Boundedness of Solutions for Duffing’s Equations with Semilinear Potentials

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In this paper we will prove the boundedness of all the solutions and the existence of quasiperiodic solutions for the second order scalar differential equation
\[ \ddot{x} + \arctan x = e(t), \]
where \( e \) is a small parameter and the 1-periodic function \( e(t) \) is a smooth function.

Key Words: boundedness of solutions; Duffing’s equation; Moser’s twist theorem.

1. INTRODUCTION

We consider the time dependent nonlinear scalar differential equation
\[ \ddot{x} + V_x(x, t) = 0, \]
where \( V(x, t+1) = V(x, t) \), which is a Hamiltonian system with Hamiltonian
\[ \tilde{H}(x, \dot{x}, t) = \frac{\dot{x}^2}{2} + V(x, t). \]

According to the growth speed of \( V(x, t) \) with respect to the variable \( x \) nearby infinite point, Eq. (1.1) is classified into the following three cases:

1. Superlinear case. \( V_x(x, t)/x \to +\infty \) as \( x \to \pm \infty \);
2. Semilinear case. \( 0 < k \leq V_x(x, t)/x \leq K < +\infty \);
3. Sublinear case. \( \text{Sign}(x) \cdot g(x) \to +\infty \) and \( g(x)/x \to 0 \) as \( x \to \pm \infty \),

where the above limits or inequalities are uniform with respect to the variable \( t \in \mathbb{R} \).

It is well known that the longtime behaviour of Eq. (1.1) can be very intricate. For example, there are equations having unbounded solutions but with infinitely many zeros and with nearby unbounded solutions having

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randomly prescribed number of zeros and also periodic solutions; see [18]. In contrast to such unbounded phenomenon Littlewood [8] suggested to study the boundedness of all the solutions of
\[ \ddot{x} + g(x) = p(t) \] (1.2)
in superlinear and sublinear cases where \( p(t + 1) = p(t) \).

The first result in superlinear case is due to Morris [17], who proved that all the solutions of
\[ \ddot{x} + 2x^3 = p(t) \] (1.3)
are bounded, where \( p(t) \in C^0(S^1) \). In 1987, Dieckerhoff and Zehnder [2] proved that all the solutions of
\[ \ddot{x} + x^{2n+1} + \sum_{i=0}^{2n} p_i(t) x^i = 0, \quad (n \in \mathbb{N}) \] (1.4)
are bounded where \( p_i(t + 1) = p_i(t) \in C^0(\mathbb{R}) \). Subsequently, this result was extended to more general cases for a large class of superlinear function \( V(x, t) \) in Eq. (1.1) by several authors; we refer to \([6, 7, 10, 26, 28, 29]\) and references therein. Recently, the boundedness of all the solutions for the following sublinear equation
\[ \ddot{x} + |x|^{\alpha-1} \cdot x = e(t) \] (1.5)
has been studied in \([5, 14]\), where \( 0 < \alpha < 1 \). They proved that every solution of Eq. (1.5) is bounded if \( e(t) = e(t + 1) \) is a smooth function. Liu [9] studied the general form Eq. (1.2) in sublinear case and under some reasonable assumptions, gave an affirmative answer to Littlewood’s problem in the sublinear case.

However, the boundedness in the semilinear case is quite different and very delicate. Some of the difficulties are related to the phenomenon of linear resonance. For example, the linear equation
\[ \ddot{x} + n^2 x = \cos nt \quad (n \in \mathbb{N}) \]
has no bounded solutions. Another interesting example was constructed by Ding [3]. He proved that the equation
\[ \ddot{x} + n^2 x + \arctan x = 4 \cos nt \] (1.6)
has no \( 2\pi \)-periodic solutions. In this case, from Massera’s theorem [16] with an observation that every solution of Eq. (1.6) exists on the whole \( t \)-axis, we conclude that all the solutions are unbounded.
In 1996, Ortega [20] investigated a special form of Eq. (1.2) in the semilinear case
\[ \ddot{x} + ax^+ - bx^- = 1 + \varepsilon h(t), \quad (1.7) \]
where \( \varepsilon \) is a small parameter, \( a \) and \( b \) are positive constants \((a \neq b)\), \( x^+ = \max\{x, 0\} \), and \( x^- = \max\{-x, 0\} \). He proved that all the solutions of Eq. (1.7) are bounded if the 1-periodic function \( h(t) \in C^4(\mathbb{R}) \) and \( |\varepsilon| \) is sufficiently small. Later Liu [11] and Ortega [21] improved the result for the cases \( 1/\sqrt{a} + 1/\sqrt{b} \in Q \) and \( 1/\sqrt{a} + 1/\sqrt{b} \in \mathbb{R} \setminus Q \), respectively. Further results have appeared in [12, 13, 22, 27] for semilinear Duffing’s equation.

In 1998, Ortega [23] proposed the problem whether all solutions of
\[ \ddot{x} + \arctan x = \varepsilon e(t) \quad (1.8) \]
are bounded or not, where \( \varepsilon \) is a small parameter and \( e(t + 1) = e(t) \). In this paper, using the method in [9] we will give an affirmative answer to this problem.

We denote by \( c < 1 \) and \( C > 1 \), respectively, two universal positive constants.

The main result is

**Theorem 1.** Assume that \( \varepsilon \) is a small parameter, \( e(t) \in C^\omega(S^1) \) and \( \int_0^1 e(t) \, dt = 0 \). Then every solution of (1.8) is bounded, that is, if \( x = x(t) \) is a solution of (1.8), then it is defined on \(( -\infty, +\infty )\) and
\[ \sup_{t \in \mathbb{R}} (|x(t)| + |\dot{x}(t)|) < +\infty. \]

**Remark.** The condition that \( \int_0^1 e(t) \, ds = 0 \) can be deleted, see Section 5, Remark 2.

The proofs are based on Moser’s twist theorem [4, 19, 25], by means of the following steps. Using transformation theory, (1.8) is, outside of a large disc \( \mathcal{D}_r = \{(x, \dot{x}) \in \mathbb{R}^2 : x^2 + \dot{x}^2 \leq r^2\} \) in the \((x, \dot{x})\)-plane, reduced to a perturbation of an integrable system. The transformation is chosen in such a way that the Poincaré map of the new system is close to a so-called twist map in \( \mathbb{R}^2 \setminus \mathcal{D}_r \). Then Moser’s twist theorem guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the \((x, \dot{x})\)-plane. Every such a curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space \((x, \dot{x}, t) \in \mathbb{R}^2 \times \mathbb{R} \), which confines the solutions in the interior and which leads to a bound of these solutions.
The rest of the paper is organized as follows. Some technical lemmas which are useful for our proof are stated in Section 2 and Section 3. We will give the proof of Theorem 1 in Section 4. Some remarks and another theorem about the existence of quasiperiodic solutions, the Aubry–Mather set, and unlinked periodic solutions are given in Section 5.

2. ACTION-ANGLE VARIABLES AND SOME ESTIMATES

In this section, we first introduce action-angle variables for Eq. (1.8) and state some technical lemmas which will be used in the proof of our main result. From now on we assume that the conditions of Theorem 1 hold.

For brevity, we set $g(x) = \arctan x$ and $E(t) = \int_0^t e(s) \, ds$. Therefore we have

$$G(x) = \int_0^x \arctan s \, ds = x \cdot \arctan x - \frac{1}{2} \ln(1 + x^2).$$

Since $g(x)$ is odd and $\int_0^\infty e(t) = 0$, $G(x)$ is even and $E(t)$ is 1-periodic. Let $W(x) = \frac{G(x)}{s(x)}$. Then it is easy to verify that for all $|x| \geq d_0$ with some fixed constant $d_0 > 0$

$$\frac{1}{2} < 1 - a \leq W'(x) \leq 1,$$

and

$$|x^k G^{(k)}(x)| \leq C \cdot G(x) \quad (\forall k \in \mathbb{N})$$

Equation (1.8) is equivalent to the planar Hamiltonian system

$$\dot{x} = \partial_y h(x, y, t), \quad \dot{y} = -\partial_x h(x, y, t),$$

with Hamiltonian

$$h(x, y, t) = \frac{y^2}{2} + G(x) + \varepsilon y E(t).$$

Consider an auxiliary autonomous system

$$\dot{x} = y, \quad \dot{y} = -g(x),$$

which is an integrable Hamiltonian system with Hamiltonian

$$H^*(x, y) = \frac{1}{2} y^2 + G(x).$$
The closed curves \( C_h : H^*(x, y) = h > 0 \) are just the integral curves of (2.5). It is well known from [1] that (2.5) has action and angle variables \((\theta, I)\). To construct the map \((x, y) \mapsto (\theta, I)\), we let \( H_0(I) \) be the value of \( H^*(x, y) = \frac{1}{2} y^2 + G(x) \) on that level curve which encloses area \( I \) in the \((x, y)\)-plane; i.e., we define implicitly by

\[
\int_{C_h} \! \left( \frac{1}{2} y^2 + G(x) \right) \, dx = I. \tag{2.6}
\]

We define now the generating function \( S(x, I) \) as the area

\[
S(x, I) = \int_{C_h} \! y \, dx, \tag{2.7}
\]

where \( C \) is the part of the level curve \( H^*(x, y) = H_0(I) \) connecting the \( y \)-axis with the point \((x, y)\), oriented clockwise. This defines \( S \) up to an integer multiple of \( I = \frac{1}{2} y \, dx \) since \( C \) is defined up to an integer number of full trips around the level curve. We define the map \((\theta, I) \mapsto (x, y)\) via

\[
S_x(x, I) = y, \quad S_I(x, I) = \theta. \tag{2.8}
\]

Then it is symplectic because

\[
dx \wedge dy = dx \wedge (S_{xx} \, dx + S_{xt} \, dI) = S_{xt} \, dx \wedge dI,
\]

\[
d\theta \wedge dI = (S_{rt} \, dx + S_{tt} \, dI) \wedge dI = S_{rt} \, dx \wedge dI.
\]

Equation (2.3) in the new variables \((\theta, I, t)\) retains its Hamiltonian character with the new Hamiltonian

\[
H(\theta, I, t) = H_0(I) + H_1(\theta, I, t), \tag{2.9}
\]

where \( H_1(\theta, I, t) = \xi \nu E(t) \), and \( y = y(\theta, I) \) is defined implicitly by (2.8).

Now we give an expression for \( H_0(I) \). Denote by \( G^{-1}_+ \) and \( G^{-1}_- \) the right and left inverse of \( G \), respectively. Assume \((x_+, 0)\) and \((x_-, 0)\) are the intersection points of \( I_h \) with the \( x \)-axis, i.e.,

\[
x_- = G^{-1}_-(H_0(I)) < 0 < G^{-1}_+(H_0(I)) = x_+.
\]

Since \( G(x) \) is even, we have \( x_+ = -x_- \). Rewriting (2.6) we obtain the implicit definition of \( H_0(I) \):

\[
I = 4 \sqrt{2} \int_0^{x_+} \sqrt{H_0(I) - G(\xi)} \, d\xi. \tag{2.10}
\]
Restating this slightly, $H_0(I)$ is defined as the inverse function of

$$I_0(H) = 4 \sqrt{2 \int_0^{x_*} \sqrt{H-G(\xi)} \, d\xi}, \quad (2.11)$$

where $x_*(H) = G^{-1}_*(H)$.

In the following, we state some lemmas which will be used in Sections 3 and 4.

**Lemma 2.1.** For all $H$ large enough, we have

$$2 \sqrt{2H} \cdot G^{-1}_*(H) \leq I_0(H) \leq 4 \sqrt{2H} \cdot G^{-1}_*(H), \quad (2.12)$$

$$I_0^{(k)}(H) \geq c \cdot H^{-k} \cdot I_0(H) \quad (k = 1, 2) \quad (2.13)$$

**Proof.** (1) One can prove it by comparing the area bounded by $\Gamma_h$ respectively with the area of the triangle or rectangle with sides $\sqrt{2H}$ and $G^{-1}_*(H)$.

(2) Similar to the proof of [7, A3.2], one can prove that

$$I_0(H) = \frac{4}{H} \int_0^{x_*} \left(\frac{1}{2} + W'(\xi)\right) \cdot \sqrt{2(H-G(\xi))} \, d\xi.$$

It is easy to verify that for all $x \in \mathbb{R}$

$$\frac{1}{2} \leq W'(x) \leq 1.$$

Therefore we have

$$I_0'(H) \geq H^{-1} \cdot I_0(H)$$

which leads to

$$I_0(H) \geq C \cdot H.$$

Let $H$ be large enough such that

$$4 \sqrt{2} \left(\frac{1}{2} - a\right) d_0 \cdot H^{-\frac{1}{2}} < c_0 < \frac{1}{2} - \frac{1}{4} a.$$

Set $a_0 = a + c_0$. Then $0 < a_0 < \frac{1}{2}$. By (2.1), we have for all $H$ large enough

$$I_0'(H) = \frac{4}{H} \left(\int_0^{d_0} + \int_{d_0}^{x_*}\right) \left(\frac{1}{2} + W'(\xi)\right) \sqrt{2(H-G(\xi))} \, d\xi$$

$$\geq 4 \sqrt{2} d_0 \cdot H^{-\frac{1}{2}} + \left(\frac{3}{2} - a\right) \cdot H^{-1} I_0(H) - 4 \sqrt{2} \left(\frac{3}{2} - a\right) d_0 \cdot H^{-\frac{1}{2}}.$$
\[
\begin{align*}
&= \left( \frac{3}{2} - a \right) \cdot H^{-1} I_0(H) - 4 \sqrt{2} \left( \frac{1}{2} - a \right) d_0 \cdot H^{-\frac{3}{2}} \\
&\geq \left( \frac{3}{2} - a \right) \cdot H^{-1} I_0(H) - c_0 \\
&= \left( \frac{3}{2} - a - c_0 \right) \cdot H^{-1} I_0(H) + c_0 \cdot H^{-1} I_0(H) - c_0 \\
&\geq \left( \frac{3}{2} - a_0 \right) \cdot H^{-1} I_0(H).
\end{align*}
\]

On the other hand, we obtain

\[
I'_0(H) \leq \frac{3}{2} \cdot H^{-1} I_0(H).
\]

Therefore we get for all \( H \) large enough

\[
\left( \frac{1}{2} - a_0 \right) \cdot H^{-1} I_0(H) \leq I'_0(H) \leq \frac{3}{2} \cdot H^{-1} I_0(H),
\]

which yields that

\[
c \cdot H^{\frac{1}{2} - a_0} \leq I_0(H) \leq C \cdot H^{\frac{3}{2}}.
\]

Using the same trick in computing \( I'_0(H) \), one can see that

\[
I''_0(H) = \frac{4}{H} \int_0^{\xi_+} \left( \frac{W''(\xi)}{2} - \frac{1}{2} \right) \cdot \frac{1}{\sqrt{2(H - G(\xi))}} d\xi.
\]

Hence, we have

\[
c \cdot H^{-2} I_0(H) \leq \left( \frac{1}{2} - a_0 \right) \cdot H^{-1} I'_0(H) \leq I''_0(H) \leq \frac{3}{2} \cdot H^{-2} I'_0(H) \leq C \cdot H^{-2} I_0(H).
\]

From Lemma 2.1, it is easy to obtain the following

**Lemma 2.2.** For all \( I \) large enough we have

\[
c \cdot I^\frac{3}{2} \leq H_0(I) \leq C \cdot I^\frac{3}{2},
\]

\[
|H''_0(I)| \geq c \cdot I^{-k} \cdot H_0(I). \quad (k = 1, 2)
\]

|H''_0(I)| \geq c \cdot I^{-k} \cdot H_0(I). \quad (k = 1, 2)
Lemma 2.3 [7]. The following inequalities hold for all nonnegative integers $k$

\[
|\partial^2_{\theta}x(\theta, I)| \leq C \cdot I^{-k} \cdot |x(\theta, I)|, \quad |\partial^2_{\theta}y(\theta, I)| \leq C \cdot I^{-k} \cdot |y(\theta, I)|, \quad (2.18)
\]

\[
|I^{(k)}_0(H)| \leq C \cdot H^{-k} \cdot I_0(H), \quad |H^{(k)}_0(I)| \leq C \cdot I^{-k} \cdot H_0(I). \quad (2.19)
\]

By the definition of $\theta$, we have

\[
\partial_\theta x(\theta, I) = I'_0(H_0(I)) \cdot y(\theta, I),
\]

\[
\partial_\theta y(\theta, I) = -I'_0(H_0(I)) \cdot g(x(\theta, I)). \quad (2.20)
\]

Lemma 2.4.

\[
|\partial^i_\theta \partial^j_\theta \partial^l_\theta H_i(\theta, I, t)| \leq C \cdot \varepsilon \cdot I^{-k} \cdot \sqrt{H_0(I)}, \quad (i = 0, 1) \quad (2.21)
\]

for all nonnegative integers $k$ and $l$.

Proof. From the definition of $H_i(\theta, I, t)$, we have

\[
\partial^i_\theta \partial^j_\theta H_i(\theta, I, t) = \varepsilon \cdot \partial^i_\theta y(\theta, I) \cdot E^{i0}(t).
\]

The conclusion for $i = 0$ follows easily from (2.18) and $|y(\theta, I)| \leq \sqrt{2H_0(I)}$.

For $i = 1$, from (2.20) we have

\[
\partial_\theta \partial^i_\theta H_i(\theta, I, t) = -\varepsilon \cdot I'_0(H_0(I)) \cdot g(x(\theta, I)) \cdot E^{i0}(t).
\]

By (2.1), (2.2), and (2.18), it follows that

\[
|\partial^i_\theta g(x)| \leq \sum_{k_1 + \cdots + k_s = k} C \cdot |G^{(s+1)}(x)| \cdot \partial_{k_1}^i x \cdots \partial_{k_s}^i x
\]

\[
\leq \sum_{k_1 + \cdots + k_s = k} C \cdot |G^{(s+1)}(x)| \cdot I^{-k_1} |x| \cdots I^{-k_s} |x|
\]

\[
\leq C \cdot I^{-k} \left| \frac{G(x)}{x} \right|
\]

\[
\leq C \cdot I^{-k} |g(x)|
\]

\[
\leq C \cdot I^{-k}.
\]

BOUNDEDNESS OF SOLUTIONS 255
Similarly, by (2.13), (2.17) and (2.19), it follows that
\[ \left| \frac{d^k}{dI^k} I_0^k(H_0(I)) \right| \leq C \cdot I^{-k-1} \cdot [H_0(I)]^{-1}. \]

Therefore, by (2.16) we have
\[ |\partial_I^k \partial_\theta^l H_1(H, I, t)| \leq C \cdot e \cdot I^{-k-1} \cdot [H_0(I)]^{-1} \leq C \cdot e \cdot I^{-k} \cdot \sqrt{H_0(I)}. \]

This completes the proof.

3. NEW ACTION-ANGLE VARIABLES

Now we are concerned with the Hamiltonian system with Hamiltonian
\[ H(\theta, I, t) \] given by (2.9). Note that
\[ I \, d\theta - H(\theta, I, t) \, dt = -(H \, dt - I(t, H, \theta) \, d\theta). \]

This means that if one can solve \( I = I(t, H, \theta) \) from (2.9) (\( \theta \) and \( t \) as parameters) as a function of \( t, H, \) and \( \theta, \) then
\[ \frac{dt}{d\theta} = \partial_\theta I(t, H, \theta), \quad \frac{dH}{d\theta} = -\partial_\theta I(t, H, \theta). \quad (3.1) \]

That is, (3.1) is also a Hamiltonian system with Hamiltonian \( I(t, H, \theta) \) and now the action, angle and time variables are \( H, t \) and \( \theta, \) respectively. This trick has been used in \([6, 7]\).

Because of \( \partial_H \partial_\theta^2 H(\theta, I, t) \neq 0 \) for \( I \) large enough by Lemmas 2.2 and 2.4, we define \( I(t, H, \theta) \) as the inverse function of \( H(\theta, I, t) \) with \( t, \theta \) playing the role of parameters; thus we define \( I_1(t, H, \theta): \)
\[ I(t, H, \theta) = I_0(H) + I_1(t, H, \theta). \]

Now we give some estimates on the function \( I_1(t, H, \theta). \)

**Lemma 3.1.** \( I_1(t, H, \theta) \) possesses the estimates for all nonnegative integers \( k, l \) and \( H \gg 1 \)
\[ |\partial_H^k \partial_\theta^l \partial_I I_1(t, H, \theta)| \leq C \cdot e \cdot H^{-k-1} \cdot I_0(H) \quad (i = 0, 1) \quad (3.2) \]

**Proof.** We will prove it in the following two cases: \( i = 0 \) and \( i = 1. \)
Case 1. \( i = 0 \). We classify it into the three cases:

(i) \( i = 0, l = 0, k = 1 \). From the definition of \( I(t, H, \theta) \),

\[
H_0(I(H)) + H_1(I(H)) = H, \tag{3.3}
\]

where we treat \( H \) as the independent variable and \( \theta, t \) as parameters, we obtain

\[
I(H) = I_0(H - H_1(I(H))) \tag{3.4}
\]

and finally, expanding

\[
I_1(H) = I(H) - I_0(H) = I_0(H - H_1(I(H))) - I_0(H)
\]

in Taylor’s series,

\[
I_1(H) = -I_0'(H)H_1 + \int_0^H sI_0''(H - H_1 + s) \, ds, \quad H_1 = H_1(I(H)). \tag{3.5}
\]

We will now estimate \( H_1(I(H)) \) via (3.3), thus use it in (3.4) and finally use this in (3.5) to estimate \( I_1(H) \). To estimate \( H_1(I(H)) \), by Lemmas 2.1 and 2.4, note that \( I(H) \to \infty \) as \( H \to \infty \), and that \( |H_1(I)| < \frac{1}{2}H_0(I) \) for all \( I \) large enough and for all \( \theta, t \); consequently

\[
|H_1(I(H))| < \frac{1}{2}H_0(I(H)) \tag{3.6}
\]

for all \( H \) large enough and for all \( t, \theta \). To estimate \( I(H) \) we use (3.4), (3.6) and the monotonicity of \( I_0 \) in \( H \), obtaining

\[
I_0\left(\frac{1}{2}H\right) < I(H) < I_0\left(\frac{3}{2}H\right). \tag{3.7}
\]

By Lemma 2.1, it follows that

\[
I_0\left(\frac{1}{2}H\right) > c \cdot I_0(H), \quad I_0\left(\frac{3}{2}H\right) < C \cdot I_0(H), \tag{3.8}
\]

which leads to

\[
c \cdot I_0(H) < I(H) < C \cdot I_0(H). \tag{3.9}
\]

By Lemmas 2.2 and 2.4, we obtain

\[
|H_1(I(H))| \leq C \cdot e \cdot \sqrt{H_0(I(H))} \leq C \cdot e \cdot \sqrt{H_0(C \cdot I_0(H))} \leq C \cdot e \cdot \sqrt{H}, \tag{3.10}
\]

which we now finally use to estimate \( I_1(H) \) from (3.5).
Estimating the first term in (3.5), we get
\[ |I_0'(H) \cdot H_1(I(H))| \leq C \cdot e \cdot H^{-1} \cdot I_0(H) \]
as desired. The second term in (3.5) is bounded by
\[ H_1^2 \sup_{H-H_1 < H < H} I_0''(H) \leq C \cdot H_1 \cdot H^{-2} \cdot I_0(H) \leq C \cdot e^2 \cdot H^{-1} \cdot I_0(H). \]

This completes the proof of (i).

(ii) \( i = 0, l = 0, k > 1 \). Differentiating (3.5) \( k-1 \) times by \( H \), we obtain
\[ \partial^k_I I_1(H) = -\sum_{i=0}^{k} C_{ki} \cdot \partial^{i+1}_H I_0(H) \cdot \partial^{k-i}_H H_1 + \partial^k_I \int_0^{H_1} s I_0''(H-H_1+s) \, ds, \]
where \( H_1 = H_1(I(H)) \) and \( C_{ki} \) is an integer which is only dependent on \( k \) and \( i \). Note that
\[ \partial^k_I H_1(I(H)) = \sum_{k_1 + \ldots + k_j = k} C_{k_1} \cdot \partial^{k_1}_H H_1(I(H)) \cdot \ldots \cdot (\partial^{k_j}_H I(H)) \]
and
\[ \partial^k_I \int_0^{H_1} s I_0''(H-H_1+s) \, ds \]
\[ = \sum_{i+j = k} C_{k_1} \cdot \partial^{i+1}_H H_1 \cdot \partial^{j}_H H_1 \cdot \partial^{k-i-j+2}_H I_0(H) \]
\[ + \sum_{i+j+1 = k} C_{k_1} \cdot \partial^{i+1}_H H_1 \cdot \partial^{j+1}_H H_1 \cdot \partial^{k-i-j+2}_H I_0(H) \cdot \partial^{i}_H H_0 \cdot \ldots \cdot \partial^{k}_H H_0 \]
\[ + \sum_{i+1 = k} C_{k_1} \cdot \partial^{i}_H H_0 \cdot \ldots \cdot \partial^{k}_H H_0 \int_0^{H_1} s \partial^{i+1}_H I_0(H-H_1+s) \, ds. \]

Therefore, the proof of (ii) reduces to the proof of
\[ |\partial^k_I I(H)| \leq C \cdot I(H) \quad (H \gg 1, k \in \mathbb{N}) \quad (3.11) \]

Differentiating (3.3) by \( H \) yields
\[ I'(H) = I_0'(H-H_1) \cdot [1 - H_1' \cdot I'(H)]. \]
We express

\[ I'(H) = \frac{I'_0(H-H_i)\cdot H_1'}{1+I'_0(H-H_i)\cdot H_1'}. \]  

(3.12)

The denominator above is close to one for large \( H \), indeed,

\[
\begin{align*}
I'_0(H-H_i)\cdot H_1' &= I'_0(H-H_i)\cdot H_0'(H-H_i)) \cdot \frac{H_1'(I(H))}{H'_0(I_0(H-H_i))} \\
&= 1 \cdot \frac{H_1'(I(H))}{H'_0(I(H))} \to 0 \quad \text{as} \quad H \to \infty.
\end{align*}
\]

For \( H \) large enough we get

\[ I'(H) < 2I'_0(H-H_i) < C \cdot (H-H_i)^{-1} \cdot I_0(H-H_i) < C \cdot H^{-1} \cdot I(H) \]

proving the case \( k=1 \) in (3.11).

Assume inductively that (3.11) holds for \( 1 \leq k \leq n \), now we prove it for \( k=n+1 \). Differentiating (3.12) \( n \) times by \( H \) yields that

\[
\partial^n_{H} I(H) = \sum_{i=0}^{n} C_{n} \cdot \partial^n_{H} I'_0(H-H_i) \cdot \partial^{n-i}_{H} A,
\]

where \( A = [1+I'_0(H-H_i)\cdot H_1']^{-1} \). The proof of (3.11) for \( k=n+1 \) reduces to the proof of

\[ |\partial^n_{H} I'_0(H-H_i)| \leq C \cdot H^{-m} I'_0(H-H_i) \quad (m \leq n, H \gg 1) \]  

(3.13)

and

\[ |\partial^n_{H} A| \leq C \cdot H^{-m} \quad (m \leq n, H \gg 1) \]  

(3.14)

**Proof of (3.13).** Note that

\[
\partial^m_{H} I'_0(H-H_i) = \sum_{m_1+\cdots+m_j=m} C_{m_j} \cdot I'_0(H-H_i) \cdots \partial^m_{H} (H-H_i) \]

(3.15)

and

\[
\partial^n_{H} (H-H_i) = \partial^n_{H} H_0(I(H))
\]

(3.16)
Using inductive assumptions in (3.16), we get for \( l \leq n \)
\[
|\partial_H^l (H - H_i)| \leq C \cdot H^{-l} \cdot (H - H_i),
\]
which by (3.15) and Lemma 2.2 implies that for \( m \leq n \)
\[
|\partial_H^m \partial_H^0 (H - H_i)| \leq C \cdot H^{-m} \partial_H^0 (H - H_i)
\]
as desired.

**Proof of (3.14).** It is easy to see that
\[
\partial_H^m A = \sum C_{m_0} \cdot [1 + \partial_H^0 (H - H_i) \cdot H'_i]^{-1 - j} \\
\cdot \partial_H^m [1 + \partial_H^0 (H - H_i) \cdot H'_i] \cdot \partial_H^m [1 + \partial_H^0 (H - H_i) \cdot H'_i],
\]
(3.17)
where \( 0 \leq j \leq m, m_1 + \cdots + m_j = m \). By (3.13) and Lemma 2.4, it follows that for \( 1 \leq j \leq n \)
\[
|\partial_H^j [1 + \partial_H^0 (H - H_i) \cdot H'_i]| \leq C \cdot H^{-j} \cdot |\partial_H^0 (H - H_i) \cdot H'_i|.
\]
Since
\[
1 + \partial_H^0 (H - H_i) \cdot H'_i \leq 1 + C \cdot H^{-0} \cdot I(H) \cdot I(H)^{-1} \cdot H_i \rightarrow 1, \quad \text{as} \quad H \rightarrow \infty,
\]
we have for all \( H \) large enough and \( 1 \leq j \leq n \)
\[
1 + \partial_H^0 (H - H_i) \cdot H'_i \leq C
\]
and
\[
|\partial_H^j [1 + \partial_H^0 (H - H_i) \cdot H'_i]| \leq C \cdot H^{-j}.
\]
Therefore by (3.17), we obtain for all \( H \) large enough and \( m \leq n \)
\[
|\partial_H^m A| \leq C \cdot H^{-m}
\]
as desired. This completes the proof of (ii)

(iii) \( i = 0, k \geq 0, l > 1 \). Similar to the above analysis, the proof of (iii) reduces to the proof of
\[
|\partial_H^k \partial_H^l I(H)| \leq C \cdot H^{-k} \cdot I(H)
\]
(3.18)
for \( k \geq 0, l > 1 \) and \( H \gg 1 \). In fact, according to (3.12), Lemmas 2.1 and 2.4, we conclude that the differentiation of \( I^{(i)}(H) \) \( l \) times with respect to the variable \( t \) does not increase the order of growth of the upper bound.
Case 2. \( i = 1 \). Differentiating the equality \( I(t, H, I, t, \theta) = I \) by \( \theta \) yields that

\[
\partial_{H} I_{1}(H) = -\partial_{H} I(H) \cdot \partial_{\theta} H_{1}(I(H)). \tag{3.19}
\]

It is easy to prove that

\[
\partial^{k}_{H} \partial_{\theta} H_{1}(I(H)) = C \cdot \partial^{k}_{H} I_{1}(H) \cdot \partial^{k}_{\theta} I_{1}(H).
\]

By (3.11), (3.18), (3.19), and Lemma 2.4, we obtain for \( H \gg 1 \)

\[
|\partial^{k}_{H} \partial_{\theta} I_{1}(t, H, \theta)| \leq C \cdot e^{-H^{k-1}} \cdot I_{0}(H).
\]

This completes the proof of Lemma 3.1. \( \blacksquare \)

4. PROOF OF THEOREM 1

Up to now, we have given an equivalent form of Eq. (1.1), that is, the system (3.1), which is expressed in the action and angle variables \((H, t)\). However, its Poincaré mapping is far from a small perturbation of the standard twist mapping \((t, H) \mapsto (t + H, H)\). Hence, one cannot use Moser’s twist theorem directly.

In this section, we first introduce some transformations such that in the transformed system, the terms depending on the new angle variable are very small if the new action variable is sufficiently large and prove, based on Moser’s twist theorem, the statement of Theorem 1.

Lemma 4.1. There is a canonical transformation \( \Psi \): \((\lambda, \tau) \mapsto (H, t)\) of the form

\[
\Psi: \quad H = \lambda + U(\tau, \lambda, \theta), \quad t = \tau + V(\tau, \lambda, \theta), \tag{4.1}
\]

where the functions \( U \) and \( V \) are 1-periodic in \( \theta \) and satisfy

\[
\frac{U(\tau, \lambda, \theta)}{\lambda}, \quad V(\tau, \lambda, \theta) \to 0 \quad \text{as} \quad \lambda \to \infty
\]

uniformly for \((\tau, \theta) \in T^{2}\) such that under this mapping, the system (3.1) with Hamiltonian function \( I(t, H, \theta) \) is changed into the form

\[
\frac{d\lambda}{d\theta} = -\partial_{\lambda} \mathcal{K}(\tau, \lambda, \theta), \quad \frac{d\tau}{d\theta} = \partial_{\tau} \mathcal{K}(\tau, \lambda, \theta), \tag{4.2}
\]
where
\[ \mathcal{K}(\tau, \lambda, \theta) = I_0(\lambda) + [I_1](\lambda, \theta) + \mathcal{K}_1(\tau, \lambda, \theta) \] (4.3)

with
\[ [I_1](\lambda, \theta) = \int_0^1 I_1(t, \lambda, \theta) \, dt. \]

Moreover, the new perturbation \( \mathcal{K}_1 \) possesses the estimate:
\[ |\partial_1^k \partial_2^l \mathcal{K}_1(\tau, \lambda, \theta)| \leq C \cdot e^{-k + \frac{1}{2}} \] (4.4)
for all nonnegative integers \( k \) and \( l \).

**Proof.** We will look for the required transformation \( \Psi \) determined by a generating function \( \mathcal{F}(t, \lambda, \theta) \) in the following way,
\[ H = \lambda + \partial_1 \mathcal{F}(t, \lambda, \theta), \quad \tau = t + \partial_1 \mathcal{F}(t, \lambda, \theta), \] (4.5)
where the function \( \mathcal{F} \) will be given later. Under \( \Psi \), the transformed system of (3.1) is of the form
\[ \frac{d\lambda}{d\theta} = -\partial_1 \mathcal{K}(\tau, \lambda, \theta), \quad \frac{d\tau}{d\theta} = \partial_2 \mathcal{K}(\tau, \lambda, \theta), \]
where
\[ \mathcal{K}(\tau, \lambda, \theta) = I_0(\lambda + \partial_1 \mathcal{F}) + I_1(t, \lambda + \partial_1 \mathcal{F}, \theta) + \partial_0 \mathcal{F}. \]

By Taylor’s formula, we have
\[ \mathcal{K}(\tau, \lambda, \theta) = I_0(\lambda) + I_0'(\lambda) \cdot \partial_1 \mathcal{F} + I_1(t, \lambda, \theta) + \mathcal{K}_1(\tau, \lambda, \theta), \]
where
\[ \mathcal{K}_1(\tau, \lambda, \theta) = \partial_0 \mathcal{F} + \int_0^1 (1-s) I_1(\lambda + s \partial_1 \mathcal{F}) \cdot \partial_1 \mathcal{F} \cdot ds \]
\[ + \int_0^1 \partial_1 I_1(t, \lambda + s \partial_1 \mathcal{F}, \theta) \cdot \partial_1 \mathcal{F} \cdot ds. \]

We now choose \( \mathcal{F} \):
\[ \mathcal{F}(\theta, \lambda, \theta) = -\int_0^1 \frac{1}{I_0'(\lambda)} \cdot (I_1(t, \lambda, \theta) - [I_1](\lambda, \theta)) \, dt. \]

Then \( \mathcal{K} \) is of the form (4.3).
From Lemmas 2.1 and 3.1, it follows that
\[ |\partial_t^k \partial_x^l \partial_y \mathcal{F}(t, \lambda, \theta)| \leq C \cdot e \cdot \lambda^{-k+\frac{1}{2}} \] (4.6)
for all nonnegative integers \( k, l \) and \( i = 0, 1 \). In particular,
\[ |\partial_t \partial_x \mathcal{F}(t, \lambda, \theta)| \leq C \cdot e \cdot \lambda^{-\frac{1}{2}} \]
if \( \lambda \gg 1 \). So one can solve the second equation of (4.5) for \( t \),
\[ t = \tau + V(\tau, \lambda, \theta), \]
where the function \( V \) satisfies
\[ V(\tau, \lambda, \theta) = -\partial_\lambda \mathcal{F}(\tau + V, \lambda, \theta). \]
Set
\[ U(\tau, \lambda, \theta) = \partial_x \mathcal{F}(\tau + V, \lambda, \theta). \]
Then the canonical transformation \( \Psi \) is of the form (4.1). Moreover, similar to the proof of [2, Lemma 2], one can verify that
\[ |\partial_x^k \partial^l U(\tau, \lambda, \theta)| \leq C \cdot e \cdot \lambda^{-k+\frac{1}{2}}, \quad |\partial_x^k \partial^l V(\tau, \lambda, \theta)| \leq C \cdot e \cdot \lambda^{-k+\frac{1}{2}} \] (4.7)
for all nonnegative integers \( k, l \) and \( \lambda^{-1} U, V \rightarrow 0 \) as \( \lambda \rightarrow +\infty \).
Let
\[ \phi_1(\tau, \lambda, \theta) = \partial_\lambda \mathcal{F}(\tau + v, \lambda, \theta), \]
\[ \phi_2(\tau, \lambda, \theta) = \int_0^1 (1 - s) I_0^v(\lambda + sU) \cdot U^2 \ ds, \]
\[ \phi_3(\tau, \lambda, \theta) = \int_0^1 \partial_\lambda I_1(\tau + V, \lambda + sU, \theta) \cdot U \ ds. \]
Similar to the proof of (3.13) in Section 3, it is not difficult to prove that
\[ |\partial_x^k \partial^l \phi_1(\tau, \lambda, \theta)| \leq C \cdot e \cdot \lambda^{-k+\frac{1}{2}}, \]
\[ |\partial_x^k \partial^l \phi_2(\tau, \lambda, \theta)| \leq C \cdot e^2 \cdot \lambda^{-k-1} \cdot I_0(\lambda), \]
\[ |\partial_x^k \partial^l \phi_3(\tau, \lambda, \theta)| \leq C \cdot e^2 \cdot \lambda^{-k-1} \cdot I_0(\lambda), \]
for all nonnegative integers \( k \) and \( l \). Hence, from (2.14) we have

\[
|\partial_\tau \partial_\lambda \lambda_1(\tau, \lambda, \theta)| \leq C \cdot e \cdot \lambda^{-k+1/2}
\]

for all nonnegative integers \( k \) and \( l \). The proof is finished.

For \( \lambda_0 > 0 \), we define the domain

\[
A_{\lambda_0} = \{(\lambda, \tau, \theta) | \lambda \geq \lambda_0, (\theta, t) \in T^2 \}, \quad T^2 = S^1 \times S^1.
\]

In order to apply Moser’s twist theorem, we need the following

**Lemma 4.2** [2]. The Poincaré mapping \( \Phi \) of (4.2) has the intersection property on \( A_{\lambda_0} \), i.e., if \( \Gamma \) is an embedded circle in \( A_{\lambda_0} \) homotopic to a circle \( \lambda = \text{constant} \) in \( A_{\lambda_0} \), then \( \Phi(\Gamma) \cap \Gamma \neq \emptyset \).

Under the diffeomorphism \( \Psi_1 \) on \( A_{\lambda_0} \) given by

\[
\mu = I'_\lambda(\lambda), \quad \tau = \tau, \quad \theta = \theta,
\]

(4.8)

the transformed system of (4.2) is of the form

\[
\frac{d\mu}{d\theta} = f_1(\tau, \mu, \theta), \quad \frac{d\tau}{d\theta} = \mu + f_2(\tau, \mu, \theta),
\]

(4.9)

where

\[
f_1(\tau, \mu, \theta) = -I'_\lambda(\lambda) \cdot \partial_\mu \lambda_1(\tau, \lambda, \theta),
\]

\[
f_2(\tau, \mu, \theta) = \partial_\lambda [I_1](\lambda, \theta) + \partial_\tau \lambda_1(\tau, \lambda, \theta),
\]

(4.10)

with \( \lambda = \lambda(\mu) \) defined by (4.8).

Now we estimate the functions \( f_1 \) and \( f_2 \). By (2.14) and Lemma 3.1, we have

\[
c \cdot \lambda^{1-n} \leq I'_\lambda(\lambda) \leq C \cdot \lambda^{1/2}, \quad |\partial_\lambda [I_1](\lambda, \theta)| \leq C \cdot e.
\]

Hence

\[
c \cdot \mu^2 \leq |\lambda(\mu)| \leq C \cdot \mu^{-n/2}.
\]

(4.11)

Obviously, \( \lambda \gg 1 \iff \mu \gg 1 \). Moreover, by Lemmas 2.1 and 2.2, we have

\[
|\lambda^{(\theta)}(\mu)| \leq C \cdot \lambda(\mu).
\]

(4.12)
From (4.4), (4.12), Lemmas 2.2 and 3.1, it follows that for all nonnegative integers $k$ and $l$
\[
|\partial_k^p \partial_l^l f_1(\tau, \mu, \theta) \leq C \cdot \mu^{-k} \cdot \lambda^{-l} I_0(\lambda) \cdot e \cdot \lambda^2 \leq C \cdot \varepsilon
\]
and
\[
|\partial_k^p \partial_l^l f_2(\tau, \mu, \theta) \leq C \cdot e \cdot \mu^{-k} \cdot (C + \lambda^{-l}) \leq C \cdot \varepsilon.
\]

Now we are in a position to prove the statement of Theorem 1.

Proof of Theorem 1. Since the functions $f_1$ and $f_2$ are sufficiently small if $\mu \gg 1$, one can verify that the solutions of (4.9) do exist for $0 \leq \theta \leq 1$ if the initial value $\mu(0) = \mu$ is sufficiently large. Integrating Eq. (4.9) from $\theta = 0$ to $\theta = 1$, we obtain that the Poincaré mapping $\Phi$ of (4.9) is of the form
\[
\Phi: \quad \tau_1 = \tau_0 + \mu_0 + \Xi_1(\tau_0, \mu_0), \quad \mu_1 = \mu_0 + \Xi_2(\tau_0, \mu_0),
\]
where $\Xi_1$ and $\Xi_2$ possess the estimates as well as $f_1$ and $f_2$, that is, for all nonnegative integers $k$ and $l$,
\[
|\partial_k^p \partial_l^l \Xi_i| \leq \varepsilon \quad (i = 1, 2).
\]

Because $\Phi$ is a diffeomorphism, $\Phi$ possess the intersection property on $A_{\mu_0}$. Hence $\Phi$ satisfies all the assumptions of Moser's twist theorem [4, 19, 25]. From this theorem, it follows that for any $\omega \gg 1$ satisfying
\[
\left| \omega - \frac{p}{q} \right| \geq C_0 \cdot |q|^{-2}, \quad (4.13)
\]
there is an invariant curve $I'$ of $\Phi$ and on which $\Phi$ is of the form
\[
\tau_1 = \tau_0 + \omega.
\]

One can conclude that there exist invariant curves of the Poincaré mapping of the system (2.3), which surrounding the origin $(x, y) = (0, 0)$ and arbitrarily far from the origin. So every solution of (1.8) is bounded.

5. FINAL REMARKS

Let us consider the more general equation
\[
\ddot{x} + g(x) = \varepsilon e(t), \quad (5.1)
\]
where \( e(t+1) = e(t) \). We also let \( G(x) = \int_{0}^{x} g(s) \, ds \) and \( W(x) = \frac{g(x)}{e_{0}} \). For the reader’s convenience, we still use same notations without ambiguousness. From the proof of Theorem 1, it is not difficult to conclude

**Theorem 2.** Assume that \( e \) is a small parameter, \( e(t) \in C^{\infty}(S^{1}) \), \( \int_{0}^{1} e(t) \, dt = 0 \) and \( g(x) \in C^{\infty}(\mathbb{R}) \) satisfies that for all \( x \neq 0 \),

1. \( x \cdot g(x) > 0 \) and there is a positive constant \( a \)
   \[
   \frac{1}{2} < 1 - a \leq W'(x) \leq 1;
   \]
2. \( |x^{k} \cdot G^{(k)}(x)| \leq C \cdot G(x) \) \( (\forall k \in \mathbb{N}) \);
3. \( c \cdot G(x) \leq G(-x) \leq C \cdot G(x) \).

Then every solution of (5.1) is bounded, that is, if \( x = x(t) \) is a solution of (5.1), then it is defined in \( (-\infty, +\infty) \) and

\[
\sup_{t \in \mathbb{R}} (|x(t)| + |\dot{x}(t)|) < +\infty.
\]

**Remark 1.** The assumptions in this theorem can be weakened to the requirement that they hold for \( x \geq d \) for any fixed constant \( d > 0 \).

**Remark 2.** Because of Remark 1, the condition that the average value of \( e(t) \) vanishes can be deleted. In fact, if \( \int_{0}^{1} e(t) \, dt \neq 0 \), we can use \( \bar{g}(x) = g(x) - \bar{e} \int_{0}^{1} e(t) \, dt \) and \( \bar{e}(t) = e(t) - \frac{1}{\bar{e}} \int_{0}^{1} e(t) \, dt \) instead of \( g(x) \) and \( e(t) \), respectively. It is easy to verify that \( \bar{g}(x) \) satisfies all the conditions of Theorem 2 for any \( |x| \geq d \) with some positive constant \( d \).

**Remark 3.** It is enough to assume that the functions \( g(x) \) and \( e(t) \) are finitely smooth. Indeed, we can assume that \( g(x) \in C^{4}(\mathbb{R}) \) and \( e(t) \in C^{5}(\mathbb{R}) \).

Applying the Aubry–Mather theory [15, 24], one can prove the following conclusions.

**Theorem 3.** Under the conditions of Theorem 2, there is \( \epsilon_{0} > 0 \) such that

1. for any rational \( \frac{p}{q} \in (0, \epsilon_{0}) \), Eq. (5.1) possesses an unlinked periodic solution (Birkhoff type) with period \( q \);
2. for any irrational \( \omega \in (0, \epsilon_{0}) \), Eq. (5.1) has generalized quasi-periodic solutions with frequency \( (1, \omega) \) corresponding to the Mather set \( M_{\omega} \);
3. for any irrational \( \omega \in (0, \epsilon_{0}) \) with \( \frac{1}{\omega} \) satisfying (4.13), there is a quasi-periodic solution of (5.1) with frequency \( (1, \omega) \).
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