmutation operators (see [19]), especially

— We prove that $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are permutation operators between $P_{\alpha,\beta}$ and a second order difference operator Q.

— We define and characterize spaces of functions on which the transforms $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are topological isomorphisms.

— We prove the following inversion formulas for $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ when applied to some spaces of functions,

$$f = \chi_{\alpha,\beta}\, \mathsf{M}_0\, {}^t\chi_{\alpha,\beta} f; \qquad f = {}^t\chi_{\alpha,\beta}\, \chi_{\alpha,\beta}\, \mathsf{M}_0 f,$$

$$f = \chi_{\alpha,\beta}\, {}^t\chi_{\alpha,\beta}\, \mathsf{M} f; \qquad f = {}^t\chi_{\alpha,\beta}\, \mathsf{M} \chi_{\alpha,\beta} f,$$

where M_0 and M are pseudo-differential operators.

We conclude this introduction with a summary of the content of this paper. In the first section we recall the properties of the Jacobi functions. The second section contains the addition formula for Jacobi functions, the dual Mehler representation of the function $\varphi_\lambda^{(\alpha,\beta)}(t)$, and the properties of the kernel $b_{\alpha,\beta}(\lambda_0, \lambda)$. In the third section we define and study the pair of dual integral transforms $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$. The next section contains the definition of two second order difference operators $P_{\alpha,\beta}$ and Q such that $\varphi_\lambda^{(\alpha,\beta)}(t)$ is an eigenfunction of the operator $P_{\alpha,\beta}$ with respect to the dual variable λ; moreover we establish that $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are permutation operators between $P_{\alpha,\beta}$ and Q. In the last section we give inversion formulas for the transforms $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$.

1. THE JACOBI FUNCTIONS

In this section we recall the properties of the Jacobi functions; for more details see [6, 7, 13].

For $\alpha > -\frac{1}{2}$, $\beta \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $t \geq 0$, the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ is defined by

$$\varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\left(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -\mathrm{sh}^2 t\right), \tag{1.1}$$

where $\rho = \alpha + \beta + 1$ and ${}_2F_1$ is the Gauss hypergeometric function.

The function $t \to \varphi_\lambda^{(\alpha,\beta)}(t)$ is the unique solution of the differential equation

$$\begin{cases} \Delta_{\alpha,\beta}\varphi_\lambda^{(\alpha,\beta)}(t) = -(\lambda^2 + \rho^2)\varphi_\lambda^{(\alpha,\beta)}(t), \\ \varphi_\lambda^{(\alpha,\beta)}(0) = 1, \qquad \dfrac{d}{dt}\varphi_\lambda^{(\alpha,\beta)}(0) = 0, \end{cases} \tag{1.2}$$

where $\Delta_{\alpha,\beta}$ is the Jacobi differential operator

$$\Delta_{\alpha,\beta} = \frac{1}{A_{\alpha,\beta}(t)} \frac{d}{dt}\left[A_{\alpha,\beta}(t) \frac{d}{dt} \right] \tag{1.3}$$

with

$$A_{\alpha,\beta}(t) = 2^{2\rho}(\operatorname{sh} t)^{2\alpha+1}(\operatorname{ch} t)^{2\beta+1}. \tag{1.4}$$

In this paper we assume that $\alpha \geq \beta \geq -1/2$.

We have the following properties of the function $\varphi_\lambda^{(\alpha,\beta)}(t)$.

(i) For $t \geq 0$, $\lambda \in \mathbb{C}$ with $|\mathrm{Im}\,\lambda| \leq \rho$,

$$\left| \varphi_\lambda^{(\alpha,\beta)}(t) \right| \leq 1. \tag{1.5}$$

(ii) For all $n \in \mathbb{N}$, there exists $k_n > 0$ such that for all $\lambda \in \mathbb{C}$ and $t \geq 0$,

$$\left| \frac{d^n}{dt^n} \varphi_\lambda^{(\alpha,\beta)}(t) \right| \leq k_n(1+t)(1+|\lambda|)^n e^{(|\mathrm{Im}\,\lambda|-\rho)t}. \tag{1.6}$$

(iii) There exists $k > 0$ such that for all $\lambda \in \mathbb{C}$, $t \geq 0$, and $n \in \mathbb{N}$,

$$\left| \frac{d^n}{d\lambda^n} \varphi_\lambda^{(\alpha,\beta)}(t) \right| \leq k(1+t)^{n+1} e^{(|\mathrm{Im}\,\lambda|-\rho)t}. \tag{1.7}$$

(iv)

$$\begin{aligned} (\Gamma(\alpha+1))^{-1} \frac{d}{dt} \varphi_\lambda^{(\alpha,\beta)}(t) &= -\frac{1}{4}\left[(\alpha+\beta+1)^2 + \lambda^2 \right] \\ &\quad \times (\Gamma(\alpha+2))^{-1} \operatorname{sh} 2t\, \varphi_\lambda^{(\alpha+1,\beta+1)}(t). \end{aligned} \tag{1.8}$$

(v)

$$\begin{aligned} (\Gamma(\alpha+2))^{-1} \frac{d}{dt}\left[(\operatorname{sh} 2t)^{-1} A_{\alpha+1,\beta+1}(t)\, \varphi_\lambda^{(\alpha+1,\beta+1)}(t) \right] \\ = 16(\Gamma(\alpha+1))^{-1} A_{\alpha,\beta}(t)\, \varphi_\lambda^{(\alpha,\beta)}(t). \end{aligned} \tag{1.9}$$

(vi) The function $t \to \varphi_\lambda^{(\alpha,\beta)}(t)$ possesses the Mehler representation

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \int_0^t K_{\alpha,\beta}(t,u) \cos \lambda u \, du, \qquad t > 0,\ \lambda \in \mathbb{C}, \tag{1.10}$$

where $K_{\alpha,\beta}(t,.)$ is a nonnegative even function continuous on $]-t,t[$ and of support $[-t,t]$.

1.1. *The Fourier–Jacobi Transform and Its Inverse*

Let us define the following functions spaces (see [6, pp. 146–147]).

— $\mathcal{D}_{*,R}(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$ with support in $[-R,R]$ for $R>0$. This space is provided with the topology of uniform convergence of functions and their derivatives.

— $\mathcal{D}_*(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$ with compact support,

$$\mathcal{D}_*(\mathbb{R}) = \bigcup_{R\geq 0} \mathcal{D}_{*,R}(\mathbb{R});$$

this space shall be given the inductive limit topology.

— $S_*(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$, rapidly decreasing, equipped with the usual Schwartz-topology.

— $S_*^r(\mathbb{R}) = \{(\operatorname{ch} t)^{-2\rho/r} S_*(\mathbb{R})\}$, for $0<r\leq 2$. This space shall be equipped with the topology given by $S_*(\mathbb{R})$.

Clearly $S_*^r(\mathbb{R})$ is invariant under $\Delta_{\alpha,\beta}$ and we notice that the semi-norms defined by

$$Q_{n,m}(f) = \sup_{t\geq 0}\left|(\operatorname{ch} t)^{2\rho/r}(1+t)^n \Delta_{\alpha,\beta}^m f(t)\right| \tag{1.11}$$

are continuous on $S_*^r(\mathbb{R})$.

— $D_r = \{\lambda\in\mathbb{C}/|\operatorname{Im}\lambda| \leq (2/r-1)\rho\}$.

— $\mathbb{H}_{*,R}(\mathbb{C})$ the space of even, entire, rapidly decreasing functions ψ of exponential type R, that is, for all $n\in\mathbb{N}$, $P_n(\psi) = \sup_{\lambda\in\mathbb{C}}|(1+\lambda^n)e^{-R|\operatorname{Im}\lambda|}\psi(\lambda)| < +\infty$.

$\mathbb{H}_{*,R}(\mathbb{C})$ is equipped with the topology defined by the semi-norms P_n.

— $\mathbb{H}_*(\mathbb{C}) = \bigcup_{R\geq 0}\mathbb{H}_{*,R}(\mathbb{C})$, equipped with the inductive limit topology.

— H_*^r the space of even holomorphic functions ψ in the interior of D_r, C^∞ in D_r, and rapidly decreasing, that is,

$$\forall m,n\in\mathbb{N},\qquad P_{n,m}(\psi) = \sup_{\lambda\in D_r}\left|(1+\lambda^n)\frac{d^m}{d\lambda^m}\psi(\lambda)\right| < +\infty.$$

Notice that $H_*^2 = S_*(\mathbb{R})$ and if $r\leq s$ then $H_*^r \subseteq H_*^s$.

Give H_*^r the topology defined by the semi-norms $P_{n,m}$ which can equivalently be defined by the semi-norms

$$P_{n,m}^0(\psi) = \sup_{\lambda \in D_r} \left\{ \left| \frac{d^n}{d\lambda^n} \left(\left(\lambda^2 + \rho^2 \right)^m \psi(\lambda) \right) \right| \right\}. \tag{1.12}$$

We note that all these spaces are Fréchet spaces.

Notations. We denote by

$$d\mu_{\alpha,\beta}(t) = (2\pi)^{-1/2} 2^{2\rho} (\operatorname{sh} t)^{2\alpha+1} (\operatorname{ch} t)^{2\beta+1}\, dt, \tag{1.13}$$

$$d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-1/2} |c_{\alpha,\beta}(\lambda)|^{-2}\, d\lambda, \tag{1.14}$$

where

$$c_{\alpha,\beta} = \frac{2^{\rho - i\lambda} \Gamma(i\lambda) \Gamma(\alpha+1)}{\Gamma((\alpha+\beta+1+i\lambda)/2)\Gamma((\alpha-\beta+1+i\lambda)/2)}. \tag{1.15}$$

— $L^p([0,+\infty[, d\mu_{\alpha,\beta})$, $p \geq 1$, the space of measurable functions f on $[0,+\infty[$, such that

$$\int_0^{+\infty} |f(t)|^p \, d\mu_{\alpha,\beta}(t) < +\infty.$$

— $L^p([0,+\infty[, \mathrm{d}\nu_{\alpha,\beta})$, $p \geq 1$, the space of measurable functions h on $[0,+\infty[$, such that

$$\int_0^{\infty} |h(\lambda)|^p \, d\nu_{\alpha,\beta}(\lambda) < +\infty.$$

DEFINITION 1.1. (i) The Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}$ of $f \in L^1([0,+\infty[, \mathrm{d}\mu_{\alpha,\beta})$ is defined by

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_0^{\infty} f(t) \varphi_\lambda^{(\alpha,\beta)}(t)\, d\mu_{\alpha,\beta}(t). \tag{1.16}$$

(ii) The inverse Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}^{-1}$ of $h \in L^1([0,+\infty[, \mathrm{d}\nu_{\alpha,\beta})$ is defined by

$$\mathcal{F}_{\alpha,\beta}^{-1}(h)(t) = \int_0^{\infty} h(\lambda) \varphi_\lambda^{(\alpha,\beta)}(t)\, d\nu_{\alpha,\beta}(\lambda). \tag{1.17}$$

THEOREM 1.1. *The Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}$ is an isomorphism*

(i) *from* $\mathcal{D}_*(\mathbb{R})$ *onto* $\mathbb{H}_*(\mathbb{C})$,

(ii) *from* $S_*^r(\mathbb{R})$ *onto* H_*^r.

(See [6, p. 146].)

1.2. *The Convolution Structure*

The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}$ satisfies the following product formula

$$\forall t, s > 0, \qquad \varphi_\lambda^{(\alpha,\beta)}(t)\,\varphi_\lambda^{(\alpha,\beta)}(s) = \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(r) W_{\alpha,\beta}(t,s,r)\, d\mu_{\alpha,\beta}(r), \tag{1.18}$$

where $W_{\alpha,\beta}(t,s,.)$ is a nonnegative function compactly supported and such that

$$\int_0^\infty W_{\alpha,\beta}(t,s,r)\, d\mu_{\alpha,\beta}(r) = 1. \tag{1.19}$$

The formula (1.18) allows us to define the generalized translation operator T_r, $r \geq 0$, by

$$\begin{aligned} \forall r, s > 0, \qquad & T_r f(s) = \int_0^\infty f(t) W_{\alpha,\beta}(t,s,r)\, d\mu_{\alpha,\beta}(t), \\ \forall r \geq 0, \qquad & T_r f(0) = f(r), \end{aligned} \tag{1.20}$$

and the convolution associated with $\Delta_{\alpha,\beta}$ of two functions f and g in $D_*(\mathbb{R})$ by the relation

$$f * g(t) = \int_0^\infty T_t f(s) g(s)\, d\mu_{\alpha,\beta}(s). \tag{1.21}$$

Notice that for $\alpha > \beta > -1/2$, we can also write this in the form

$$f * g(t) = \int_0^\infty f(s) \int_0^1 \int_0^\pi g(\Lambda(t,s,r,\psi))\, d\tilde{m}_{\alpha,\beta}(r,\psi)\, d\mu_{\alpha,\beta}(s), \tag{1.22}$$

where

$$\Lambda(t,s,r,\psi) = \operatorname{Arc\,ch}(|\operatorname{ch} t \operatorname{ch} s + r e^{i\psi} \operatorname{sh} t \operatorname{sh} s|) \tag{1.23}$$

and

$$\begin{aligned} d\tilde{m}_{\alpha,\beta}(r,\psi) = {} & \frac{2\Gamma(\alpha+1)}{\sqrt{\Pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+1/2)} r^{2\beta+1} \\ & \times (1-r^2)^{\alpha-\beta-1} (\sin\psi)^{2\beta}\, dr\, d\psi. \end{aligned} \tag{1.24}$$

Remark. It is well known that the formula (1.18) generates a hypergroup structure on $\mathbb{R}_+ = [0, +\infty[$ in the sense of Jewett (see [5, 10]), called the Jacobi hypergroup of non-compact type. Its dual is identified with $K = \mathbb{R}_+ \cup i[0, \rho]$.

1.3. *The Dual Convolution Structure*

It was established (see [3, 8]) that the function $\varphi_\lambda^{(\alpha,\beta)}(t)$ satisfies a product formula with respect to the argument λ in $\mathbb{R}_+ \cup i([0, s_0] \cup \{\rho\})$, $s_0 = \min(\alpha + \beta + 1, \alpha - \beta + 1)$. More precisely we have the following result: for $\lambda_1, \lambda_2 \in \mathbb{R}_+ \cup i([0, s_0] \cup \{\rho\})$, there exist unique finite nonnegative measures μ_1 on $\mathbb{R}_+$ and μ_2 on $i]0, \rho]$ such that for every $t \geq 0$, we have

$$\varphi_{\lambda_1}^{(\alpha,\beta)}(t)\,\varphi_{\lambda_2}^{(\alpha,\beta)}(t) = \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(t)\,d\mu_1(\lambda) + \int_{i]0,\rho]} \varphi_\lambda^{(\alpha,\beta)}(t)\,d\mu_2(\lambda). \tag{1.25}$$

The measures μ_1 and μ_2 satisfy the following properties.

(i) If $|\mathrm{Im}\,\lambda_1| + |\mathrm{Im}\,\lambda_2| < \rho$ then $\mu_2 = 0$ and $d\mu_1(\lambda) = a_{\alpha,\beta}(\lambda_1, \lambda_2, \lambda)\,d\nu_{\alpha,\beta}(\lambda)$, where $a_{\alpha,\beta}(\lambda_1, \lambda_2, \lambda)$ is a nonnegative function and $\nu_{\alpha,\beta}$ is the measure defined by the relation (1.14).

(ii) If $|\mathrm{Im}\,\lambda_1| + |\mathrm{Im}\,\lambda_2| \geq \rho$, we have

$$\operatorname{supp}\mu_1 \subset \mathbb{R}_+ \qquad \text{and} \qquad \lambda_1 + \lambda_2 - i\rho \in \operatorname{supp}\mu_2 \subset i\left[0, \frac{\lambda_1 + \lambda_2}{i} - \rho\right].$$

M. Flensted-Jensen and T. H. Koornwinder have given some properties of the kernel $a_{\alpha,\beta}(\lambda_1, \lambda_2, \lambda)$; in particular we have:

$$\int_0^\infty a_{\alpha,\beta}(\lambda_1, \lambda_2, \lambda)\,d\nu_{\alpha,\beta}(\lambda) = 1, \tag{1.26}$$

— the function $a_{\alpha,\beta}$ is analytic on $\{(\lambda_1, \lambda_2, \lambda) \in \mathbb{C}^3 / |\mathrm{Im}\,\lambda_1| + |\mathrm{Im}\,\lambda_2| + |\mathrm{Im}\,\lambda| < \rho\}$;

— for fixed λ_1 and $\lambda_2 \in \mathbb{R}$, $\lambda \to a_{\alpha,\beta}(\lambda_1, \lambda_2, \lambda)$ is an analytic function on the strip $\{\lambda \in \mathbb{C} / |\mathrm{Im}\,\lambda| < \rho\}$.

The formula (1.25) bears a dual convolution structure on the subset S of K defined by $S = \mathbb{R}_+ \cup i([0, s_0] \cup \{\rho\})$. This dual convolution is given for a continuous and bounded function f on K with support in S, by

$$\delta_{\lambda_1} \# \delta_{\lambda_2}(f) = \int_K f(\lambda)\,d\mu_{\lambda_1,\lambda_2}(\lambda) \tag{1.27}$$

where δ_{λ_j}, $j = 1, 2$, is the point measure at λ_j and $\mu_{\lambda_1,\lambda_2}$ is the measure on K defined by the dual product formula (1.25), that is,

$$\int_K f(\lambda)\,d\mu_{\lambda_1,\lambda_2}(\lambda) = \int_0^\infty f(\lambda)\,d\mu_1(\lambda) + \int_{i]0,\rho]} f(\lambda)\,d\mu_2(\lambda).$$

The formula (1.27) defines the generalized dual translation operators $\mathcal{T}_\lambda$, $\lambda \in S$, defined for a continuous and bounded function f on S by

$$\mathcal{T}_{\lambda_1} f(\lambda_2) = \delta_{\lambda_1} \# \delta_{\lambda_2}(f). \tag{1.28}$$

In particular for $f, g \in L^1([0, \infty[, d\nu_{\alpha,\beta})$ the dual convolution product is given by

$$f\#g(\lambda_1) = \int_0^\infty \int_0^\infty f(\lambda_2) g(\lambda_3) a_{\alpha,\beta}(\lambda_1, \lambda_2, \lambda_3)\, d\nu_{\alpha,\beta}(\lambda_2)\, d\nu_{\alpha,\beta}(\lambda_3). \tag{1.29}$$

It is clear that if $f, g \in L^1([0, +\infty[, d\nu_{\alpha,\beta})$, then $f\#g \in L^1([0, +\infty[, d\nu_{\alpha,\beta})$ and

$$\mathcal{F}_{\alpha,\beta}^{-1}(f\#g) = \mathcal{F}_{\alpha,\beta}^{-1}(f)\mathcal{F}_{\alpha,\beta}^{-1}(g), \tag{1.30}$$

where $\mathcal{F}_{\alpha,\beta}^{-1}$ is given by the formula (1.17).

2. DUAL MEHLER REPRESENTATION FOR JACOBI FUNCTIONS

M. Flensted-Jensen and T. H. Koornwinder have announced without proof in Remark 4.12 of [8, p. 150] the existence of a Mehler type representation for the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the argument λ. In this section we shall deal with this representation. Especially we shall prove the existence of a nonnegative kernel $b_{\alpha,\beta}(\lambda_0, \lambda)$ such that

$$\forall t \geq 0, \forall \lambda_0 \in \mathbb{R}, \qquad \varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) \cos \lambda t\, d\lambda. \tag{2.1}$$

Next we give some properties of this kernel.

Proposition 2.1. *Let $\alpha \geq \beta \geq -1/2$, $\alpha + \beta + 1 > 0$, $\gamma \geq \delta \geq -1/2$, and $\gamma + \delta < \alpha + \beta$. Then for all $\lambda \in \mathbb{R}$, the function $t \to \varphi_\lambda^{(\alpha,\beta)}(t)$ belongs to $L^p([0, +\infty[, d\mu_{\gamma,\delta})$ for some p in* $[1, 2[$.

Proof. From the relation (1.6) there exists $k_0 > 0$ such that for all $\lambda \in \mathbb{R}$, $t \geq 0$,

$$\left|\varphi_\lambda^{(\alpha,\beta)}(t)\right| \leq k_0(1 + t)e^{-(\alpha+\beta+1)t}.$$

Let $p = (\alpha + \beta + \gamma + \delta + 2)/(\alpha + \beta + 1)$; it is clear that p is in $[1, 2[$. Then the function $t \to \varphi_\lambda^{(\alpha,\beta)}(t)$ belongs to $L^p([0, +\infty[, d\mu_{\gamma,\delta})$.

PROPOSITION 2.2. *Let* $\alpha \geq \beta \geq -1/2$, $\alpha+\beta+1>0$, $\gamma \geq \delta \geq -1/2$, *and* $\gamma+\delta<\alpha+\beta$. *Then the kernel* $b_{\alpha,\beta,\gamma,\delta}$ *given by*

$$b_{\alpha,\beta,\gamma,\delta}(\lambda_0,\lambda)=\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\,\varphi_{\lambda}^{(\gamma,\delta)}(t)\,du_{\gamma,\delta}(t) \tag{2.2}$$

is well defined for $\lambda_0,\ \lambda \in \mathbb{C}$ *such that* $|\mathrm{Im}\,\lambda_0|+|\mathrm{Im}\,\lambda|<\alpha+\beta-\gamma-\delta$.

Proof. From Proposition 2.1, there exists $p\in[1,2[$ such that for $\lambda_0\in\mathbb{R}$ the function $t\to\varphi_{\lambda_0}^{(\alpha,\beta)}(t)$ is in $L^p([0,\infty[,d\mu_{\gamma,\delta})$. Then for all $\lambda_0\in\mathbb{R}$ the function $\mathcal{F}_{\gamma,\delta}(\varphi_{\lambda_0}^{(\alpha,\beta)})$ given by

$$\forall\lambda\in\mathbb{R},\qquad \mathcal{F}_{\gamma,\delta}\left(\varphi_{\lambda_0}^{(\alpha,\beta)}\right)(\lambda)=\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\,\varphi_{\lambda}^{(\gamma,\delta)}(t)\,d\mu_{\gamma,\delta}(t)$$

is well defined.

We put

$$b_{\alpha,\beta,\gamma,\delta}(\lambda_0,\lambda)=\mathcal{F}_{\gamma,\delta}\left(\varphi_{\lambda_0}^{(\alpha,\beta)}\right)(\lambda). \tag{2.3}$$

Then the kernel $b_{\alpha,\beta,\gamma,\delta}(\lambda_0,\lambda)$ is well defined for $\lambda_0,\ \lambda\in\mathbb{R}$.

Now using the fact that the function $(\lambda_0,\lambda)\to\varphi_{\lambda_0}^{(\alpha,\beta)}(t)\varphi_{\lambda}^{(\gamma,\delta)}(t)$, $t\geq 0$, is analytic in $\mathbb{C}^2$ and the relation

$$\begin{aligned} &A_{\gamma,\delta}(t)\left|\varphi_{\lambda_0}^{(\alpha,\beta)}(t)\,\varphi_{\lambda}^{(\gamma,\delta)}(t)\right| \\ &\qquad \leq k_0(1+t)^2\exp\left[\left(|\mathrm{Im}(\lambda)|+|\mathrm{Im}(\lambda_0)|-(\alpha+\beta-\gamma-\delta)\right)t\right] \end{aligned}$$

we can extend by analytic continuation the function $b_{\alpha,\beta,\gamma,\delta}$ for $\lambda_0,\ \lambda\in\mathbb{C}$ such that $|\mathrm{Im}\,\lambda|+|\mathrm{Im}\,\lambda_0|<\alpha+\beta-\gamma-\delta$.

In the following, with the help of an addition formula for the Jacobi functions given by M. Flensted-Jensen and T. H. Koornwinder in [8], we shall prove the nonnegativity of the kernel $b_{\alpha,\beta,\gamma,\delta}(\lambda_0,\lambda)$ given by formula (2.3) for $\lambda_0,\ \lambda\in\mathbb{R}$. Before establishing this property we shall recall some results related to the Jacobi functions (see [2, 8]).

The associated Jacobi function $\varphi_{\lambda,k,l}^{(\alpha,\beta)}$ is defined by

$$\varphi_{\lambda,k,l}^{(\alpha,\beta)}(t)=(\mathrm{sh}\,t)^{k+l}(\mathrm{ch}\,t)^{k+l}\varphi_{\lambda}^{(\alpha+k+l,\,\beta+k-l)},$$
$$\text{for } k,l\in\mathbb{Z},\ k\geq l\geq 0. \tag{2.4}$$

We denote for $n, m \in \mathbb{Z}$, $n \geq m \geq 0$,

$$\Pi_{n,m}^{(\alpha,\beta)} = \frac{(2n - 2m + 2\beta + 1)(n + m + \alpha + \beta + 3/2) \times (\alpha + 1)_m (2\beta + 2)_{n-m} (\alpha + \beta + 5/2)_n}{(n - m + 2\beta + 1)(m + \alpha + \beta + 3/2) \times m!(n - m)!(\beta + 3/2)_n} \tag{2.5}$$

and

$$R_{n,m}^{(\alpha,\beta)}(x, y) = R_m^{(\alpha,\beta+n-m+1/2)}(2y - 1) y^{(n-m)/2} R_{n-m}^{(\beta,\beta)}(y^{-1/2} x), \tag{2.6}$$

where $R_n^{(\alpha,\beta)}$ is the Jacobi polynomial such that $R_n^{(\alpha,\beta)}(1) = 1$.

We have the following addition formula (see [8]).

THEOREM 2.1. *Let* $\alpha > \beta > -1/2$. *For* $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $r \in [0,1]$, $\psi \in [0, \pi]$, *we have*

$$\begin{aligned} &\varphi_\lambda^{(\alpha,\beta)}(\Lambda(-x, y, r, \psi)) \\ &\quad = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \gamma_{k,l}^{(\alpha,\beta)}(\lambda) \varphi_{\lambda,k,l}^{(\alpha,\beta)}(x) \varphi_{\lambda,k,l}^{(\alpha,\beta)}(y) \Pi_{k,l}^{(\alpha-\beta-1,\beta-1/2)} \\ &\qquad \times R_{k,l}^{(\alpha-\beta-1,\beta-1/2)}(r \cos \psi, r^2). \end{aligned} \tag{2.7}$$

The function $\Lambda(-x, y, r, \psi)$ *is given by the relation* (1.23) *and*

$$\gamma_{k,l}^{(\alpha,\beta)}(\lambda) = \frac{((1/2)(\alpha + \beta + 1 + i\lambda))_k ((1/2)(\alpha + \beta + 1 - i\lambda))_k \times ((1/2)(\alpha - \beta + 1 + i\lambda))_l ((1/2)(\alpha - \beta + 1 - i\lambda))_l}{[(\alpha + 1)_{k+l}]^2}.$$

The series converge absolutely and uniformly for (x, y, r, ψ) *in all compact subsets of* $\mathbb{R}^2 \times [0,1] \times [0, \pi]$.

Remarks. (i) From the relation $\varphi_\lambda^{(\alpha,-1/2)}(2x) = \varphi_{2\lambda}^{(\alpha,\alpha)}(x)$ we deduce that for $\alpha = \beta > -1/2$ or $\alpha > \beta = -1/2$, Theorem 2.1 still holds, but it degenerates to a single series.

(ii) Let $\lambda \in \mathbb{C}$, $\alpha \geq \beta > -1/2$, or $\alpha > \beta = -1/2$. Then $\gamma_{k,l}^{(\alpha,\beta)}(\lambda) \geq 0$, for all k, l, if and only if $\lambda \in \mathbb{R} \cup i([-s_0, s_0] \cup \{\pm\rho\})$, where $s_0 = \min(\alpha + \beta + 1, \alpha - \beta + 1)$.

LEMMA 2.1. *Let* $F \in L^p([0, +\infty[, d\mu_{\gamma,\delta})$, $1 \leq p < 2$. *Then* $\mathcal{F}_{\gamma,\delta}(F)(\lambda) \geq 0$ *for all* $\lambda \in [0, +\infty[$ *if and only if for all* $g \in \mathcal{D}_*(\mathbb{R})$,

$$\int_0^\infty F * g(t) \bar{g}(t) \, d\mu_{\gamma,\delta}(t) \geq 0. \tag{2.8}$$

Here $*$ *denotes the convolution associated with the Jacobi operator* $\Delta_{\gamma,\delta}$.

(See [8, p. 147].)

LEMMA 2.2. *Let* $\alpha > \beta > -1/2$, $\gamma > \delta > -1/2$. *Then for all* $k, l \in \mathbb{Z}$, $k \geq l \geq 0$, *we have*

$$a_{k,l,\alpha,\beta,\gamma,\delta} = (-1)^{k-l} \int_0^1 \int_0^{\pi} R_{k,l}^{(\alpha-\beta-1,\,\beta-1/2)}(r\cos\psi, r^2)\, d\tilde{m}_{\gamma,\delta}(r,\psi) \geq 0, \tag{2.9}$$

where $d\tilde{m}_{\gamma,\delta}$ *is defined as in* (1.24).

Proof. From [1, 12], the integral $\int_0^1 \int_0^{\Pi} R_{k,l}^{(\alpha-\beta-1,\,\beta-1/2)}(r\cos\psi, r^2)$ $d\tilde{m}_{\gamma,\delta}(r,\psi)$ is zero if $(k-l)$ is odd and has the same sign as $(\beta-\delta)_{(1/2)(k-l)}$ if $(k-l)$ is even.

THEOREM 2.2. *Let* $\alpha > \beta > -1/2$, $\gamma > \delta > -1/2$, *and* $\gamma + \delta < \alpha + \beta$. *Then for all* λ_0, $\lambda \in \mathbb{R}$, *the kernel* $b_{\alpha,\beta,\gamma,\delta}$ *is nonnegative*.

Proof. From the relation (2.3), we have for all λ_0, $\lambda \in \mathbb{R}$

$$b_{\alpha,\beta,\gamma,\delta}(\lambda_0, \lambda) = F_{\gamma,\delta}\left(\varphi_{\lambda_0}^{(\alpha,\beta)}\right)(\lambda).$$

In view of Proposition 2.1 and Lemma 2.1, we have to prove that

$$\forall \lambda_0 \in \mathbb{R}, \qquad \int_0^{\infty} \left(\varphi_{\lambda_0}^{(\alpha,\beta)} * g\right)(t)\, \bar{g}(t)\, d\mu_{\gamma,\delta}(t) \geq 0,$$

for all $g \in D_*(\mathbb{R})$.

We write

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} g(t_1)\overline{g(t_2)}\varphi_{\lambda_0}^{(\alpha,\beta)}(t_3) W_{\alpha,\beta}(t_1,t_2,t_3)\, d\mu_{\gamma,\delta}(t_1) \times d\mu_{\gamma,\delta}(t_2)\, d\mu_{\gamma,\delta}(t_3).$$

Using (1.22), we deduce that

$$\int_0^{\infty} \varphi_{\lambda_0}^{(\alpha,\beta)}(t_3) W_{\alpha,\beta}(t_1,t_2,t_3)\, d\mu_{\gamma,\delta}(t_3) = \int_0^1 \int_0^{\Pi} \varphi_{\lambda_0}^{(\alpha,\beta)}(\Lambda(t_1,t_2,r,\psi))\, d\tilde{m}_{\gamma,\delta}(r,\psi).$$

Now by applying the addition formula (2.7) we find that

$$\int_0^1\int_0^\Pi \varphi_{\lambda_0}^{(\alpha,\beta)}(\Lambda(t_1,t_2,r,\psi))\,d\tilde m_{\gamma,\delta}(r,\psi)$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\gamma_{k,l}^{(\alpha,\beta)}(\lambda_0)\,\varphi_{\lambda_0,k,l}^{(\alpha,\beta)}(t_1)\,\varphi_{\lambda_0,k,l}^{(\alpha,\beta)}(t_2)$$
$$\times\,\Pi_{k,l}^{(\alpha-\beta-1,\beta-1/2)}(-1)^{k-l}$$
$$\times\int_0^1\int_0^\Pi R_{k,l}^{(\alpha-\beta,\beta-1/2)}(r\cos\psi,r^2)\,d\tilde m_{\gamma,\delta}(r,\psi).$$

Consequently, we deduce that

$$\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)} * g(t)\bar g(t)\,d\mu_{\gamma,\delta}(t)$$
$$= \int_0^\infty\int_0^\infty g(t_1)\overline{g(t_2)}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\gamma_{k,l}^{(\alpha,\beta)}(\lambda_0)\,\varphi_{\lambda_0,k,l}^{(\alpha,\beta)}(t_1)\,\varphi_{\lambda_0,k,l}^{(\alpha,\beta)}(t_2)$$
$$\times\,\Pi_{k,l}^{(\alpha-\beta-1,\beta-1/2)}a_{k,l,\alpha,\beta,\gamma,\delta}\,d\mu_{\gamma,\delta}(t_1)\,d\mu_{\gamma,\delta}(t_2)$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\gamma_{k,l}^{(\alpha,\beta)}(\lambda_0)\Pi_{k,l}^{(\alpha-\beta-1,\beta-1/2)}a_{k,l,\alpha,\beta,\gamma,\delta}$$
$$\times\left|\int_0^\infty g(t)\,\varphi_{\lambda_0,k,l}^{(\alpha,\beta)}(t)\,d\mu_{\gamma,\delta}(t)\right|^2.$$

By Lemma 2.2 and the fact that $\gamma_{k,l}^{(\alpha,\beta)}(\lambda_0)\geq 0$ for all $\lambda_0\in\mathbb{R}$, we deduce that

$$\forall\lambda_0\in\mathbb{R},\qquad \int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)} * g(t)\bar g(t)\,d\mu_{\gamma,\delta}(t)\geq 0, \tag{2.10}$$

for all $g\in \mathcal{D}_*(\mathbb{R})$ and we obtain that $b_{\alpha,\beta,\gamma,\delta}(\lambda_0,\lambda)$ is nonnegative for all $\lambda_0,\ \lambda\in\mathbb{R}$.

LEMMA 2.3. *Let $F\in L^p([0,+\infty[,d\mu_{\alpha,\beta})$, $p\in[1,2[$, such that F is essentially bounded in some neighbourhood of* 0 *and $\mathcal{F}_{\alpha,\beta}(F)\geq 0$. Then $\mathcal{F}_{\alpha,\beta}(F)\in L^1([0,+\infty[,d\nu_{\alpha,\beta})$ and we have*

$$F(t) = \int_0^\infty \mathcal{F}_{\alpha,\beta}(F)(\lambda)\,\varphi_\lambda^{(\alpha,\beta)}(t)\,d\nu_{\alpha,\beta}(\lambda),$$

almost everywhere on $[0,+\infty[$.

(See [8, pp. 147–148].)

PROPOSITION 2.3. *Let* $\alpha > \beta > -1/2$, $\gamma > \delta > -1/2$, *and* $\gamma + \delta < \alpha + \beta$. *Then we have*

$$\forall \lambda_0 \in \mathbb{R}, \forall t \geq 0, \qquad \varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \int_0^\infty b_{\alpha,\beta,\gamma,\delta}(\lambda_0, \lambda)\, \varphi_\lambda^{(\gamma,\delta)}(t)\, d\nu_{\gamma,\delta}(\lambda). \tag{2.11}$$

Proof. The result is a consequence of Theorem 2.2 and Lemma 2.3.

PROPOSITION 2.4. *Let* $\alpha > \beta > -1/2$. *Then for all* λ_0, $\lambda \in \mathbb{R}$, *we have*

$$\lim_{(\gamma,\delta)\to(-1/2,-1/2)} b_{\alpha,\beta,\gamma,\delta}(\lambda_0, \lambda) = (2\pi)^{-1/2} \int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t) \cos \lambda t\, dt. \tag{2.12}$$

Proof. For all λ_0, $\lambda \in \mathbb{R}$, we have

$$b_{\alpha,\beta,\gamma,\delta}(\lambda_0, \lambda) = (2\pi)^{-1/2} 2^{2(\gamma+\delta+1)} \times \int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\, \varphi_\lambda^{(\gamma,\delta)}(t) (\operatorname{sh} t)^{2\gamma+1} (\operatorname{ch} t)^{2\delta+1}\, dt.$$

If we impose that $-1/2 \leq \delta \leq \gamma$ and $\gamma + \delta + 1 < (1/2)(\alpha + \beta + 1)$, we deduce that there exists a constant $k > 0$ such that for all $t \geq 0$

$$\left| \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\, \varphi_\lambda^{(\gamma,\delta)}(t) (\operatorname{sh} t)^{2\gamma+1} (\operatorname{ch} t)^{2\delta+1} \right| \leq k^2 (1 + t)^2\, e^{-(1/2)(\alpha+\beta+1)t}.$$

Now by applying the dominated convergence theorem we deduce the result.

Remark. We recall that the Fourier-cosine $\mathcal{F}_0$ is defined for an even and integrable function f on $\mathbb{R}$ (with respect to the Lebesgue measure) by

$$\forall \lambda \in \mathbb{R}, \qquad \mathcal{F}_0(f)(\lambda) = \int_0^\infty f(t) \cos \lambda t\, dt.$$

Its inverse $\mathcal{F}_0^{-1}$ is given for a regular function h by

$$\forall t \geq 0, \qquad \mathcal{F}_0^{-1}(h)(t) = \frac{2}{\pi} \int_0^\infty h(\lambda) \cos \lambda t\, d\lambda.$$

Hence we see that

$$\lim_{(\gamma,\delta)\to(-1/2,-1/2)} b_{\alpha,\beta,\gamma,\delta}(\lambda_0, \lambda) = (2\pi)^{-1/2} \mathcal{F}_0\left(\varphi_{\lambda_0}^{(\alpha,\beta)} \right)(\lambda).$$

THEOREM 2.3. *Let $\alpha \geq \beta \geq -1/2$, $(\alpha, \beta) \neq (-1/2, -1/2)$. Then the function $\varphi_{\lambda_0}^{(\alpha,\beta)}(t)$ possesses the following dual Mehler representation*

$$\forall t \geq 0, \forall \lambda_0 \in \mathbb{R}, \qquad \varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda)\cos \lambda t \, d\lambda, \quad (2.13)$$

where $b_{\alpha,\beta}(\lambda_0, \lambda)$ is a nonnegative kernel.

Proof. (i) Let $\alpha > \beta > -1/2$, and $\lambda_0 \in \mathbb{R}$. We put for all $\lambda \in \mathbb{R}$

$$b_{\alpha,\beta}(\lambda_0, \lambda) = \mathcal{F}_0\left(\varphi_{\lambda_0}^{(\alpha,\beta)}\right)(\lambda). \quad (2.14)$$

By using the nonnegativity of the function $\lambda \to b_{\alpha,\beta}(\lambda_0, \lambda)$ and Corollary 1.26 of [14], we deduce from the inversion formula of the Fourier-cosine transform $\mathcal{F}_0$ that

$$\varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda)\cos \lambda t \, d\lambda.$$

(ii) For $\beta = -1/2$, we proceed in the same way as in (i).

(iii) For $\alpha = \beta > -1/2$, the result is deduced from the relation $\varphi_{\lambda}^{(\alpha,-1/2)}(2t) = \varphi_{2\lambda}^{(\alpha,\alpha)}(t)$.

EXAMPLE. For $\alpha = 1/2$ and $\beta = -1/2$, a computation gives the kernel $b_{1/2,-1/2}(\lambda_0, \lambda)$. We have

$$\forall \lambda_0, \lambda \in \mathbb{R}, \qquad b_{1/2,-1/2}(\lambda_0, \lambda)$$
$$= \frac{\pi \operatorname{sh} \pi\lambda_0}{2\lambda_0}\left[\frac{\operatorname{ch} \pi\lambda + \operatorname{ch} \pi\lambda_0}{(\operatorname{ch} \pi(\lambda_0 + \lambda) + 1)(\operatorname{ch} \pi(\lambda_0 - \lambda) + 1)}\right].$$

Remark. There is a striking contrast between the kernel $K_{\alpha,\beta}(t, u)$ given in (1.10) defining the Mehler representation of $\varphi_{\lambda}^{(\alpha,\beta)}(t)$ with respect to the variable t and the kernel $b_{\alpha,\beta}(\lambda_0, \lambda)$ defining the dual Mehler representation of $\varphi_{\lambda_0}^{(\alpha,\beta)}(t)$ with respect to the argument λ. In particular the function $\lambda \to b_{\alpha,\beta}(\lambda_0, \lambda)$ has no compact support in contrast with the function $u \to K_{\alpha,\beta}(t, u)$.

In the following we shall give the main properties of the kernel $b_{\alpha,\beta}$.

Properties of the Kernel $b_{\alpha,\beta}$

THEOREM 2.4. *The kernel $b_{\alpha,\beta}$ satisfies the following properties.*

(i) *For all $\lambda_0, \lambda \in \mathbb{R}$, $b_{\alpha,\beta}$ is nonnegative and we have*

$$\frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda)\, d\lambda = 1.$$

(ii) *The kernel* $b_{\alpha,\beta}(\lambda_0, \lambda)$ *can be extended to an even analytic function on the set* $\{(\lambda_0, \lambda) \in \mathbb{C}^2 / |\mathrm{Im}\ \lambda_0| + |\mathrm{Im}\ \lambda| < \rho\}$.

(iii) *For all* $\lambda \in \mathbb{R}$, *the function* $\lambda_0 \to b_{\alpha,\beta}(\lambda_0, \lambda)$ *is even and analytic on the strip* $\{\lambda_0 \in \mathbb{C} / |\mathrm{Im}\ \lambda_0| < \rho\}$.

(iv) *There exists a constant* $c > 0$, *such that for all* $\lambda \in \mathbb{C}$ *and* $\lambda_0 \in \mathbb{C}$, $|\mathrm{Im}\ \lambda_0| + |\mathrm{Im}\ \lambda| < \rho$, *we have*

$$\left|b_{\alpha,\beta}(\lambda_0, \lambda)\right| \le c\left(\frac{1}{\rho - |\mathrm{Im}\ \lambda_0|} + \frac{1}{(\rho - |\mathrm{Im}\ \lambda_0|)^2}\right).$$

(v) *For all* $\lambda_0 \in \mathbb{C}$, $|\mathrm{Im}\ \lambda_0| < \rho$, *the function* $\lambda \to b_{\alpha,\beta}(\lambda_0, \lambda)$ *is in* $S_*(\mathbb{R})$.

Proof. (i) The nonnegativity of $b_{\alpha,\beta}$ is already shown. On the other hand we have

$$\frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda)\, d\lambda = \varphi_{\lambda_0}^{(\alpha,\beta)}(0) = 1.$$

(ii) We have

$$\forall \lambda,\ \lambda_0 \in \mathbb{R}, \qquad b_{\alpha,\beta}(\lambda_0, \lambda) = \int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t) \cos \lambda t\, dt.$$

By using (1.6), there exists $k_0 > 0$ such that

$$\forall \lambda,\ \lambda_0 \in \mathbb{C}, \forall t \ge 0, \qquad \left|\varphi_{\lambda_0}^{(\alpha,\beta)}(t)\cos \lambda t\right| \le k_0(1+t)e^{(|\mathrm{Im}\ \lambda_0| + |\mathrm{Im}\ \lambda| - \rho)t}.$$

Hence we deduce that $b_{\alpha,\beta}$ is well defined on the set $\{(\lambda_0, \lambda) \in \mathbb{C}^2 / |\mathrm{Im}\ \lambda_0| + |\mathrm{Im}\ \lambda| < \rho\}$ and it is analytic on this set.

(iii) It is a consequence of (ii).

(iv) Using (1.6) we obtain for all $\lambda_0 \in \mathbb{C}$, $|\mathrm{Im}\ \lambda_0| < \rho$, and $\lambda \in \mathbb{R}$

$$\left|b_{\alpha,\beta}(\lambda_0, \lambda)\right| \le \int_0^\infty \left|\varphi_{\lambda_0}^{(\alpha,\beta)}(t)\cos \lambda t\right| dt \le k_0 \int_0^\infty (1+t)e^{(|\mathrm{Im}\ \lambda_0| - \rho)t}\, dt$$

and integration by parts gives the result.

(v) Let $\lambda_0 \in \mathbb{C}$, $|\mathrm{Im}\ \lambda_0| < \rho$, and $\lambda \in \mathbb{R}$. We have for all $n, m \in \mathbb{N}$

$$\begin{aligned}\lambda^{2m}\frac{d^{2n}}{d\lambda^{2n}} b_{\alpha,\beta}(\lambda_0, \lambda) &= (-1)^n \int_0^\infty t^{2n}\varphi_{\lambda_0}^{(\alpha,\beta)}(t)\, \lambda^{2m} \cos \lambda t\, dt \\ &= (-1)^{n+m} \int_0^\infty t^{2n}\varphi_{\lambda_0}^{(\alpha,\beta)}(t)\left(\frac{d^{2m}}{dt^{2m}}\cos \lambda t\right) dt.\end{aligned}$$

By integration by parts we obtain

$$\lambda^{2m} \frac{d^{2n}}{d\lambda^{2n}} b_{\alpha,\beta}(\lambda_0, \lambda) = (-1)^{n+m} \int_0^\infty \frac{d^{2m}}{dt^{2m}} \left(t^{2n} \varphi_{\lambda_0}^{(\alpha,\beta)}(t) \right) \cos \lambda t \, dt.$$

Taking into account the relation (1.6) we conclude that the function $\lambda \to b_{\alpha,\beta}(\lambda_0, \lambda)$ is in $S_*(\mathbb{R})$.

3. DUAL MEHLER TRANSFORMS ASSOCIATED WITH THE JACOBI FUNCTIONS

In this section, using the dual Mehler representation of the functions $\varphi_\lambda^{(\alpha,\beta)}(t)$ given in Theorem 2.3, we shall define integral transforms with the kernel $b_{\alpha,\beta}$ which we call the dual Mehler transform and its transposed. Next we give some properties of these transforms.

Notations. We denote by

- $C_*^\infty(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$.
- $C_{*,b}(\mathbb{R})$ the space of even, continuous, and bounded functions on $\mathbb{R}$.
- For $f \in C_{*,b}(\mathbb{R})$, $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$.

DEFINITION 3.1. The dual Mehler transform denoted by $\chi_{\alpha,\beta}$ is defined by

$$\forall \lambda_0 \in \mathbb{R}, \qquad \chi_{\alpha,\beta}(f)(\lambda_0) = \frac{2}{\pi} \int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) f(\lambda) \, d\lambda, \qquad (3.1)$$

where $b_{\alpha,\beta}$ is the kernel given by the formula (2.14).

Remark. We have for all $\lambda_0 \in \mathbb{R}$ and $t \geq 0$,

$$\varphi_{\lambda_0}^{(\alpha,\beta)}(t) = \chi_{\alpha,\beta}(\cos(.t))(\lambda_0). \qquad (3.2)$$

PROPOSITION 3.1. (i) *For* $f \in C_{*,b}(\mathbb{R})$, $\chi_{\alpha,\beta}(f) \in C_{*,b}(\mathbb{R})$ *and we have* $\|\chi_{\alpha,\beta}(f)\|_\infty \leq \|f\|_\infty$.

(ii) *If* $f \in L^1([0, +\infty[, d\lambda)$ (*the space of integrable functions on* $[0, +\infty[$ *with respect to the Lebesgue measure* $d\lambda$), *then the function* $\lambda_0 \to \chi_{\alpha,\beta}(f)(\lambda_0)$ *is well defined on the strip* $|\mathrm{Im}\, \lambda_0| < \rho$ *and it is analytic on this strip*.

Proof. These results are deduced easily from the properties of the kernel $b_{\alpha,\beta}$.

DEFINITION 3.2. The transposed of $\chi_{\alpha,\beta}$ denoted by ${}^t\chi_{\alpha,\beta}$ is defined for $g \in L^1([0, +\infty[, d\nu_{\alpha,\beta})$ by

$$\forall \lambda \in \mathbb{R}, \qquad {}^t\chi_{\alpha,\beta}(g)(\lambda) = \int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0). \quad (3.3)$$

From the analyticity of the kernel $b_{\alpha,\beta}$ we deduce the following result.

PROPOSITION 3.2. *For $g \in L^1([0, +\infty[, d\nu_{\alpha,\beta})$, the function ${}^t\chi_{\alpha,\beta}(g)$ is analytic on the strip $|\mathrm{Im}\, \lambda| < \rho$.*

PROPOSITION 3.3. *For $f \in L^1([0, +\infty[, d\lambda)$ and $g \in L^1([0, +\infty[, d\nu_{\alpha,\beta})$, we have the following duality relation*

$$\int_0^\infty \chi_{\alpha,\beta}(f)(\lambda_0) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0) = \frac{2}{\pi} \int_0^\infty f(\lambda)\, {}^t\chi_{\alpha,\beta}(g)(\lambda)\, d\lambda. \quad (3.4)$$

Proof. The relation (3.4) is obtained by using Definitions 3.1 and 3.2 and Fubini's theorem.

PROPOSITION 3.4. *For g in $S_*(\mathbb{R})$, the function ${}^t\chi_{\alpha,\beta}(g)$ belongs to $S_*(\mathbb{R})$.*

Proof. Let $g \in S_*(\mathbb{R})$. We have

$$\forall \lambda \in \mathbb{R}, \qquad {}^t\chi_{\alpha,\beta}(g)(\lambda) = \int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0).$$

It is clear that the function ${}^t\chi_{\alpha,\beta}(g) \in C_*^\infty(\mathbb{R})$.

For all $n \in \mathbb{N}$, we have

$$\forall \lambda \in \mathbb{R}, \qquad \lambda^{2n}\, {}^t\chi_{\alpha,\beta}(g)(\lambda) = \int_0^\infty \lambda^{2n} b_{\alpha,\beta}(\lambda_0, \lambda) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0).$$

Now by using the expression of $b_{\alpha,\beta}$, the relation (1.6), and the last formula of the proof of Theorem 2.4, we deduce that there exists $C_n > 0$, such that

$$\forall \lambda, \lambda_0 \in \mathbb{R}, \qquad \left| \lambda^{2n} b_{\alpha,\beta}(\lambda_0, \lambda) \right| \le C_n \left(1 + |\lambda_0|\right)^{2n}.$$

Since $g \in S_*(\mathbb{R})$ and the measure $\nu_{\alpha,\beta}$ is tempered we deduce that $\lambda^{2n}\, {}^t\chi_{\alpha,\beta}(g)(\lambda)$ is bounded. We proceed in the same way to prove that for all $n, m \in \mathbb{N}$, the function

$$\lambda \to \lambda^{2n} \frac{d^{2m}}{d\lambda^{2m}}\, {}^t\chi_{\alpha,\beta}(g)(\lambda)$$

is bounded.

COROLLARY 3.1. *For all $g \in S_*(\mathbb{R})$, we have the formula*

$$\mathcal{F}_{\alpha,\beta}^{-1}(g) = \mathcal{F}_0^{-1} \circ {}^t\chi_{\alpha,\beta}(g), \tag{3.5}$$

where $\mathcal{F}_{\alpha,\beta}^{-1}$ denotes the inverse Fourier–Jacobi transform defined by (1.15) *and $\mathcal{F}_0$ the Fourier-cosine transform.*

Proof. In the same way as in Proposition 3.3, for $f \in C_{*,b}(\mathbb{R})$ and $g \in S_*(\mathbb{R})$ the duality relation (3.4) holds, that is,

$$\int_0^\infty \chi_{\alpha,\beta}(f)(\lambda_0) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0) = \frac{2}{\pi}\int_0^\infty f(\lambda)\, {}^t\chi_{\alpha,\beta}(g)(\lambda)\, d\lambda.$$

We apply this relation to the function $f(\lambda) = \cos \lambda u$, $u \geq 0$, and $g \in S_*(\mathbb{R})$. So that we obtain

$$\int_0^\infty \chi_{\alpha,\beta}(\cos(.u))(\lambda_0) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda_0) = \frac{2}{\pi}\int_0^\infty (\cos \lambda u)\, {}^t\chi_{\alpha,\beta}(g)(\lambda)\, d\lambda.$$

We deduce

$$\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(u) g(\lambda_0)\, d\nu_{\alpha,\beta}(\lambda) = \frac{2}{\pi}\int_0^\infty {}^t\chi_{\alpha,\beta}(g)(\lambda) \cos \lambda u\, d\lambda,$$

which gives the relation (3.5).

LEMMA 3.1. *The Fourier-cosine transform $\mathcal{F}_0$ is a topological isomorphism from $S_*^r(\mathbb{R})$ onto $H_*^{2r/(r+2)}$*, $0 < r \leq 2$.

Proof. Let f be in $S_*^r(\mathbb{R})$. There exists g in $S_*(\mathbb{R})$ such that $f(t) = (\operatorname{ch} t)^{-2\rho/r} g(t)$. Then we have

$$\mathcal{F}_0(f)(\lambda) = \int_0^\infty g(t)(\operatorname{ch} t)^{-2\rho/r} \cos \lambda t\, dt.$$

So we deduce that the function $\lambda \to \mathcal{F}_0(f)(\lambda)$ is even, holomorphic in the strip $|\operatorname{Im} \lambda| < (2/r)\rho$, and C^∞ in the closed strip $|\operatorname{Im} \lambda| \leq (2/r)\rho$.

Moreover, for all $n, m \in \mathbb{N}$, there exists a positive constant $k_{n,m}$ such that for all $\lambda \in D_{2r/(r+2)}$, we have

$$\begin{aligned}
&\left|(1+\lambda^n)\frac{d^m}{d\lambda^m}\mathcal{F}_0(f)(\lambda)\right| \\
&\quad \leq k_{n,m} \sum_{j=0}^n \int_0^{+\infty} \left((1+t)^{m+2} f^{(j)}(t)(\operatorname{ch} t)^{2\rho/r}\right)(\operatorname{ch} t)^{-2\rho/r} \\
&\qquad \times e^{(2\rho/r)t}(1+t)^{-2}\, dt, \\
&\quad \leq 2^\rho k_{n,m} \sum_{j=0}^n N_{j,m+2}(f),
\end{aligned}$$

where

$$N_{j,m}(f) = \sup_{t \geq 0} (1+t)^m (\operatorname{ch} t)^{2\rho/r} |f^{(j)}(t)|$$

($N_{j,m}$ are the semi-norms on $S_*^r(\mathbb{R})$).

Thus

$$P_{n,m}(\mathcal{F}_0(f)) \leq 2^{\rho} k_{n,m} \sum_{j=0}^{n} N_{j,m+2}(f),$$

where $P_{n,m}$ are the semi-norms defined in Subsection 1.1.

Hence $\mathcal{F}_0(S_*^r(\mathbb{R}))$ is included continuously in $H_*^{2r/(r+2)}$.

Now let $h \in H_*^{2r/(r+2)}$. We have

$$\mathcal{F}_0^{-1}(h)(t) = \frac{2}{\pi}\int_0^\infty h(\lambda)\cos \lambda t \, d\lambda$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} h(\lambda) e^{i\lambda t}\, d\lambda.$$

By using the Cauchy integral theorem, we obtain

$$\mathcal{F}_0^{-1}(h)(t) = \frac{e^{-(2/r)\rho t}}{\pi}\int_{-\infty}^{+\infty} h\left(\sigma + i\frac{2}{r}\rho\right) e^{i\sigma t}\, d\sigma.$$

Thus we obtain that $\mathcal{F}_0^{-1}(h) \in S_*^r(\mathbb{R})$.

We conclude by observing that $S_*^r(\mathbb{R})$ and $H_*^{2r/(r+2)}$ are Fréchet spaces.

THEOREM 3.1. *For $0 < r \leq 2$, the transform ${}^t\chi_{\alpha,\beta}$ is a topological isomorphism from H_*^r onto $H_*^{2r/(r+2)}$.*

Proof. From Fubini's theorem and Theorem 2.3, we have for all f in H_*^r and $\lambda_0 \in D_r$

$$\mathcal{F}_0 \circ \mathcal{F}_{\alpha,\beta}^{-1}(f)(\lambda_0) = \int_0^\infty \left(\int_0^\infty f(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)\, d\nu_{\alpha,\beta}(\lambda)\right)\cos \lambda_0 t\, dt$$
$$= \int_0^\infty \left(\int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(t)\cos \lambda_0 t\, dt\right) f(\lambda)\, d\nu_{\alpha,\beta}(\lambda)$$
$$= \int_0^\infty b_{\alpha,\beta}(\lambda,\lambda_0) f(\lambda)\, d\nu_{\alpha,\beta}(\lambda)$$
$$= {}^t\chi_{\alpha,\beta}(f)(\lambda_0).$$

Hence

$$ {}^t\chi_{\alpha,\beta}(f) = \mathcal{F}_0 \circ \mathcal{F}_{\alpha,\beta}^{-1}(f). \tag{3.6}$$

The result is obtained by using Theorem 1.1(ii) and the previous lemma.

THEOREM 3.2. *The transform $\chi_{\alpha,\beta}$ is defined on H_*^1 and it is injective on this space.*

Proof. Let $h \in H_*^1$. For $\lambda_0 \in \mathbb{R}$ we have

$$\chi_{\alpha,\beta}(h)(\lambda_0) = \int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) h(\lambda)\, d\lambda.$$

It is evident that $\chi_{\alpha,\beta}(h)$ is well defined.

By using the duality relation (3.4) we deduce that if $\chi_{\alpha,\beta}(h) = 0$, then for all $g \in S_*(\mathbb{R})$,

$$\int_0^\infty h(\lambda)\, {}^t\chi_{\alpha,\beta}(g)(\lambda)\, d\lambda = 0.$$

From Theorem 3.1, for $g = ({}^t\chi_{\alpha,\beta})^{-1}(h)$, we conclude that $h = 0$.

4. PERMUTATION RELATIONS BETWEEN DIFFERENCE OPERATORS

It is well known that the Jacobi function $t \to \varphi_\lambda^{(\alpha,\beta)}(t)$ is an eigenfunction of the operator $\Delta_{\alpha,\beta}$. Also, it is well known that the Mehler representation of the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the variable t permits us to define the Abel transform and its dual which are permutation operators between the differential operator $\Delta_{\alpha,\beta}$ given in (1.3) and d^2/dt^2 (see [18, 19]. Natural questions arise: whether or not the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ is an eigenfunction of an operator with respect to the dual variable λ and whether or not the operators $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are permutation operators with some operators to define. In this section we shall give an answer to these questions. Especially we shall define a second order difference operator $P_{\alpha,\beta}$ such that the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ is an eigenfunction of $P_{\alpha,\beta}$ corresponding to the eigenvalue $-\operatorname{ch} 2t$. Next we shall define another second order difference operator Q such that $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are permutation operators between $P_{\alpha,\beta}$ and Q.

PROPOSITION 4.1. *For all $\lambda \in \mathbb{C}$ and $t \geq 0$, the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ satisfies the equation*

$$\begin{aligned}(i\lambda+\rho)(i\lambda-1)\left[\frac{1}{2}(i\lambda-\rho)+\alpha+1\right]\varphi_{\lambda-2i}^{(\alpha,\beta)}(t)\\ +(i\lambda-\rho)(i\lambda+1)\left[\frac{1}{2}(i\lambda-\rho)+\beta\right]\varphi_{\lambda+2i}^{(\alpha,\beta)}(t)\\ -i\lambda(\alpha^2-\beta^2)\varphi_\lambda^{(\alpha,\beta)}(t) = -i\lambda(\lambda^2+1)\operatorname{ch}2t\,\varphi_\lambda^{(\alpha,\beta)}(t). \quad (4.1)\end{aligned}$$

Proof. We have for all $\lambda \in \mathbb{C}$ and $t \geq 0$

$$\varphi_\lambda^{(\alpha,\beta)}(t) = R_{1/2(i\lambda-\rho)}^{(\alpha,\beta)}(\operatorname{ch}2t),$$

where

$$R_\mu^{(\alpha,\beta)}(z) = {}_2F_1\left(-\mu, \mu+\rho; \alpha+1, \frac{1}{2}(1-z)\right),$$

${}_2F_1$ is the Gauss hypergeometric function.

For $n \in \mathbb{Z}_+$, $R_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n, normalized by $R_n^{(\alpha,\beta)}(1) = 1$ (see [13]). It satisfies the following three-term recurrence formula (see [15, p. 71])

$$\begin{aligned}2(n+\rho)(2n+\rho-1)(n+\alpha+1)R_{n+1}^{(\alpha,\beta)}(x)\\ = (2n+\rho)\left[(2n+\rho)^2-1\right]xR_n^{(\alpha,\beta)}(x)\\ +(\alpha^2-\beta^2)(2n+\rho)R_n^{(\alpha,\beta)}(x)\\ -2n(n+\beta)(2n+\rho+1)R_{n-1}^{(\alpha,\beta)}(x).\end{aligned}$$

Using the Carlson Lemma (see [16, p. 186]), we obtain the relation (4.1) by analytic continuation in n.

DEFINITION 4.1. For an even and continuous function f on $\mathbb{C}$, we define the second order difference operators $P_{\alpha,\beta}$ and Q by

$$P_{\alpha,\beta}(f)(\lambda) = \begin{cases} U_{\alpha,\beta}(\lambda)f(\lambda-2i) + V_{\alpha,\beta}(\lambda)f(\lambda) + U_{\alpha,\beta}(-\lambda)f(\lambda+2i), \\ \qquad \text{if } \lambda \neq 0, \lambda \neq \pm i \\ (\alpha^2-\beta^2-1)f(2i) - (\alpha^2-\beta^2)f(0), \\ \qquad \text{if } \lambda = 0, \\ \frac{1}{4}(\alpha^2-\beta^2+4\alpha)f(i) - \frac{1}{4}(\alpha^2-\beta^2+4\alpha+4)f(3i), \\ \qquad \text{if } \lambda = \pm i, \end{cases}$$

$$Q(f)(\lambda) = -\frac{1}{2}\left[f(\lambda+2i) + f(\lambda-2i)\right],$$

where

$$U_{\alpha,\beta}(\lambda) = \frac{(i\lambda+\rho)(i\lambda+\alpha-\beta+1)}{2\lambda(\lambda-i)} \quad \text{and}$$

$$V_{\alpha,\beta}(\lambda) = -\frac{\alpha^2-\beta^2}{\lambda^2+1}.$$

THEOREM 4.1. *The function* $\lambda \to \varphi_\lambda^{(\alpha,\beta)}(t)$ *is an eigenfunction of the operator* $P_{\alpha,\beta}$ *corresponding to the eigenvalue* $-\operatorname{ch} 2t$, *that is,*

$$\forall\lambda \in \mathbb{C},\ t \geq 0, \qquad P_{\alpha,\beta}\left(\varphi_\lambda^{(\alpha,\beta)}(t)\right) = -(\operatorname{ch} 2t)\,\varphi_\lambda^{(\alpha,\beta)}(t). \tag{4.2}$$

Proof. It is a consequence of Proposition 4.1 and the definition of the operator $P_{\alpha,\beta}$.

Remark. For all g in $\mathcal{D}_*(\mathbb{R})$, we have

$$Q\mathcal{F}_0(g) = -\mathcal{F}_0((\operatorname{ch} 2t)g) \tag{4.3}$$

$$P_{\alpha,\beta}\left(\mathcal{F}_{\alpha,\beta}(g)\right) = -\mathcal{F}_{\alpha,\beta}((\operatorname{ch} 2t)g). \tag{4.4}$$

THEOREM 4.2. *Let f be in $\mathbb{H}_*(\mathbb{C})$. We have the following permutation relations*

$$P_{\alpha,\beta}\,\chi_{\alpha,\beta}(f) = \chi_{\alpha,\beta}Q(f), \tag{4.5}$$

$${}^t\chi_{\alpha,\beta}P_{\alpha,\beta}(f) = Q\,{}^t\chi_{\alpha,\beta}(f). \tag{4.6}$$

Proof. Let f be in $\mathbb{H}_*(\mathbb{C})$. By the Paley–Wiener theorem for the Fourier-cosine transform $\mathcal{F}_0$ there exists $g \in \mathcal{D}_*(\mathbb{R})$ such that $f = \mathcal{F}_0(g)$. Then, from the relation (4.3) and Definition 3.1, we have for all $\lambda_0 \in \mathbb{R}$

$$\begin{aligned}\chi_{\alpha,\beta}Q(f)(\lambda_0) &= \chi_{\alpha,\beta}Q(\mathcal{F}_0(g))(\lambda_0)\\ &= -\chi_{\alpha,\beta}\mathcal{F}_0((\operatorname{ch} 2t)g)(\lambda_0)\\ &= -\frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0,\lambda)\,\mathcal{F}_0((\operatorname{ch} 2t)g)(\lambda)\,d\lambda.\end{aligned}$$

By using Fubini's theorem, we obtain

$$\chi_{\alpha,\beta}Q(f)(\lambda_0) = -\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\,g(t)\operatorname{ch} 2t\,dt.$$

On the other hand, we have for all $\lambda_0 \in \mathbb{R}$

$$P_{\alpha,\beta}\chi_{\alpha,\beta}(f)(\lambda_0) = P_{\alpha,\beta}\chi_{\alpha,\beta}\big(\mathcal{F}_0(g)\big)(\lambda_0).$$

By using Fubini's theorem, we have for all $\lambda_0 \in \mathbb{R}$

$$\begin{aligned}\chi_{\alpha,\beta}\mathcal{F}_0(g)(\lambda_0) &= \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0,\lambda)\mathcal{F}_0(g)(\lambda)\,d\lambda\\ &= \int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)g(t)\,dt.\end{aligned}$$

Hence using Theorem 4.1 we deduce that

$$P_{\alpha,\beta}\chi_{\alpha,\beta}(f)(\lambda_0) = -\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)g(t)\operatorname{ch}2t\,dt.$$

Thus we obtain the relation (4.5).

For the relation (4.6), let f be in $\mathbb{H}_*(\mathbb{C})$. By Theorem 1.1(i) there exists $g \in \mathcal{D}_*(\mathbb{R})$ such that $f = \mathcal{F}_{\alpha,\beta}(g)$. So that from the relations (4.4) and (3.5), we have for all $\lambda_0 \in \mathbb{R}$

$$\begin{aligned}{}^t\chi_{\alpha,\beta}P_{\alpha,\beta}(f)(\lambda_0) &= {}^t\chi_{\alpha,\beta}P_{\alpha,\beta}\big(\mathcal{F}_{\alpha,\beta}(g)\big)(\lambda_0)\\ &= -{}^t\chi_{\alpha,\beta}\big(\mathcal{F}_{\alpha,\beta}((\operatorname{ch}2t)g)\big)(\lambda_0)\\ &= -\mathcal{F}_0((\operatorname{ch}2t)g)(\lambda_0).\end{aligned}$$

On the other hand, we have for all $\lambda_0 \in \mathbb{R}$

$$\begin{aligned}Q{}^t\chi_{\alpha,\beta}(f)(\lambda_0) &= Q{}^t\chi_{\alpha,\beta}\mathcal{F}_{\alpha,\beta}(g)(\lambda_0)\\ &= Q\mathcal{F}_0(g)(\lambda_0)\\ &= -\mathcal{F}_0((\operatorname{ch}2t)g)(\lambda_0).\end{aligned}$$

Thus we complete the proof of the relation (4.6).

PROPOSITION 4.2. *Let f be an even analytic uniformly almost periodic function on $\mathbb{C}$. Then we have*

$$\forall\lambda \in \mathbb{R}, \qquad P_{\alpha,\beta}\chi_{\alpha,\beta}(f)(\lambda) = \chi_{\alpha,\beta}Q(f)(\lambda).$$

Proof. From the Polynomial Approximation Theorem, see [4, p. 148], the function f is an uniform limit of a sequence of even exponential polynomials functions, that is,

$$f = \lim_{n\to\infty} f_n,$$

where $f_n(\lambda)$, $\lambda \in \mathbb{C}$, is given by

$$f_n(\lambda) = \sum_{k=1}^{n} a_k \cos t_k \lambda, \qquad a_k \in \mathbb{C} \text{ and } t_k \geq 0.$$

But from the relations (3.2) and (4.2), we have

$$\forall \lambda \in \mathbb{R}, \qquad P_{\alpha,\beta}\chi_{\alpha,\beta}(f_n)(\lambda) = \chi_{\alpha,\beta}Q(f_n)(\lambda).$$

Thus we obtain the result by taking $n \to +\infty$.

5. INVERSION FORMULAS FOR THE OPERATORS $\chi_{\alpha,\beta}$ AND ${}^t\chi_{\alpha,\beta}$

In this section we shall define some functions spaces on which we can invert the operators $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$.

Notations. We denote by

— $S_{*,0}(\mathbb{R})$ the subspace of $S_*(\mathbb{R})$ consisting of functions f satisfying

$$\forall n \in \mathbb{N}, \qquad \frac{d^{2n}}{d\lambda^{2n}} f(0) = 0.$$

— $S_{*,0}^{\perp}(\mathbb{R})$ the subspace of $S_*(\mathbb{R})$ consisting of functions f satisfying

$$\forall n \in \mathbb{N}, \qquad \int_0^{\infty} f(\lambda)\lambda^{2n}\,d\lambda = 0.$$

— $H^r_{*,0}$ the subspace of H^r_* consisting of functions h satisfying

$$\forall n \in \mathbb{N}, \qquad \int_0^{\infty} h(\lambda)\lambda^{2n}\,d\lambda = 0.$$

— $H^r_{*,\perp}$ the subspace of H^r_* consisting of functions h satisfying

$$\forall n \in \mathbb{N}, \qquad \int_0^{\infty} h(\lambda)c_n(\lambda)\,d\nu_{\alpha,\beta}(\lambda) = 0,$$

where

$$c_n(\lambda) = \int_0^{\infty} b_{\alpha,\beta}(\lambda, s)s^{2n}\,ds.$$

LEMMA 5.1. *The Fourier-cosine transform* $\mathcal{F}_0$ *is a topological isomorphism from*

(i) $S^{\perp}_{*,0}(\mathbb{R})$ *onto* $S_{*,0}(\mathbb{R})$.

(ii) $(\operatorname{ch} t)^{-2\rho/r} S_{*,0}(\mathbb{R})$ *onto* $H^{2r/(r+2)}_{*,0}$, $0 < r \leq 2$.

Proof. (i) We know that $\mathcal{F}_0$ is a topological isomorphism from $S_*(\mathbb{R})$ onto itself. Furthermore for any $f \in S_*(\mathbb{R})$ and $n \in \mathbb{N}$, we have

$$\frac{d^{2n}}{d\lambda^{2n}} \mathcal{F}_0(f)(0) = (-1)^n \int_0^{\infty} f(\lambda) \lambda^{2n} \, d\lambda.$$

We deduce that $f \in S^{\perp}_{*,0}(\mathbb{R})$ if and only if $\mathcal{F}_0(f) \in S_{*,0}(\mathbb{R})$.

(ii) From Lemma 3.1, we know that $\mathcal{F}_0$ is a topological isomorphism from $S^r_*(\mathbb{R})$ onto $H^{2r/(r+2)}_*$. We obtain the result by observing that $f \in (\operatorname{ch} t)^{-2\rho/r} S_{*,0}(\mathbb{R})$ if and only if $\mathcal{F}_0(f) \in H^{2r/(r+2)}_{*,0}$.

LEMMA 5.2. *Let* $\varepsilon \in [2, +\infty[$. *Then the operator* $\mathbb{A}_{\alpha,\beta}$ *defined by*

$$\mathbb{A}_{\alpha,\beta}(f)(t) = (2\pi)^{-1/2} A_{\alpha,\beta}(t) f(t)$$

is a topological isomorphism from $(\operatorname{ch} t)^{-\varepsilon\rho} S_{*,0}(\mathbb{R})$ *onto* $(\operatorname{ch} t)^{-(\varepsilon-2)\rho} \times S_{*,0}(\mathbb{R})$.

Proof. It suffices to remark that for all $\gamma \in \mathbb{R}$ and $f \in S_{*,0}(\mathbb{R})$ the function $t \to (\tanh t)^{\gamma} f(t)$ belongs to $S_{*,0}(\mathbb{R})$.

PROPOSITION 5.1. *The operator* $\mathcal{M}_0$ *defined by*

$$\mathcal{M}_0(f) = (2\pi)^{-1/2} \mathcal{F}_0\left[A_{\alpha,\beta} \mathcal{F}_0^{-1}(f) \right] \tag{5.1}$$

is a topological isomorphism from $H^{2r/(r+2)}_{*,0}$ *onto* $H^{2r/(2-r)}_{*,0}$, *for* $0 < r \leq 1$.

Proof. Since $\mathcal{M}_0 = \mathcal{F}_0 \circ \mathbb{A}_{\alpha,\beta} \circ \mathcal{F}_0^{-1}$, then the result is deduced from Lemmas 5.1 and 5.2.

LEMMA 5.3. *The Fourier–Jacobi transform* $\mathcal{F}_{\alpha,\beta}$ *is a topological isomorphism from* $(\operatorname{ch} t)^{-2\rho/r} S_{*,0}(\mathbb{R})$ *onto* $H^r_{*,\perp}$, $0 < r \leq 2$.

Proof. From Theorem 1.1(ii), $\mathcal{F}_{\alpha,\beta}$ is a topological isomorphism from $(\operatorname{ch} t)^{-2\rho/r} S_*(\mathbb{R})$ onto H^r_*. Let f be in $(\operatorname{ch} t)^{-2\rho/r} S_{*,0}(\mathbb{R})$. By using the inversion formula for $F_{\alpha,\beta}$ we have

$$f(t) = \int_0^\infty F_{\alpha,\beta}(f)(\lambda)\,\varphi_\lambda^{(\alpha,\beta)}(t)\,d\nu_{\alpha,\beta}(\lambda).$$

Then

$$\forall n \in \mathbb{N}, \qquad \frac{d^{2n}}{dt^{2n}} f(0) = \int_0^\infty F_{\alpha,\beta}(f)(\lambda)\,\frac{d^{2n}}{dt^{2n}}\varphi_\lambda^{(\alpha,\beta)}(t)\Big|_{t=0}\,d\nu_{\alpha,\beta}(\lambda).$$

But from the relation (2.13) we have

$$\forall n \in \mathbb{N}, \qquad \frac{d^{2n}}{dt^{2n}}\varphi_\lambda^{(\alpha,\beta)}(t)\Big|_{t=0} = (-1)^n \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda,s)\,s^{2n}\,ds.$$

Hence we obtain

$$\forall n \in \mathbb{N}, \qquad \int_0^\infty F_{\alpha,\beta}(f)(\lambda)\,c_n(\lambda)\,d\nu_{\alpha,\beta}(\lambda) = 0.$$

Thus we conclude that $f \in (\operatorname{ch} t)^{-2\rho/r} S_{*,0}(\mathbb{R})$ if and only if $F_{\alpha,\beta}(f) \in H^r_{*,\perp}$.

THEOREM 5.1. (i) *For* $0 < r \le 1$, *the dual Mehler transform* $\chi_{\alpha,\beta}$ *is a topological isomorphism from* $H^{2r/(2-r)}_{*,0}$ *onto* $H^r_{*,\perp}$.

(ii) *For* $0 < r \le 2$, *the transform* ${}^t\chi_{\alpha,\beta}$ *is a topological isomorphism from* $H^r_{*,\perp}$ *onto* $H^{2r/(r+2)}_{*,0}$.

Proof. (i) It is deduced from the relation $\chi_{\alpha,\beta} \circ F_0 \circ \mathbb{A}_{\alpha,\beta} = F_{\alpha,\beta}$.

(ii) It is obtained by using Lemmas 5.1 and 5.3.

PROPOSITION 5.2. *The operator* M *defined by*

$$M(f) = (2\pi)^{-1/2} F_{\alpha,\beta}\left[A_{\alpha,\beta} F^{-1}_{\alpha,\beta}(f)\right] \tag{5.2}$$

is a topological isomorphism from $H^r_{*,\perp}$ *onto* $H^{r/(1-r)}_{*,\perp}$, $0 < r < 1$.

Proof. The operator M can also be written in the form

$$M = F_{\alpha,\beta} \circ \mathbb{A}_{\alpha,\beta} \circ F^{-1}_{\alpha,\beta}.$$

Then the result is a consequence of Lemmas 5.2 and 5.3.

COROLLARY 5.1. *We have*

$$M = \left({}^t\chi_{\alpha,\beta}\right)^{-1} \circ M_0 \circ {}^t\chi_{\alpha,\beta}. \tag{5.3}$$

Proof. It is obtained by remarking that

$$ {}^t\chi_{\alpha,\beta} = \mathcal{F}_0 \circ \mathcal{F}_{\alpha,\beta}^{-1} $$

and using Proposition 5.1.

PROPOSITION 5.3. *We have the following inversion formula for the operator* $\chi_{\alpha,\beta}$:

$$ \text{(i)} \quad \forall f \in H^r_{*,\perp},\ 0 < r \leq 1,\ f = \chi_{\alpha,\beta} \circ \mathcal{M}_0 \circ {}^t\chi_{\alpha,\beta}(f), \tag{5.4} $$

$$ \text{(ii)} \quad \forall f \in H^r_{*,\perp},\ 0 < r < 1,\ f = \chi_{\alpha,\beta} \circ {}^t\chi_{\alpha,\beta} \circ \mathcal{M}(f). \tag{5.5} $$

Proof. From the inversion formula for the transform $\mathcal{F}_{\alpha,\beta}^{-1}$ we have

$$ \begin{aligned} \forall \lambda_0 \in \mathbb{R}, \qquad f(\lambda_0) &= \int_0^\infty \mathcal{F}_{\alpha,\beta}^{-1}(f)(t)\, \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\, d\mu_{\alpha,\beta}(t) \\ &= (2\pi)^{-1/2} \int_0^\infty \mathcal{F}_{\alpha,\beta}^{-1}(f)(t)\, \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\, A_{\alpha,\beta}(t)\, dt. \end{aligned} $$

Using Corollary 3.1 and the relation (2.13) we obtain

$$ \begin{aligned} \forall \lambda_0 \in \mathbb{R}, \qquad f(\lambda_0) &= 2^{1/2}\pi^{-3/2} \int_0^\infty \mathcal{F}_0^{-1} \circ {}^t\chi_{\alpha,\beta}(f)(t) \\ &\qquad \times \left(\int_0^\infty b_{\alpha,\beta}(\lambda_0, \lambda) \cos \lambda t\, d\lambda \right) A_{\alpha,\beta}(t)\, dt \\ &= (2\pi)^{-1/2} \chi_{\alpha,\beta}\left[\mathcal{F}_0\left(A_{\alpha,\beta} \mathcal{F}_0^{-1}\left({}^t\chi_{\alpha,\beta}(f) \right) \right) \right](\lambda_0) \\ &= \chi_{\alpha,\beta} \circ \mathcal{M}_0 \circ {}^t\chi_{\alpha,\beta}(f)(\lambda_0). \end{aligned} $$

The relation (5.5) is obtained by using (5.3).

PROPOSITION 5.4. *We have the following inversion formula for the operator* ${}^t\chi_{\alpha,\beta}$:

$$ \text{(i)} \quad \forall f \in H^{2r/(r+2)}_{*,0},\ 0 < r \leq 1,\ f = {}^t\chi_{\alpha,\beta} \circ \chi_{\alpha,\beta} \circ \mathcal{M}_0(f), \tag{5.6} $$

$$ \text{(ii)} \quad \forall f \in H^{2r/(2-r)}_{*,0},\ 0 < r < 1,\ f = {}^t\chi_{\alpha,\beta} \circ \mathcal{M} \circ \chi_{\alpha,\beta}(f). \tag{5.7} $$

Proof. The relation (5.6) is obtained by applying (5.4) to the function $({}^t\chi_{\alpha,\beta})^{-1}(f)$. The relation (5.7) is obtained by applying (5.5) to the function $\chi_{\alpha,\beta}(f)$.

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