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REALIZABILITY OF REPRESENTATIONS OF FINITE GROUPS

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A complex character of a finite group G is called orthogonal if it is the character of a real representation. If all characters of G are orthogonal, then G is called totally orthogonal. Totally orthogonal groups are generated by involutions. Necessary and sufficient conditions for total orthogonality are obtained for 2-groups, for split extensions of elementary abelian 2-groups, for Frobenius groups, and for groups whose irreducible character degrees are bounded by 2. Sylow 2-subgroups of alternating groups and finite reflection groups are observed to be totally orthogonal.

1. Introduction

A complex character of a finite group G is called orthogonal if it is the character of a real representation. The determination of the number of complex irreducible characters of G that are orthogonal was cited by Brauer in 1963 as an old question. We introduce here the notion of total orthogonality, the extreme case in which all characters of G are orthogonal.

The only substantial progress to date toward a solution of Brauer's problem seems to be in the two papers [6, 7] of Gow, in which answers are obtained for groups with a cyclic Sylow 2-subgroup and for 2-nilpotent groups with an abelian Sylow 2-subgroup.

We develop below some basic facts about totally orthogonal groups, perhaps the most useful being that such a group is generated by involutions. Necessary and sufficient conditions for total orthogonality are obtained for split extensions of elementary abelian 2-groups and for groups whose character degrees are bounded by 2. It is established that Sylow 2-subgroups of alternating groups and of finite reflection groups are totally orthogonal. Also, Brauer's question for Frobenius groups is reduced to the corresponding questions for Frobenius kernels and complements.

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2. Background

All groups considered will be finite. The real and complex fields will be denoted by $\mathbb{R}$ and $\mathbb{C}$. The set of all characters of irreducible $\mathbb{C}$-representations of a group G will be denoted by $\mathrm{Irr}(G)$, and the set of all characters of $\mathbb{R}$-irreducible $\mathbb{R}$-representations by $\mathrm{Irr}_{\mathbb{R}}(G)$. We will write $\mathscr{L}(G)$ and $\mathscr{N}(G)$ for the subsets of linear (degree 1) and nonlinear characters in $\mathrm{Irr}(G)$, respectively.

Suppose $\chi \in \mathrm{Irr}(G)$. If χ is the character of a real representation, it is said to be of type 1. If χ is real-valued but not of type 1, it is of type 2; if it is not real-valued, it is of type 3. The Frobenius–Schur *indicator* of χ is $\nu(\chi) = |G|^{-1} \sum \{\chi(x^2) \colon x \in G\}$. The Frobenius–Schur Theorem asserts that $\nu(\chi)$ is 1 if χ is of type $1, -1$ if χ is of type 2, and 0 if χ is of type 3. Motivated by Brauer's question and the Frobenius–Schur Theorem we define $\nu^+(G)$, $\nu^-(G)$, and $\nu^0(G)$ to be the number of characters of types 1, 2, and 3, respectively, in $\mathrm{Irr}(G)$. Thus $\nu^+(G) + \nu^-(G) + \nu^0(G) = |\mathrm{Irr}(G)| = c(G)$, the class number of G.

An element x of G is called *real* if it is conjugate to x^{-1}, and a conjugacy class is *real* if it consists of real elements. The following result is well known; it is an easy consequence of Brauer's Lemma [5, p. 142] on orbits in character tables.

Theorem 2.1. *The number $\nu^+(G) + \nu^-(G)$ of real-valued characters in* $\mathrm{Irr}(G)$ *is equal to the number of real conjugacy classes in G.* □

A group G is called *ambivalent* if every class of G is real. Write $\mathscr{A}$ for the class of ambivalent groups. Observe that, as an immediate consequence of Theorem 2.1, $G \in \mathscr{A}$ if and only if all characters of G are real-valued, i.e. $\nu^0(G) = 0$.

If $H \leq G$, define subsets of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$ as follows:

(1) $\mathrm{Ext}_H(G) = \{\chi \in \mathrm{Irr}(G) \colon \chi|_H \in \mathrm{Irr}(H)\}$,

(2) $\mathrm{Ind}_H(G) = \{\chi \in \mathrm{Irr}(G) \colon \chi = \varphi^G$ for some $\varphi \in \mathrm{Irr}(H)\}$,

(3) $\mathrm{Ext}^G(H) = \{\varphi \in \mathrm{Irr}(H) \colon \varphi = \chi|_H$ for some $\chi \in \mathrm{Irr}(G)\}$, and

(4) $\mathrm{Ind}^G(H) = \{\varphi \in \mathrm{Irr}(H) \colon \varphi^G \in \mathrm{Irr}(G)\}$.

3. Total orthogonality

A finite group G will be called *totally orthogonal* if every character of G is orthogonal, or equivalently if $\nu^+(G) = c(G)$. We will write $\mathscr{TO}$ for the class of totally orthogonal groups. Clearly $\mathscr{TO} \subseteq \mathscr{A}$, i.e. totally orthogonal groups are ambivalent.

For any group G and $x \in G$ we will write $J_G(x) = \{y \in G \colon y^2 = x\}$, and $\mathrm{SQRT}_G(x) = |J_G(x)|$. Note that G is totally orthogonal if and only if $\mathrm{SQRT}_G(1) = \sum \{\chi(1) \colon \chi \in \mathrm{Irr}(G)\}$, by the Frobenius–Schur count of involutions. Note also that if G is ambivalent, then both $Z(G)$ and $\mathscr{L}(G) \cong G/G'$ are elementary abelian 2-groups.

Proposition 3.1. *If* $\mathrm{SQRT}_G(1) > \frac{1}{2}|G|$, *then* $G \in \mathcal{TO}$.

Proof. For every $\chi \in \mathrm{Irr}(G)$ we have

$$\begin{aligned} v(\chi) &= |G|^{-1}[\sum \{\chi(x^2)\colon x \in J_G(1)\} + \sum \{\chi(x^2)\colon x \in G \setminus J_G(1)\}] \\ &\geq |G|^{-1}[\chi(1)\mathrm{SQRT}_G(1) - \chi(1)(|G| - \mathrm{SQRT}_G(1))] \\ &= |G|^{-1}\chi(1)[2\mathrm{SQRT}_G(1) - |G|] > 0, \end{aligned}$$

so $v(\chi) = 1$. □

Proposition 3.2. *Let* A *be an abelian group, and suppose* $\sigma \in \mathrm{Aut}(A)$ *has order* 2, *with* ${}^{\sigma}a = a^{-1}$ *for all* $a \in A \setminus J_A(1)$. *Then the semidirect product* $G = A \rtimes \langle \sigma \rangle$ *is totally orthogonal.*

Proof. Observe that $(a, \sigma)^2 = (1, 1)$ for all $a \in A \setminus J_A(1)$. Thus

$$J_G(1) \supseteq \{(a, \sigma)\colon a \in A \setminus J_A(1)\} \cup \{(b, 1)\colon b \in J_A(1)\} \cup \{(1, \sigma)\},$$

hence

$$\mathrm{SQRT}_G(1) \geq |A \setminus J_A(1)| + |J_A(1)| + 1 > \tfrac{1}{2}|G|,$$

so $G \in \mathcal{TO}$ by Proposition 3.1. □

Theorem 3.3. *If* $G \in \mathcal{TO}$, *then* G *is generated by involutions.*

Proof. Set $N = \langle J_G(1) \rangle \lhd G$. Let $\varphi_1 = 1_N, \ldots, \varphi_k$ be representatives of the G-orbits in $\mathrm{Irr}(N)$. For each φ_i choose $\chi_i \in \mathrm{Irr}(G)$ with $1 \leq (\chi_i, \varphi_i^G) = (\chi_i|_N, \varphi_i) \in \mathbb{Z}$. We may take $\chi_1 = 1_G$. Then, by Clifford's Theorem,

$$\chi_i|_N = (\chi_i|_N, \varphi_i) \sum \{\theta\colon \theta \in \mathrm{Orb}_G(\varphi_i)\},$$

and so

$$\begin{aligned} \mathrm{SQRT}_G(1) &= \sum \{\chi(1)\colon \chi \in \mathrm{Irr}(G)\} \geq \sum_{i=1}^{k} \chi_i(1) \\ &= \sum_{i=1}^{k} (\chi_i|_N, \varphi_i) \sum \{\theta(1)\colon \theta \in \mathrm{Orb}_G(\varphi_i)\} \\ &\geq \sum \{v(\theta)\theta(1)\colon \theta \in \mathrm{Irr}(N)\} = \mathrm{SQRT}_N(1) = \mathrm{SQRT}_G(1). \end{aligned}$$

Thus all are equal, and it follows that $\chi_i = \varphi_i^G$ for all i. In particular, $1 = \chi_1(1) = \varphi_1^G(1) = [G : N]$, so $N = G$. □

Corollary 3.4. *If* $G \in \mathcal{TO}$ *and* $N \leq G$, *with* $[G : N] = 2$, *then* $G = N \rtimes \langle \sigma \rangle$ *for some* σ *of order* 2 *in* G. □

The next corollary of Theorem 3.3 is well known.

Corollary 3.5. *If G is metacyclic, then $G \in \mathscr{TO}$ if and only if G is a dihedral group D_{2m} (of order $2m$) for some m.*

Proof. '$\Leftarrow$'. Apply Proposition 3.2.

'$\Rightarrow$'. Take $N \lhd G$ with $N = \langle a \rangle$ and G/N cyclic. Then $[G:N]=2$ and we may write $G = N \rtimes \langle \sigma \rangle$. Since $G \in \mathscr{A}$ we must have ${}^{\sigma}a = a^{-1}$, so G is dihedral. □

Proposition 3.6. *Suppose $G \in \mathscr{TO}$ and $[G:N]=2$. Then $v^{-}(N)=0$, i.e. N has no characters of type 2.*

Proof. Take $\varphi \in \mathrm{Irr}(N)$. If $\varphi \in \mathrm{Ext}^{G}(N)$, then $\varphi = \chi|_N$ for some $\chi \in \mathrm{Irr}(G)$, so φ is of type 1 since χ is orthogonal. Suppose then that $\varphi \in \mathrm{Ind}^{G}(N)$, say $\varphi^{G} = \chi \in \mathrm{Irr}(G)$. Again χ is orthogonal, so $\chi|_N$ is orthogonal. This time $\chi|_N = \varphi + \varphi^{\sigma}$, for $\sigma \in G \setminus N$, and $\varphi \neq \varphi^{\sigma}$. But also $\chi|_N$ is a sum of characters from $\mathrm{Irr}_{\mathbb{R}}(N)$, so we may conclude (see [15, p. 108]) that either φ and φ^{σ} are both of type 1, or else $\varphi^{\sigma} = \bar{\varphi} \neq \varphi$, and φ is of type 3. □

Corollary 3.7. *If $G \in \mathscr{TO}$ and $[G:N]=2$, then $v^{+}(N)$ is equal to the number of real conjugacy classes in N.* □

Corollary 3.8. *If $G \in \mathscr{TO}$ and $[G:N]=2$, then $N \in \mathscr{TO}$ if and only if $N \in \mathscr{A}$.* □

Proposition 3.9. *Suppose $[G:N]=2$ and $N \in \mathscr{TO}$. Then $v^{-}(G)=0$, i.e. G has no characters of type 2.*

Proof. If $\chi \in \mathrm{Ind}_N(G)$, then $v(\chi)=1$, so suppose $\chi \in \mathrm{Ext}_N(G)$, say with $\chi|_N = \psi \in \mathrm{Irr}(N)$. Thus if $N = \ker(\theta)$, $\theta \in \mathscr{L}(G)$, then $\theta\chi \neq \chi$ and $\psi^{G} = \chi + \theta\chi$. Of course ψ^{G} is orthogonal since $N \in \mathscr{TO}$. If χ is of type 2, then $2\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ [15, p. 108]. But then $2\chi = \chi + \theta\chi$, since $(2\chi, \chi + \theta\chi) > 0$ and the degrees are equal, and we may conclude that $\chi = \theta\chi$, a contradiction. □

Corollary 3.10. *If $[G:N]=2$ and $N \in \mathscr{TO}$, then $v^{+}(G)$ is equal to the number of real conjugacy classes in G.* □

Theorem 3.11. *Suppose G is a 2-group. Then $G \in \mathscr{TO}$ if and only if*

(1) *G is generated by involutions,*

(2) *G is ambivalent, and*

(3) *$v^{-}(N)=0$ whenever $[G:N]=2$.*

Proof. Only the sufficiency of the conditions remains to be proved, and we use induction on $|G|$. The result is clear if $|G| \leq 8$, so assume $|G| > 8$ and take $\chi \in \mathrm{Irr}(G)$. If χ is not faithful, choose $z \neq 1$ in $Z(G) \cap \ker(\chi)$. Then $G/\langle z \rangle$ satisfies (1)–(3), so χ is of type 1 by induction. Assume then that χ is faithful. If χ is of type 2, then

$2\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$, with centralizer algebra $\mathbb{H}$, the quaternion algebra [15, p. 108]. If 2χ is $\mathbb{H}$-imprimitive, it is induced from a character $\psi \in \mathrm{Irr}_{\mathbb{R}}(N)$ of a subgroup N, and we may assume that N is maximal, so $[G:N]=2$. It follows from Clifford's Theorem and [15, Section 13] that $\psi = 2\theta$ for some θ of type 2 in $\mathrm{Irr}(N)$, contradicting (3).

Suppose then that 2χ is $\mathbb{H}$-primitive. It is faithful since χ is faithful, so G is dihedral by [14, p. 524, Lemma], since G is generated by involutions. But then χ being of type 2 contradicts Corollary 3.5, a final contradiction. □

Theorem 3.12. *Suppose $G = E \rtimes H$, with E an elementary abelian 2-group. Then $G \in \mathcal{TO}$ if and only if $\mathrm{Stab}_H(\phi) \in \mathcal{TO}$ for all $\phi \in \mathrm{Irr}(E)$.*

Proof. The sufficiency is an immediate consequence of Mackey's 'little group' method, as discussed, e.g., in [15, p. 62]; we apply the same method to prove the necessity. If $\phi \in \mathrm{Irr}(E)$ and $\theta \in \mathrm{Irr}(\mathrm{Stab}_H(\phi))$, define $\chi_{\phi,\theta}$ on $E \rtimes \mathrm{Stab}_H(\phi)$ by setting $\chi_{\phi,\theta}(v,h) = \phi(v)\theta(h)$, so $\chi_{\phi,\theta}^G \in \mathrm{Irr}(G)$. Since $G \in \mathcal{TO}$ we have $\chi_{\phi,\theta}^G \in \mathrm{Irr}_{\mathbb{R}}(G)$ for all ϕ, θ. Also $(\chi_{\phi,\theta}^G|_E, \phi) \neq 0$ as irreducible $\mathbb{R}$-characters of G and E. By Clifford's Theorem there is a unique $\zeta \in \mathrm{Irr}_{\mathbb{R}}(E \rtimes \mathrm{Stab}_H(\phi))$ such that $\zeta^G = \chi_{\phi,\theta}^G$, so $\zeta \in \mathrm{Irr}(E \rtimes \mathrm{Stab}_H(\phi))$ as well. Thus $\zeta = \chi_{\phi,\theta}$, so $\chi_{\phi,\theta}$ is of type 1, and $\nu(\theta) = \nu(\chi_{\phi,\theta}) = 1$, as required. □

Corollary 3.13. *If E_1 and E_2 are both elementary abelian 2-groups and $G = E_1 \rtimes E_2$, then $G \in \mathcal{TO}$.* □

We close this section with a small example to show that a split extension of one totally orthogonal group by another need not be totally orthogonal. Let E be an elementary abelian 2-group with basis $\{e_1, e_2, e_3\}$ and let $D = D_8 = \langle a, b\colon a^4 = b^2 = (ba)^2 = 1\rangle$. Define an action of D on E by means of

(1) $\quad {}^a e_1 = e_1 + e_3, \qquad {}^a e_2 = e_2 + e_3, \qquad {}^a e_3 = e_3;$

(2) $\quad {}^b e_1 = e_2, \qquad {}^b e_2 = e_1, \qquad {}^b e_3 = e_3;$

and let $G = E \rtimes D$. Then there exists φ in $\mathrm{Irr}(E)$ with $\mathrm{Stab}_D(\varphi) = \langle a \rangle$, so $G \notin \mathcal{TO}$ by Theorem 3.12.

4. Total orthogonality of k-uniform groups

A group G will be called *k-uniform* ($1 \le k \in \mathbb{Z}$) if $\{1, k\}$ is the set of degrees of characters in $\mathrm{Irr}(G)$. Thus, for example, G is 1-uniform if and only if it is abelian, an extra-special 2-group of width n is 2^n-uniform, and a nonabelian group having an abelian normal subgroup of prime index p is p-uniform by Ito's theorem. Note that if $G \in \mathcal{TO}$ and G is k-uniform, then $\mathrm{SQRT}_G(1) = |\mathcal{L}(G)| + k|\mathcal{N}(G)|$ by the Frobenius–Schur count of involutions.

Theorem 4.1. *If $G \in \mathscr{A}$ and G is k-uniform, then either G is a 2-group or $k=1$ or 2. In particular k is a power of 2.*

Proof. Use induction on $|G|$, the result clearly holding for G of sufficiently low order. We may assume that G is not a 2-group and hence, by [11, Theorem 12.5], that there is an abelian normal subgroup A of index k in G. Clearly G/A is abelian (or it would have an irreducible character of degree $k=|G/A|$), and it is ambivalent. Thus G/A is an elementary abelian 2-group and k is a power of 2. If p is an odd prime divisor of $|G|$, there is a p-Sylow subgroup P of G with $P \leq A$. If $P \neq A$, there is also a nontrivial q-Sylow subgroup $Q \leq A$ for some prime $q \neq p$. Then Q is characteristic in A since A is abelian, so $Q \lhd G$. Then $G/Q \in \mathscr{A}$ and $p \mid [G:Q]$, so G/Q is not abelian, hence G/Q is k-uniform. By induction $k=2$ in that case, so we may assume that $P=A$. If T is 2-Sylow in G, then $G=PT=P \rtimes T$, so T is elementary abelian of order k. Observe that $G'=P$ since G/P is abelian and G/G' is a 2-group. Take $\varphi \in \operatorname{Irr}(P)$. Since G is $k=[G:P]$-uniform, $\operatorname{Stab}_T(\varphi)$ must be 1 or T. But if $\operatorname{Stab}(\varphi)=T$, then φ extends to G, so $\varphi=1_P$ (otherwise φ is not real-valued). It follows from Brauer's Lemma that $C_T(a)=1$, or $C_G(a) \leq P$ if $1 \neq a \in P$, so G is a Frobenius group with kernel P and complement T. But then $\operatorname{SQRT}_T(1)=2$, so $k=|T|=2$. □

Proposition 4.2. *Suppose G is ambivalent and k-uniform. Then $G=E \times H$, where E is an elementary abelian 2-group, H is ambivalent and k-uniform, $Z(H) \leq H'=G'$, and $Z(G)=E \times Z(H)$.*

Proof. Use induction on $|G|$; the result obviously holds if $|G| \leq 8$. If $Z(G) \leq G'$ the statement is true by default, with $H=G$, so suppose $Z(G) \not\leq G'$. Choose $z \in Z(G) \setminus G'$, so $|z|=2$. Choose $\theta \in \mathscr{L}(G)$ with $\theta(z) \neq 1$, and set $N=\ker(\theta)$. Then $[G:N]=2$ and $z \notin N$, so $G=\langle z \rangle \times N$. Clearly N is ambivalent and k-uniform, and $N'=G'$. By induction $N=F \times H$, where F is an elementary abelian 2-group, H is k-uniform, $Z(H) \leq H'=N'=G'$, and $Z(N)=F \times Z(H)$. Set $E=\langle z \rangle \times F$. □

Theorem 4.3. *A nonabelian group G is 2-uniform and totally orthogonal if and only if $G=A \rtimes \langle \sigma \rangle$, with A abelian, $|\sigma|=2$, and ${}^{\sigma}a=a^{-1}$ for all $a \in A \setminus J_A(1)$.*

Proof. '$\Leftarrow$'. Proposition 3.2 and Ito's Theorem.

'$\Rightarrow$'. Observe first that it is sufficient to show that G has an abelian subgroup A of index 2, for then $G=A \rtimes \langle \sigma \rangle$, with $|\sigma|=2$, by Corollary 3.4, and furthermore ${}^{\sigma}a=a^{-1}$ for all $a \in A \setminus J_A(1)$ since $G \in \mathscr{A}$.

If G is not a 2-group, then G has an abelian subgroup of index 2 by [11, Theorem 12.5], again using the fact that $G \in \mathscr{A}$. We assume then that G is a 2-group, and use induction on $|G|$. Clearly the conclusion holds if $|G|=8$. We may assume that $Z(G) \leq G'$ by Proposition 4.2 and the induction hypothesis. By [11, Theorem 12.11] we may assume that $[G:Z(G)]=8$. If $x \in G$ has order 4 modulo $Z(G)$, then $\langle x, Z(G) \rangle$

is an abelian group of index 2, a contradiction. Thus $G/Z(G)$ is elementary abelian and $G'=Z(G)$. By the Burnside Basis Theorem and Theorem 3.3 we have $G=\langle\sigma_1,\sigma_2,\sigma_3\rangle$, the σ_i being distinct involutions. It is easy to see that $G'=\langle[\sigma_1,\sigma_2],[\sigma_1,\sigma_3],[\sigma_2,\sigma_3]\rangle$, so G' is elementary abelian of order at most 8. If $x,y\in G\setminus G'$, $xG'\neq yG'$, and $xy=yx$, then $\langle x,y,Z(G)\rangle$ is abelian of index 2 in G, a contradiction. Hence $[\sigma_i,\sigma_j]=1$ for $i\neq j$, $[\sigma_i,\sigma_j][\sigma_i,\sigma_k]=[\sigma_i,\sigma_j\sigma_k]=1$ for $\{i,j,k\}=\{1,2,3\}$, and $[\sigma_1,\sigma_2][\sigma_1,\sigma_3][\sigma_2,\sigma_3]=[\sigma_1\sigma_2,\sigma_1\sigma_3]=1$, and it follows that $[\sigma_1,\sigma_2]$, $[\sigma_1,\sigma_3]$, $[\sigma_2,\sigma_3]$ are linearly independent and $|G'|=8$. Thus $|G|=64$, $|\mathcal{N}(G)|=14$, and $\mathrm{SQRT}_G(1)=36$. But $J_G(1)$ is a union of cosets of $Z(G)$, and $8\nmid 36$, so we have a contradiction and the proof is complete. □

We remark that if G is as in Theorem 4.3, then $Z(G)=\{a\in J_A(1): {}^{\sigma}a=a\}$ and $G'\leq A^2\cdot Z(G)$. If $G=L_1(G)\geq L_2(G)\geq\cdots$ is the descending central series of G, then easy calculations show that $L_k(G)$ consists of the 2^{k-1}-powers of the elements of A for each $k>2$. The next result follows easily.

Corollary 4.4. *If G is a totally orthogonal 2-uniform 2-group, and if 2^m is the maximal order of elements of G, then G is of nil class m.* □

5. Orthogonality and Frobenius groups

A Frobenius group is a finite permutation group G acting transitively on a set so that the stabilizer of a point is not trivial but only the identity fixes 2 or more points. Then G is a semidirect product $N\rtimes H$, where H (the *complement*) is the stabilizer of a point and $N\setminus 1$ (N is the *kernel*) is the set of elements with no fixed points.

Proposition 5.1. *Suppose $G=N\rtimes H$ is a Frobenius group with kernel N and complement H, and let σ be an involution in H. Then ${}^{\sigma}x=x^{-1}$ for all $x\in N$.*

Proof. Since $|H|$ is even, N is abelian [13, p. 60]. If $x\in N$, then σ centralizes $(x\sigma)^2$, and $(x\sigma)^2\in N$, so $(x\sigma)^2=1$, or equivalently ${}^{\sigma}x=x^{-1}$. □

The irreducible characters of a Frobenius group G are of two types: if $\chi\in\mathrm{Irr}(G)$, then either $\chi=\varphi^G$ for some $\varphi\neq 1$ in $\mathrm{Irr}(N)$, or else $\chi|_H\in\mathrm{Irr}(H)$ and $\chi(xh)=\chi(h)$ for all $x\in N$, $h\in H$. For each $\varphi\neq 1$ in $\mathrm{Irr}(N)$ we have $\mathrm{Stab}_H(\varphi)=1$ and $\varphi^G\in\mathrm{Irr}(G)$, and if $\varphi,\theta\neq 1$ in $\mathrm{Irr}(N)$, then $\varphi^G=\theta^G$ if and only if $\theta\in\mathrm{Orb}_H(\varphi)$. We remark that G is k-uniform if and only if N and H are both abelian, and then $k=|H|$.

Proposition 5.2. *Suppose $G=N\rtimes H$ is a Frobenius group, and $1\neq\varphi\in\mathrm{Irr}(N)$.*

(1) If $|H|$ *is even, then* $v(\varphi^G)=1$.

(2) *If* $|H|$ *is odd, then* $v_G(\varphi^G)=v_N(\varphi)$.

Proof. (1) Let σ be an involution in H and set $L=N\rtimes\langle\sigma\rangle$, so $[L:N]=2$. Then $\varphi^L|_N=\varphi+\varphi^\sigma$ and, since $|N|$ is odd, $\nu_N(\varphi)=\nu_N(\varphi^\sigma)=0$. Thus

$$\begin{aligned}\nu_L(\varphi^L)&=|L|^{-1}\sum\{\varphi(x^2)+\varphi^\sigma(x^2)+\varphi((x\sigma)^2)+\varphi^\sigma((x\sigma)^2):\ x\in N\}\\&=\tfrac{1}{2}[\nu_N(\varphi)+\nu_N(\varphi^\sigma)+1+1]=1,\end{aligned}$$

since $(x\sigma)^2=1$ for all $x\in N$ by Proposition 5.1. Thus φ^L is orthogonal, and hence $\varphi^G=(\varphi^L)^G$ is orthogonal.

(2) Since $|H|$ is odd and $G=N\cup[\bigcup\{H^x:\ x\in G\}]$, all elements of $G\setminus N$ have odd order, so if $x\notin N$, then $x^2\notin N$. Furthermore $\varphi^G|_N=\sum\{\varphi^h:\ h\in H\}$ and $\varphi^G|_{G\setminus N}=0$. Thus

$$\begin{aligned}\nu(\varphi^G)&=|G|^{-1}\sum\{\varphi^h(x^2):\ h\in H,\ x\in N\}\\&=|H|^{-1}\sum\{\nu_N(\varphi^h):\ h\in H\}=\nu_N(\varphi),\end{aligned}$$

since $\nu(\varphi^h)=\nu(\varphi)$ for all $h\in H$. □

Proposition 5.3. *Suppose $G=N\rtimes H$ is Frobenius and $\chi\in\mathrm{Irr}(G)$, with $\chi|_H\in\mathrm{Irr}(H)$. Then $\nu_G(\chi)=\nu_H(\chi|_H)$.*

Proof. We have

$$\begin{aligned}\nu(\chi)&=|G|^{-1}\sum\{\chi((xh)^2):\ x\in N,\ h\in H\}\\&=|G|^{-1}\sum\{\chi(x\cdot {}^hx\cdot h^2):\ x\in N,\ h\in H\}\\&=|G|^{-1}\sum\{\chi(h^2):\ x\in N,\ h\in H\}=\nu_H(\chi|_H).\end{aligned}$$ □

We may use the propositions above to reduce Brauer's problem for a Frobenius group $G=N\rtimes H$ to the corresponding problem for N and H.

Theorem 5.4. *Suppose G is a Frobenius group with kernel N and complement H.*

(1) *If $|H|$ is even, than $\nu^+(G)=\nu^+(H)+(|N|-1)/|H|$, $\nu^-(G)=\nu^-(H)$, and $\upsilon^0(G)=\nu^0(H)$.*

(2) *If $|H|$ is odd, then $\nu^+(G)=(\nu^+(N)-1)/|H|+1$, $\nu^-(G)=\nu^-(N)/|H|$, and $\nu^0(G)=\nu^0(N)+\nu^0(H)$.*

Proof. (1) The characters in $\mathrm{Irr}(H)$ extend to characters of the same type in $\mathrm{Irr}(G)$ by Proposition 5.3. The $|N|-1$ nonprincipal characters in $\mathrm{Irr}(N)=\mathscr{L}(N)$ lie in H-orbits of size $|H|$, and each induces to an orthogonal character in $\mathrm{Irr}(G)$ by Proposition 5.2.

(2) Only the principal character in $\mathrm{Irr}(H)$ extends to an orthogonal character in $\mathrm{Irr}(G)$, all others are of type 3. The characters in $\mathrm{Irr}(N)\setminus\{1_N\}$, lying in orbits of size $|H|$, induce to characters of the same types in $\mathrm{Irr}(G)$ by Proposition 5.2. □

J. Thompson has proved that a Frobenius kernel N is nilpotent (see e.g. [13, p.

184]). Thus $N=K\times T$, with $|K|$ odd and T a 2-group, and hence $\nu^+(N)=\nu^+(T)$.

The structure of a Frobenius complement has been described by H. Zassenhaus (again, see [13]). If H is solvable, then it has a normal split metacyclic subgroup L so that H/L is isomorphic with a subgroup of Sym(4). Thus $\nu^+(L)$ can be determined (see [8]), and $\nu^+(H/L)$ is no problem, but there is no systematic way to use that information to determine ν^+ for the extension H. If $Z(H)$ is a 2-group, then even more precise information is available regarding the structure of H (see [13, p. 200]).

If H is not solvable, then H has a subgroup K of index either 1 or 2 with $K=\mathrm{SL}(2,5)\times M$, M being of odd order (in fact of order prime to 30). Thus $\nu^+(K)=\nu^+(\mathrm{SL}(2,5))=5$.

Theorem 5.5. *If $G=N\rtimes H$ is a Frobenius group, then $G\in\mathscr{TO}$ if and only if $|H|=2$.*

Proof. '$\Leftarrow$'. If $H=\langle\sigma\rangle$, of order 2, then N is abelian of odd order, and ${}^{\sigma}x=x^{-1}$ for all $x\in N$ by Proposition 5.1. Thus $G\in\mathscr{TO}$ by Proposition 3.2.

'$\Rightarrow$'. It is clear from Theorem 5.4 that $G\in\mathscr{TO}$ if and only if $|H|$ is even and $H\in\mathscr{TO}$. But then H has a unique involution, so $|H|=2$ by Theorem 3.3. □

Corollary 5.6. *If G is a Frobenius group, then G is totally orthogonal if and only if G is 2-uniform.* □

6. Examples

It is well known that finite (real) reflection groups are totally orthogonal. Berggren showed in [2] that the alternating group Alt(n) is ambivalent if and only if $n\in\{1,2,5,6,10,14\}$, and it follows from Corollary 3.8 above that $\mathrm{Alt}(n)\in\mathscr{TO}$ in precisely those cases. If G is a finite reflection group and H is its rotation subgroup, observe that $\nu^+(H)$ is equal to the number of real conjugacy classes in H by Corollary 3.7. We establish in this section the total orthogonality of 2-Sylow subgroups of all reflection groups and all Alt(n).

Proposition 6.1. *If $G\in\mathscr{TO}$ and $|H|=2$, then the wreath product $G\,\mathrm{wr}\,H\in\mathscr{TO}$.*

Proof. This follows from the general representation theory of wreath products. See [12, Theorem 4.3.34]. □

Theorem 6.2. *The 2-Sylow subgroups of* Sym(n) *are totally orthogonal for all n.*

Proof. This follows inductively from Proposition 6.1 since a 2-Sylow subgroup T of Sym(n) is a direct product of wreath products of 2-Sylow subgroups of smaller symmetric groups by groups of order 2. □

Corollary 6.3. *Every finite 2-group is a subgroup of a totally orthogonal 2-group.* □

Corollary 6.4. *If T is a 2-Sylow subgroup in a reflection group of type A_n, then $T \in \mathcal{TO}$.*

Proof. The reflection group is isomorphic with $\mathrm{Sym}(n+1)$. □

The following explicit descriptions of 2-Sylow subgroups in $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, which follow easily from considerations of order, will be used in the proof of the next theorem.

(∗1) Suppose $k, m \in \mathbb{Z}$, with $1 \leq k$ and $1 \leq m < 2^k$, and set $n = 2^k + m$. Let $G_1 = \mathrm{Sym}(2^k) \leq \mathrm{Sym}(n)$ and $G_2 = \mathrm{Sym}(\{2^k+1, \dots, 2^k+m\}) \leq \mathrm{Sym}(n)$, so $G_2 \cong \mathrm{Sym}(m)$. Let H_1, H_2 be the alternating subgroups of G_1, G_2, let S_1, S_2 be 2-Sylow subgroups of H_1, H_2, and let T_1, T_2 be 2-Sylow subgroups of G_1, G_2, with $T_1 \geq S_1$ and $T_2 \geq S_2$. Then

(a) $T = T_1 T_2 = T_1 \times T_2$ is 2-Sylow in $\mathrm{Sym}(n)$, and

(b) $S = S_1 S_2 \cup (T_1 \setminus S_1)(T_2 \setminus S_2)$ is 2-Sylow in $\mathrm{Alt}(n)$.

(∗2) Suppose $1 \leq k \in \mathbb{Z}$ and set $n = 2^{k+1}$. Let $G_1 = \mathrm{Sym}(2^k) \leq \mathrm{Sym}(n)$ and let $G_2 = \mathrm{Sym}(\{2^k+1, \dots, 2^{k+1}\}) \leq \mathrm{Sym}(n)$, so $G_1 \cong G_2$, and let H_1, H_2 be their alternating subgroups. Let S_1 be a 2-Sylow subgroup in H_1, let T_1 be a 2-Sylow subgroup in G_1 with $T_1 \geq S_1$, and let $\sigma = (1, 2^k+1)(2, 2^k+2) \cdots (2^k, 2^{k+1})$ of order 2 in $\mathrm{Alt}(n)$. Set $T_2 = {}^{\sigma}T_1$ and $S_2 = {}^{\sigma}S_1$, 2-Sylow subgroups in G_2 and H_2. Then

(a) $T = (T_1 T_2) \rtimes \langle \sigma \rangle$ is 2-Sylow in $\mathrm{Sym}(n)$, and

(b) $S = [(S_1 S_2) \cup (T_1 \setminus S_1)(T_2 \setminus S_2)] \rtimes \langle \sigma \rangle$ is 2-Sylow in $\mathrm{Alt}(n)$.

Theorem 6.5. *The 2-Sylow subgroups of* $\mathrm{Alt}(n)$ *are totally orthogonal for all n.*

Proof. It will be sufficient, by Corollary 3.8, to show that a 2-Sylow subgroup S of $\mathrm{Alt}(n)$ is ambivalent. Use induction on n; we may assume $n > 2$ and assume the conclusion holds for $\mathrm{Alt}(n')$ if $n' < n$. Choose $k \in \mathbb{Z}$ with $2^k < n \leq 2^{k+1}$ and consider 2 cases.

(1) If $n = 2^k + m$, with $1 \leq m < 2^k$, we may take $S = S_1 S_2 \cup (T_1 \setminus S_1)(T_2 \setminus S_2)$ as indicated in (∗1) above, with $S_1, S_2 \in \mathcal{A}$ by induction and $T_1, T_2 \in \mathcal{A}$ by Theorem 6.2. If $x \in S$, then $x = x_1 x_2$, with either $x_i \in S_i$, $i = 1, 2$, or $x_i \in T_i \setminus S_i$. If $x_i \in S_i$, we may choose $y_i \in S_i$ with $y_i x_i y_i^{-1} = x_i^{-1}$ and $y_i x_j y_i^{-1} = x_j$ if $i \neq j$. Thus $y_1 y_2$ conjugates x to x^{-1} and x is real. If $x_i \in T_i \setminus S_i$ we may choose $y_i \in T_i$ with $y_i x_i y_i^{-1} = x_i^{-1}$ and $y_i x_j y_i^{-1} = x_j$ if $i \neq j$, and we may assume that each $y_i \in T_i \setminus S_i$ (if not replace it by $y_i x_i$). Thus again $y_1 y_2 \in S$ and it conjugates x to x^{-1}, so x is real.

(2) If $n = 2^{k+1}$, take S as in (∗2), with S_1, S_2, T_1, T_2 in $\mathcal{A}$ as above. Set $R = S_1 S_2 \cup (T_1 \setminus S_1)(T_2 \setminus S_2)$, so $S = R \rtimes \langle \sigma \rangle$. Each $x \in R$ is real, as in case (1) above. Thus it will suffice to conjugate $x\sigma$ to $(x\sigma)^{-1} = \sigma x^{-1}$ in S. Write $x = x_1 x_2$, with either $x_i \in S_i$ or $x_i \in T_i \setminus S_i$. Note that $x_1 \cdot {}^{\sigma}x_2 \in S_1$, and begin with $u_1 \in S_1$ that conjugates

$x_1 \cdot {}^{\sigma}x_2$ to $(x_1 \cdot {}^{\sigma}x_2)^{-1} = {}^{\sigma}x_2^{-1} \cdot x_1^{-1}$. Then set $y_1 = x_1^{-1}u_1$, so that y_1 conjugates $x_1 \cdot {}^{\sigma}x_2$ to $x_1^{-1} \cdot {}^{\sigma}x_2^{-1}$. It follows that $x_2^{-1} \cdot ({}^{\sigma}y_1)x_2^{-1} = {}^{\sigma}(x_1 y_1 x_1)$, and we may set $y_2 = x_2^{-1}({}^{\sigma}y_1)x_2^{-1} = {}^{\sigma}(x_1 y_1 x_1)$. Thus ${}^{\sigma}y_1 = x_2 y_2 x_2$, so if we set $y = y_1 y_2 \in R$ we have ${}^{\sigma}y = xyx$, or equivalently ${}^{y}(x\sigma) = x^{-1}\sigma$. Finally then ${}^{xy}(x\sigma) = \sigma x^{-1}$, and $x\sigma$ is real. □

Theorem 6.6. *Suppose G is a finite group generated by (real) reflections, and T is a 2-Sylow subgroup of G. Then* $T \in \mathcal{TO}$.

Proof. We consider the remaining irreducible reflection groups case by case. Write $T(G)$ for a 2-Sylow subgroup of G. If G is a dihedral group (including G_2), then $T(G)$ is also dihedral, hence in $\mathcal{TO}$ by Corollary 3.5. All other cases follow from either Theorem 6.2 or Theorem 6.5 because of the following isomorphisms:

$$T(B_n) \cong T(\mathrm{Sym}(2n)), \qquad T(D_n) \cong T(\mathrm{Alt}(2n)),$$

$$T(F_4) \cong T(\mathrm{Sym}(8)) \quad \text{and} \quad T(I_4) \text{ is a homomorphic image,}$$

$$T(E_6) \cong T(D_5), \qquad T(E_7) \cong T(D_6) \times Z_2, \quad \text{and} \quad T(E_8) \cong T(D_8). \qquad \square$$

As often happens, a general result about reflection groups has been established by a case-by-case analysis. As always a unified conceptual proof would be of considerable interest.

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