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A note on the Cochrane sum and its hybrid mean value formula ☆

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Abstract

In this paper, we use the properties of Gauss sums, primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of Cochrane sums and general Kloosterman sums, and give two sharp asymptotic formulae.

Keywords: Cochrane sums; General Kloosterman sums; Asymptotic formula

1. Introduction

For a positive integer k and an arbitrary integer h, the Dedekind sum $S(h,k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer,}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

Conrey et al. [1] studied the mean value distribution of $S(h,k)$ and proved the following important asymptotic formula

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$$\sum_{h=1}^{k}{}' \left|S(h,k)\right|^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{9/5} + k^{2m-1+1/(m+1)}\right)\log^3 k\right),$$

where $\sum_h'$ denotes the summation over all h such that $(k,h)=1$, and

$$\sum_{n=1}^{\infty}\frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)}\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

In the spirit of [1] and [2], the first author [3] obtained a sharper asymptotic formula for $\sum_{h=1}^{\prime k}|S(h,k)|^2$. That is,

$$\sum_{h=1}^{k}{}' \left|S(h,k)\right|^{2} = \frac{5}{144}k\phi(k)\frac{\prod_{p^\alpha\|k}\left[\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}\right]}{\prod_{p|k}\left(1+\frac{1}{p}+\frac{1}{p^2}\right)} + O\left(k\exp\left(\frac{4\ln k}{\ln\ln k}\right)\right)$$

for any integer $k>2$, where $\exp(y)=e^y$, $\phi(k)$ is the Euler function, $\prod_{p^\alpha\|k}$ denotes the production over all prime divisors of k with $p^\alpha|k$ and $p^{\alpha+1}\nmid k$.

In October 2000, during his visiting in Xi'an, Professor T. Cochrane introduced a sum analogous to the Dedekind sum as follows:

$$C(h,k) = \sum_{a=1}^{k}{}' \left(\left(\frac{\bar a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where $\bar a$ is defined by the equation $a\bar a \equiv 1 \bmod k$. He asked us to study the arithmetical properties and mean value distribution properties of $C(h,k)$. About this problem, we know very little. Recently, Zhang and Yi [4] studied the upper bound estimate of Cochrane sums, and gave the following sharper upper bound estimates

$$\left|C(h,k)\right| \ll \sqrt{k}\, d(k)\ln^2 k$$

and

$$\sum_{h=1}^{p-1} C^2(h,p) = \frac{5}{144}p^2 + O\left(p\exp\left(\frac{4\ln p}{\ln\ln p}\right)\right),$$

where $d(k)$ is the divisor function.

In Ref. [5], the first author found that there are some relationships between $C(h,k)$ and Kloosterman sums

$$K(m,n;k) = \sum_{b=1}^{k}{}' e\left(\frac{mb+n\bar b}{k}\right),$$

where $e(y)=e^{2\pi i y}$. For example, if k is a square-full number (i.e., $p|k$ if and only if $p^2|k$), then we have the following asymptotic formula:

$$\sum_{h=1}^{k}{}' K(h,1;k)C(h,k) = \frac{-1}{2\pi^2}k\phi(k) + O\left(k\exp\left(\frac{3\ln k}{\ln\ln k}\right)\right).$$

For general integer $k \geqslant 3$, the first author [6] proved the asymptotic formula

$$\sum_{h=1}^{k}{}' K(h,1;k)C(h,k) = \frac{-1}{2\pi^2} k\phi(k) \prod_{p\|k}\left(1 - \frac{1}{p(p-1)}\right) + O(k^{3/2+\epsilon}),$$

where ϵ be any fixed positive number. In [7], the first author also studied the asymptotic property of a hybrid mean value of Cochrane sums and general Kloosterman sums

$$K(m,n,r;k) = \sum_{b=1}^{k}{}' e\left(\frac{mb^r + n\bar{b}^r}{k}\right),$$

and obtained the mean value theorem

$$\sum_{h=1}^{p-1} K(h,1,r;p)C(h,p) = \frac{-1}{2\pi^2} p^2 + O(rp^{3/2}\ln^2 p).$$

Meanwhile, he conjectured that

$$\sum_{h=1}^{k}{}' K(h,1,r;k)C(h,k) \sim \frac{-1}{2\pi^2} k\phi(k), \quad \text{as } k \to \infty,$$

holds for all integer $k > 2$ and any fixed positive integer r.

In this paper, we studied this conjecture, and proved that it is true. Note that $K(h,1,r;k) = K(h,1;k)$ if $(r,\phi(k)) = 1$, so we suppose $(r,\phi(k)) > 1$. Then we shall use the properties of Gauss sums, primitive characters and the mean value theorems of Dirichlet L-functions to prove the following two main theorems.

Theorem 1. *For any integer $k \geqslant 3$, we have the asymptotic formula*

$$\sum_{h=1}^{k}{}' K(h,1;k)C(h,k) = \frac{-1}{2\pi^2} k\phi(k) \prod_{p\|k}\left(1 - \frac{1}{p(p-1)}\right) + O(k^{1+\epsilon}).$$

Theorem 2. *For any positive integers $k \geqslant 3$ and r with $d_r = (r,\phi(k)) > 1$, we have the asymptotic formula*

$$\sum_{h=1}^{k}{}' K(h,1,r;k)C(h,k) = \frac{-1}{2\pi^2} k\phi(k) \prod_{p\|k}\left(1 - \frac{1}{p(p-1)}\right) + O(d_r k^{3/2+\epsilon}).$$

It is clear that the error terms in our Theorem 1 is much better than that in Ref. [6], and our Theorem 2 is a generalization of Theorem 2 in [7].

2. Some lemmas

To prove the theorems, we need the following lemmas.

Lemma 1. *Let integer $k \geqslant 3$ and $(a,k)=1$. Then we have*

$$C(a,k) = \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \overline{\chi}(a)\left(\sum_{n=1}^{\infty}\frac{G(\chi,n)}{n}\right)^2,$$

where χ denotes a Dirichlet character modulo k with $\chi(-1)=-1$, and $G(\chi,n) = \sum_{b=1}^{k}\chi(b)e(bn/k)$ denotes the Gauss sum corresponding to χ.

Proof. This is Lemma 1 of [5]. □

Lemma 2. *For any positive integer k and $r>1$, let $d_r=(r,\phi(k))$ and χ_1 be a d_rth-order character modulo k. Then for any character χ modulo k, we have the identities*

$$\sum_{h=1}^{k}\overline{\chi}(h)K(h,1;k)=\tau^2(\overline{\chi})$$

and

$$\sum_{h=1}^{k}\overline{\chi}(h)K(h,1,r;k)=\tau^2(\overline{\chi})+\tau(\overline{\chi})\sum_{i=1}^{d_r-1}\tau\big(\overline{\chi}\chi_1^i\big),$$

where $\tau(\chi)=G(\chi,1)=\sum_{a=1}^{k}\chi(a)e(a/k)$.

Proof. From the properties of Gauss sums and note $(b,k)=1$ we have

$$\begin{aligned}\sum_{h=1}^{k}\overline{\chi}(h)K(h,1,r;k) &= \sum_{h=1}^{k}\overline{\chi}(h)\sum_{b=1}^{k}{}' e\left(\frac{hb^r+\overline{b}^r}{k}\right)\\ &= \sum_{h=1}^{k}\overline{\chi}(h)e\left(\frac{h}{k}\right)\sum_{b=1}^{k}\chi(b^r)e\left(\frac{\overline{b}^r}{k}\right)\\ &= \tau(\overline{\chi})\sum_{b=1}^{k}\overline{\chi}(b^r)e\left(\frac{b^r}{k}\right)=\tau(\overline{\chi})G(1,\overline{\chi}^r,r;k).\end{aligned}$$

It is obvious that $G(1,\overline{\chi},1;k)=\tau(\overline{\chi})$, so we have

$$\sum_{h=1}^{k}\overline{\chi}(h)K(h,1;k)=\tau^2(\overline{\chi}).$$

On the other hand, let χ_1 be a d_rth-order character modulo k, then for any integer $1\leqslant a\leqslant k$ with $(a,k)=1$, we have the identity

$$1+\chi_1(a)+\cdots+\chi_1^{d_r-1}(a)=\begin{cases} d_r, & \text{if } a \text{ is a } d_r\text{th residue mod } k,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, by the definition of Gauss sums, we have

$$\begin{aligned}G(1,\overline{\chi}^r,r;k) &= \sum_{b=1}^{k}\overline{\chi}(b^r)e\left(\frac{b^r}{k}\right) = \sum_{b=1}^{k}\overline{\chi}(b^{d_r})e\left(\frac{b^{d_r}}{k}\right)\\ &= \sum_{b=1}^{k}\big(1+\chi_1(b)+\cdots+\chi_1^{d_r-1}(b)\big)\overline{\chi}(b)e\left(\frac{b}{k}\right)\\ &= \tau(\overline{\chi})+\sum_{i=1}^{d_r-1}\tau\big(\overline{\chi}\chi_1^i\big).\end{aligned}$$

This proves Lemma 2. □

Lemma 3. *For any positive integer k, let χ be a non-primitive character modulo k, and k^* denote the conductor of χ with $\chi=\chi_1\chi^*_{k^*}$. If $(n,k)>1$, then we have*

$$G(\chi,n)=\begin{cases}\overline{\chi}^*\big(\frac{n}{(n,k)}\big)\chi^*\big(\frac{k}{k^*(n,k)}\big)\mu\big(\frac{k}{k^*(n,k)}\big)\phi(k)\phi^{-1}\big(\frac{k}{(n,k)}\big)\tau(\chi^*), & k^*=\frac{k_1}{(n,k_1)},\\ 0, & k^*\neq\frac{k_1}{(n,k_1)},\end{cases}$$

where χ_1 denotes the principal character modulo k, k_1 is the largest divisor of k that has the same prime factors with k^*.

If $(n,k)=1$, then we have

$$G(\chi,n)=\overline{\chi}^*(n)\chi^*\left(\frac{k}{k^*}\right)\mu\left(\frac{k}{k^*}\right)\tau(\chi^*).$$

If χ be a primitive character modulo k, then

$$G(\chi,n)=\overline{\chi}(n)\tau(\chi).$$

Proof. See Ref. [8]. □

Lemma 4. *Let q and r be integers with $q\geqslant 2$ and $(r,q)=1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \bmod q}\chi(r)=\sum_{d\mid(q,r-1)}\mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q)=\sum_{d\mid q}\mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \bmod q}$ denotes the summation over all primitive characters modulo q and $J(q)$ denotes the number of primitive characters modulo q.

Proof. This is Lemma 3 of [5]. □

Lemma 5. *Let $k = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then we have the asymptotic formula*

$$\sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{u^2 d^2 \mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1 d_2 \phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1 d_2) L^2(1, \overline{\chi})$$
$$= \frac{k}{2} \prod_{p\|k} \left(1 - \frac{1}{p(p-1)}\right) + O(k^{\epsilon}).$$

Proof. Let $\tau(n)$ be the divisor function. Then for any parameter $N \geqslant ud$ and non-principal character χ modulo ud, applying Abel's identity we have

$$L^2(1, \overline{\chi}) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\tau(n)}{n} = \sum_{1 \leqslant n \leqslant N} \frac{\overline{\chi}(n)\tau(n)}{n} + \int_N^{\infty} \frac{A(y, \overline{\chi})}{y^2}\, dy,$$

where $A(y, \overline{\chi}) = \sum_{N<n\leqslant y} \overline{\chi}(n)\tau(n)$. We can rewrite it as

$$A(y, \overline{\chi}) = 2 \sum_{n\leqslant\sqrt{y}} \overline{\chi}(n) \sum_{m\leqslant y/n} \overline{\chi}(m) - 2 \sum_{n\leqslant\sqrt{N}} \overline{\chi}(n) \sum_{m\leqslant N/n} \overline{\chi}(m)$$
$$- \left(\sum_{n\leqslant\sqrt{y}} \overline{\chi}(n)\right)^2 + \left(\sum_{n\leqslant\sqrt{N}} \overline{\chi}(n)\right)^2.$$

Applying Pólya–Vinogradov inequality

$$\left|\sum_{n=a}^{b} \chi(n)\right| \ll \sqrt{ud}\,\ln(ud)$$

we have

$$\sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \left|A(y, \overline{\chi})\right| \ll y^{1/2}(ud)^{3/2} \ln(ud).$$

Then

$$\sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{u^2 d^2 \mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1 d_2 \phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1 d_2) \int_N^{\infty} \frac{A(y, \overline{\chi})}{y^2}\, dy$$
$$\ll k^{\epsilon} \int_N^{\infty} \frac{1}{y^2} \left(\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \left|A(y, \overline{\chi})\right|\right) dy \ll k^{\epsilon} \int_N^{\infty} \frac{1}{y^2} \left(y^{1/2} k^{3/2}\right) dy \ll \frac{k^{3/2+\epsilon}}{N^{1/2}},$$

so we have

$$\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(d_1d_2)L^2(1,\overline{\chi})$$
$$=\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}$$
$$\times\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(d_1d_2)\overline{\chi}(n)+O\left(\frac{k^{3/2+\epsilon}}{N^{1/2}}\right).$$

Note that for $(a,k)=1$, from Lemma 4 we have

$$\sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\chi(a)=\frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod k}\big(1-\chi(-1)\big)\chi(a)=\frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod k}\chi(a)-\frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod k}\chi(-a)$$
$$=\frac{1}{2}\sum_{u|(k,a-1)}\mu\left(\frac{k}{u}\right)\phi(u)-\frac{1}{2}\sum_{u|(k,a+1)}\mu\left(\frac{k}{u}\right)\phi(u).$$

Therefore

$$\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(d_1d_2)\overline{\chi}(n)$$
$$=\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1}}\sum_{s|(ud,d_1d_2n-1)}\mu\left(\frac{ud}{s}\right)\phi(s)\frac{\tau(n)}{n}$$
$$-\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1}}\sum_{s|(ud,d_1d_2n+1)}\mu\left(\frac{ud}{s}\right)\phi(s)\frac{\tau(n)}{n}$$
$$=\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1\\ d_1d_2n\equiv 1(s)}}\frac{\tau(n)}{d_1d_2n}$$
$$-\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1\\ d_1d_2n\equiv 1(s)}}\frac{\tau(n)}{d_1d_2n}.$$

The main term in the above formula comes from the term corresponding to $d_1d_2n=1$ with $d_1=d_2=n=1$, and all the other terms are the error terms. So we have

$$\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(d_1d_2)\overline{\chi}(n)$$

$$= \frac{1}{2} \sum_{d|v} \frac{u^2 d^2}{\phi^2(k)} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s)$$

$$+ O\left(\sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{u^2 d^2}{\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)} \sum_{s|ud} \left|\mu\left(\frac{ud}{s}\right)\right| \phi(s) \sum_{1 \leqslant l \leqslant \frac{N d_1 d_2}{s}} (ls+1)^{\epsilon-1} \right)$$

$$+ O\left(\sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{u^2 d^2}{\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)} \sum_{s|ud} \left|\mu\left(\frac{ud}{s}\right)\right| \phi(s) \sum_{1 \leqslant l \leqslant \frac{N d_1 d_2}{s}} (ls-1)^{\epsilon-1} \right)$$

$$= \frac{u^2}{2\phi^2(k)} \sum_{d|v} d^2 J(ud) + O(k^\epsilon N^\epsilon) = \frac{u^2 J(u)}{2\phi^2(k)} \sum_{d|v} d^2 J(d) + O(k^\epsilon N^\epsilon)$$

$$= \frac{u\phi^2(u)}{2\phi^2(k)} \prod_{p|v} \left[p(p-1)^2 \left(1 - \frac{1}{p(p-1)}\right)\right] + O(k^\epsilon N^\epsilon)$$

$$= \frac{k}{2} \prod_{p\|k} \left(1 - \frac{1}{p(p-1)}\right) + O(k^\epsilon N^\epsilon),$$

where we have used the estimate $\tau(n) \ll n^\epsilon$, the fact that v is a square-free number and u is a square-full number, and the identity $J(u) = \phi^2(u)/u$, if u is a square-full number.

Taking $N = k^3$, we immediately get the asymptotic formula

$$\sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{u^2 d^2 \mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1 d_2 \phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1 d_2) L^2(1, \overline{\chi})$$

$$= \frac{k}{2} \prod_{p\|k} \left(1 - \frac{1}{p(p-1)}\right) + O(k^\epsilon).$$

This proves Lemma 5. □

Lemma 6. *For any positive integers $k \geqslant 3$ and r with $d_r = (r, \phi(k)) > 1$, let $k = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number, χ_1 be a d_rth-order character modulo k. Let*

$$\Psi = \sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{ud\,\mu(d_1)\mu\left(\frac{v}{dd_2}\right)}{d_1 d_2 \phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)}$$

$$\times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1) \chi\left(\frac{v}{dd_2}\right) L^2(1, \overline{\chi}) \sum_{a=1}^{ud} \chi(a) e\left(\frac{a}{ud}\right)$$

$$\times \sum_{i=1}^{d_r-1} \sum_{l=0}^{k/ud-1} \sum_{m=1}^{ud} \chi_1^i(l \cdot ud + m) \overline{\chi}(m) e\left(\frac{l \cdot ud + m}{k}\right).$$

Then we have the estimate

$$\Psi \ll d_r k^{1/2+\epsilon}.$$

Proof. From the methods of proving Lemma 5 we have

$$\begin{aligned}
\Psi = {} & \sum_{d\mid v}\sum_{d_1\mid \frac{v}{d}}\sum_{d_2\mid \frac{v}{d}} \frac{ud\mu(d_1)\mu(d_2)}{d_1\frac{v}{dd_2}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)} \\
& \times \sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sum_{m=1}^{ud}\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right) \\
& \times \sum_{a=1}^{ud} e\left(\frac{am}{ud}\right)\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(d_1nm)\chi(d_2am) + O\left(\frac{d_rk^{3/2+\epsilon}}{N^{1/2}}\right) \\
= {} & \frac{1}{2}\sum_{d\mid v}\sum_{d_1\mid \frac{v}{d}}\sum_{d_2\mid \frac{v}{d}} \frac{ud\mu(d_1)\mu(d_2)}{d_1\frac{v}{dd_2}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)} \\
& \times \sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sideset{}{'}\sum_{m=1}^{ud}\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right) \\
& \times \sideset{}{'}\sum_{a=1}^{ud} e\left(\frac{am}{ud}\right)\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1}}\frac{\tau(n)}{n}\sum_{\substack{s\mid ud\\ \frac{d_1n}{d_2a}\equiv 1(s)}}\mu\left(\frac{ud}{s}\right)\phi(s) \\
& - \frac{1}{2}\sum_{d\mid v}\sum_{d_1\mid \frac{v}{d}}\sum_{d_2\mid \frac{v}{d}} \frac{ud\mu(d_1)\mu(d_2)}{d_1\frac{v}{dd_2}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)} \\
& \times \sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sideset{}{'}\sum_{m=1}^{ud}\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right) \\
& \times \sideset{}{'}\sum_{a=1}^{ud} e\left(\frac{am}{ud}\right)\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1}}\frac{\tau(n)}{n}\sum_{\substack{s\mid ud\\ \frac{d_1n}{d_2a}\equiv -1(s)}}\mu\left(\frac{ud}{s}\right)\phi(s) + O\left(\frac{d_rk^{3/2+\epsilon}}{N^{1/2}}\right) \\
= {} & \frac{1}{2}\sum_{d\mid v}\sum_{d_1\mid \frac{v}{d}}\sum_{d_2\mid \frac{v}{d}} \frac{ud\mu(d_1)\mu(d_2)}{\frac{v}{d}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)}\sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sideset{}{'}\sum_{m=1}^{ud}\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right) \\
& \times \sideset{}{'}\sum_{a=1}^{ud} e\left(\frac{am}{ud}\right)\sum_{s\mid ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{\substack{1\leqslant n\leqslant N\\ (n,ud)=1\\ \frac{d_1n}{d_2a}\equiv 1(s)}}\frac{\tau(n)/a}{d_1n/d_2a}
\end{aligned}$$

$$-\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{ud\mu(d_1)\mu(d_2)}{\frac{v}{d}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)}$$
$$\times\sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sum_{m=1}^{ud}{}'\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right)$$
$$\times\sum_{a=1}^{ud}{}' e\left(\frac{am}{ud}\right)\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{\substack{1\leqslant n\leqslant N\\(n,ud)=1\\ \frac{d_1n}{d_2a}\equiv -1(s)}}\frac{\tau(n)/a}{d_1n/d_2a}+O\left(\frac{d_rk^{3/2+\epsilon}}{N^{1/2}}\right)$$
$$=\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{ud\mu(d_1)\mu(d_2)}{\frac{v}{d}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)}\sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sum_{m=1}^{ud}{}'\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right)$$
$$\times\sum_{\substack{a=1\\d_1|d_2a}}^{ud}{}' e\left(\frac{am}{ud}\right)\frac{\tau(d_2a/d_1)}{a}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)+O\left(\frac{d_rk^{3/2+\epsilon}}{N^{1/2}}\right)$$
$$+O\left(\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{ud}{\phi(k)\phi(ud)}d_rk\sum_{a=1}^{ud}{}'\frac{1}{a}\sum_{s|ud}\phi(s)\sum_{1\leqslant l\leqslant\frac{d_1N}{d_2as}}(ls+1)^{\epsilon-1}\right)$$
$$+O\left(\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{ud}{\phi(k)\phi(ud)}d_rk\sum_{a=1}^{ud}{}'\frac{1}{a}\sum_{s|ud}\phi(s)\sum_{1\leqslant l\leqslant\frac{d_1N}{d_2as}}(ls-1)^{\epsilon-1}\right)$$
$$=\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{ud\mu(d_1)\mu(d_2)}{\frac{v}{d}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)}\sum_{i=1}^{d_r-1}\sum_{l=0}^{k/ud-1}\sum_{m=1}^{ud}{}'\chi_1^i(l\cdot ud+m)e\left(\frac{l\cdot ud+m}{k}\right)$$
$$\times\sum_{\substack{a=1\\d_1|d_2a}}^{ud}{}' e\left(\frac{\frac{k}{ud}a(l\cdot ud+m)}{k}\right)\frac{\tau(d_2a/d_1)}{a}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)$$
$$+O\left(\frac{d_rk^{3/2+\epsilon}}{N^{1/2}}\right)+O(d_rk^{\epsilon}).$$

Taking $N=q^3$ in the above, we get

$$\Psi=\frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}}\frac{ud\mu(d_1)\mu(d_2)}{\frac{v}{d}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)}\sum_{i=1}^{d_r-1}\sum_{b=1}^{k}\chi_1^i(b)e\left(\frac{b}{k}\right)$$
$$\times\sum_{\substack{a=1\\d_1|d_2a}}^{ud}{}' e\left(\frac{\frac{av}{d}b}{k}\right)\frac{\tau(d_2a/d_1)}{a}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)+O(d_rk^{\epsilon})$$

$$= \frac{1}{2} \sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{udJ(ud)\mu(d_1)\mu(d_2)}{\frac{v}{d}\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{kdd_2}{v}\right)}$$

$$\times \sum_{i=1}^{d_r-1} \sideset{}{'}\sum_{\substack{a=1\\ d_1|d_2a}}^{ud} \frac{\tau(d_2a/d_1)}{a} G\left(\chi_1^i, 1+av/d\right) + O(d_r k^\epsilon)$$

$$\equiv \Omega + O(d_r k^\epsilon).$$

Since χ_1 is a d_rth-order character modulo k and $i < d_r$, χ_1^i cannot be principal character modulo k. From the properties of Gauss sums we get

$$\left|G\left(\chi_1^i, 1+av/d\right)\right| \ll (1+av/d, k)k^{1/2+\epsilon}.$$

So we have

$$\Omega \ll \sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{(ud)^2}{\phi^2(ud)} d_r k^{1/2+\epsilon} \sum_{a=1}^{ud} \frac{(1+av/d, k)}{av/d}$$

$$\ll \sum_{d|v} d_r k^{1/2+\epsilon} \sum_{s|k} \sum_{a=1}^{ud} \sum_{s|1+av/d} \frac{s}{av/d}$$

$$\ll \sum_{d|v} d_r k^{1/2+\epsilon} \sum_{s|k} \sum_{1\leqslant l\leqslant \frac{k+1}{s}} \frac{s}{ls-1} \ll d_r k^{1/2+\epsilon}.$$

This proves Lemma 6. □

3. Proofs of the theorems

In this section, we complete the proofs of the theorems. First we prove Theorem 1. For any positive integer $k \geqslant 3$, by Lemmas 1 and 2 we have

$$\sideset{}{'}\sum_{h=1}^{k} K(h,1;k)C(h,k) = \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \left(\sum_{h=1}^{k} \overline{\chi}(h)K(h,1;k)\right)\left(\sum_{n=1}^{\infty} \frac{G(\chi,n)}{n}\right)^2$$

$$= \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \tau^2(\overline{\chi})\left(\sum_{n=1}^{\infty} \frac{G(\chi,n)}{n}\right)^2.$$

Let $k = uv$, where $(u,v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Note that if χ^* is a primitive character modulo m, from the properties of Gauss sums we have

$$\tau(\chi^*)\tau(\overline{\chi}^*) = -m \quad \text{if } \chi(-1) = -1.$$

It is obvious that $\chi^*(k/m)\mu(k/m) \neq 0$ if and only if $m = ud$, where $d|v$. So for any non-primitive character χ modulo k with $\chi = \chi_1\chi^*$, from Lemma 3 we know that

$$\tau(\chi) = \chi^*\left(\frac{k}{m}\right)\mu\left(\frac{k}{m}\right)\tau(\chi^*) \neq 0 \quad \text{if and only if} \quad m = ud,$$

where $d|v$. On the other hand, from Lemma 3 we also have

$$\sum_{n=1}^{\infty} \frac{G(\chi_1\chi^*, n)}{n} = \sum_{\substack{n=1\\(n,ud)=1}}^{\infty} \frac{\overline{\chi}^*\left(\frac{n}{(n,uv)}\right)\chi^*\left(\frac{uv}{ud(n,uv)}\right)\mu\left(\frac{uv}{ud(n,uv)}\right)\phi(k)\tau(\chi^*)}{n\phi\left(\frac{uv}{(n,uv)}\right)}$$
$$= \sum_{d_1|\frac{v}{d}} \frac{\phi(k)\chi^*\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_1}\right)\tau(\chi^*)L(1, \overline{\chi^*})}{d_1\phi\left(\frac{k}{d_1}\right)},$$

where χ_1 is the principal character modulo k $(= uv)$.

Therefore, by Lemmas 3 and 5 we have

$$\begin{aligned}
&\sum_{h=1}^{k}{}' K(h, 1; k)C(h, k)\\
&= \frac{-1}{\pi^2\phi(k)} \sum_{d|v} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} \overline{\chi}^2\left(\frac{v}{d}\right)\mu^2\left(\frac{v}{d}\right)\tau^2(\overline{\chi})\\
&\quad\times \left(\sum_{d_1|\frac{v}{d}} \frac{\phi(k)\chi\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_1}\right)\tau(\chi)L(1, \overline{\chi})}{d_1\phi\left(\frac{k}{d_1}\right)}\right)^2\\
&= -\frac{\phi(k)}{\pi^2} \sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{u^2d^2\mu\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_2}\right)}{d_1d_2\phi\left(\frac{k}{d_1}\right)\phi\left(\frac{k}{d_2}\right)} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} \overline{\chi}(d_1d_2)L^2(1, \overline{\chi})\\
&= \frac{-1}{2\pi^2} k\phi(k) \prod_{p\|k}\left(1 - \frac{1}{p(p-1)}\right) + O(k^{1+\epsilon}).
\end{aligned}$$

This proves Theorem 1.

For any positive integer $r > 1$, let $d_r = (r, \phi(k))$ and χ_1 be a d_rth-order character modulo k. Then by Lemmas 1 and 2 we have

$$\begin{aligned}
&\sum_{h=1}^{k}{}' K(h, 1, r; k)C(h, k)\\
&= \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \left(\sum_{h=1}^{k} \overline{\chi}(h)K(h, 1, r; k)\right)\left(\sum_{n=1}^{\infty} \frac{G(\chi, n)}{n}\right)^2
\end{aligned}$$

$$= \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \left(\tau^2(\overline{\chi}) + \tau(\overline{\chi}) \sum_{i=1}^{d_r-1} \tau\big(\overline{\chi}\chi_1^i\big)\right)\left(\sum_{n=1}^{\infty} \frac{G(\chi,n)}{n}\right)^2$$

$$= {\sum_{h=1}^{k}}' K(h,1;k)C(h,k)$$

$$+ \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \tau(\overline{\chi}) \sum_{i=1}^{d_r-1} \tau\big(\overline{\chi}\chi_1^i\big)\left(\sum_{n=1}^{\infty} \frac{G(\chi,n)}{n}\right)^2.$$

Let $k = uv$, where $(u,v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. By Lemmas 3, 6 and the method of proving Theorem 1 we can also get

$$\frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \tau(\overline{\chi}) \sum_{i=1}^{d_r-1} \tau\big(\overline{\chi}\chi_1^i\big)\left(\sum_{n=1}^{\infty} \frac{G(\chi,n)}{n}\right)^2$$

$$= \frac{\phi(k)}{\pi^2} \sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \frac{ud\mu(d_1)\mu\big(\frac{v}{dd_2}\big)}{d_1d_2\phi\big(\frac{k}{d_1}\big)\phi\big(\frac{k}{d_2}\big)}$$

$$\times {\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}}^{*} \overline{\chi}(d_1)\chi\left(\frac{v}{dd_2}\right) L^2(1,\overline{\chi}) \sum_{a=1}^{ud} \chi(a) e\left(\frac{a}{ud}\right)$$

$$\times \sum_{i=1}^{d_r-1} \sum_{l=0}^{k/ud-1} \sum_{m=1}^{ud} \chi_1^i(l\cdot ud + m)\overline{\chi}(m) e\left(\frac{l\cdot ud + m}{k}\right)$$

$$\ll d_r k^{3/2+\epsilon}.$$

Now from Theorem 1 and the above estimate we immediately obtain

$${\sum_{h=1}^{k}}' K(h,1,r;k)C(h,k) = \frac{-1}{2\pi^2} k\phi(k) \prod_{p\|k} \left(1 - \frac{1}{p(p-1)}\right) + O\big(d_r k^{3/2+\epsilon}\big).$$

This completes the proof of Theorem 2.

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