On Group Inverses and the Sharp Order

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ABSTRACT

In this work the group inverse of a matrix is used to define the #-order on square matrices of index 1. The #-order is similar to the $*$-order of Drazin [2] and the minus order of Hartwig [6, 10] and Nambooripad [17]. The #-order and the $*$-order are compared and contrasted. Many conditions are given which assure the equivalence of the various partial orders studied.

1. INTRODUCTION AND PRELIMINARIES

Matrices are denoted by capital letters, column vectors by lowercase letters. For a matrix A, the symbols $\mathcal{M}(A)$, $\mathcal{N}(A)$, and A' denote the column span, null space, and transpose of A. $\mathcal{F}^n$ represents the vector space of n-tuples (column vectors), and $\mathcal{F}^{m\times n}$ the vector space of matrices of order $m\times n$, defined on a field $\mathcal{F}$. For a complex matrix A, A^* denotes its complex conjugate transpose. Two subspaces of a vector space are said to be virtually disjoint if they have only the null vector in common. $B = A\oplus(B - A)$ means Rank $B =$ Rank $A +$ Rank$(B - A)$ and is read as "A and $B - A$ are disjoint." A^- denotes a generalized inverse (g-inverse) of A, that is, a solution G of the matrix equation $AGA = A$. The reflexive g-inverse A_r^- of A is a solution G of the pair of equations $AGA = A$, $GAG = G$. A g-inverse of A which commutes with A is denoted by A_{com}^-. For a complex matrix A, a minimum norm g-inverse A_m^- is a matrix G that satisfies the pair of equations $AGA = A$, $(GA)^* = GA$. A least square g-inverse A_l^- is similarly defined through the equations $AGA = A$, $(AG)^* = AG$. The Moore-Penrose inverse A^+ is the unique solution G of the simultaneous matrix equations

$$AGA = A,\qquad GAG = G,\qquad (AG)^* = AG,\qquad (GA)^* = GA.$$

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$\{A^-\}$ represents the class of all g-inverses of A; $\{A_m^-\}$, $\{A_l^-\}$, etc. are similarly interpreted.

A_{mr}^-, A_{lr}^-, A_{lm}^- are defined as follows:

$$\{A_{mr}^-\} = \{A_m^-\} \cap \{A_r^-\},$$

$$\{A_{lr}^-\} = \{A_l^-\} \cap \{A_r^-\},$$

$$\{A_{lm}^-\} = \{A_l^-\} \cap \{A_m^-\}.$$

When a square matrix is of index 1, that is

$$\text{Rank } A = \text{Rank}(A^2), \tag{1}$$

it was shown by Englefield [4] that there exists an unique reflexive g-inverse of A which commutes with A. This was denoted by A_R. The same result was independently obtained by Erdelyi [5], who named this unique g-inverse the group inverse because the conditions imposed in the definition of his unique inverse are precisely those that ensure that the matrix A belongs to a multiplicative group. The group inverse of A is denoted by $A^{\#}$. Arghiriade [1] and Pearl [18] gave necessary and sufficient conditions for a matrix A to commute with its Moore-Penrose inverse A^+. When it does, A^+ will thus coincide with $A^{\#}$.

The author [14], who was interested in obtaining g-inverses G of A with specified row and column spans, discovered the same g-inverse. In [14] it is shown that $A^{\#}$ satisfies the conditions

$$\mathcal{M}(G) \subset \mathcal{M}(A) \tag{2}$$

$$\mathcal{M}(G') \subset \mathcal{M}(A'). \tag{3}$$

This g-inverse is denoted A_{RC}^-, R and C signifying row and column restrictions. One can adopt the equations

$$AGA = A, \qquad GAG = G, \qquad AG = GA \tag{4}$$

as the definition of the group inverse, as is done by Englefield [4] and Erdelyi [5], and deduce (2) and (3). Alternatively, one can use $AGA = A$ together with (2) and (3) as the definition and deduce (4), as is done in [14]. It was shown in [14] that a g-inverse satisfying any one of the two restrictions (2) and (3) exists if and only if the matrix A is of index 1. These inverses (denoted by A_C^- and A_R^- respectively), though not unique, share several properties with A_{RC}^- and are computationally somewhat simpler. In [14], further, explicit algebraic expressions for A_R^-, A_C^- are given; in particular it is shown that

$$A_{RC}^- = A(A^3)^- A. \tag{5}$$

Robert [21] proposes the following equivalent definition for the group inverse. Let X be a linear space, and A a linear transformation of X into itself. Let $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote respectively the range and nullspace of A. When $X = \mathscr{R}(A) \oplus \mathscr{N}(A)$, $A^{\#}$ is the linear transformation mapping X onto $\mathscr{R}(A)$ such that $AA^{\#}$ is the projector on $\mathscr{R}(A)$ along $\mathscr{N}(A)$. Thus some of the results in [14] and [21] have been developed in parallel.

In a star semigroup with a proper involution (denoted by $*$), Drazin [2] introduced the concept of a star order which in the context of complex matrices of order $m \times n$ could be stated as follows: we define $A \overset{*}{<} B$ to mean

$$BA^* = AA^*, \qquad A^*B = A^*A. \tag{6}$$

It was shown that (6) is equivalent to the following definition:

$$A \overset{*}{<} B \quad \text{if} \quad BA^+ = AA^+, \qquad A^+B = A^+A. \tag{7}$$

Hartwig [6] and independently Nambooripad [17] introduced another partial order by weakening the requirement given in (6) or (7). This partial order, first called the plus order in [6] and later renamed the minus order in [10], is defined as follows: We write $A \overset{-}{<} B$ whenever

$$BA^- = AA^-, \qquad A^-B = A^-A \tag{8}$$

for some generalized inverse A^- of A. Clearly

$$A \overset{*}{<} B \quad \Rightarrow \quad A \overset{-}{<} B. \tag{9}$$

Hartwig and Styan [10] state ten conditions, each one of which, together with $A \overset{-}{<} B$, implies $A \overset{*}{<} B$. Additional conditions will be presented in this paper. Another goal of this paper is to introduce the sharp order through the unique group inverse.

We write $A \overset{\#}{<} B$ if A and B are square matrices of index 1 and

$$BA^{\#} = AA^{\#}, \qquad A^{\#}B = A^{\#}A. \tag{10}$$

It is shown in Hartwig and Luh [9, p. 12] that in a strongly regular ring (10) and (8) are equivalent. We note that the minus order and the sharp order could be defined for matrices on any field, unlike the star order, which as given in (6) or (7) requires the field to be real or complex. This distinction must be kept in mind when stating and proving results about the partial orders. Normally the choice of field is clear from the context.

DEFINITION [19]. A pair of matrices A and B of the same order is said to be parallel summable (p.s.) if $A(A+B)^{-}B$ is invariant under the choice of the g-inverse $(A+B)^{-}$. When A and B are p.s., $A(A+B)^{-}B$ is called the parallel sum of A and B and denoted by the symbol $P(A, B)$.

2. PROPERTIES OF THE STAR AND SHARP ORDERS

The equivalence of (a) and (b) in Lemma 2.1 below is due to Hartwig [6]. The remaining equivalences are trivial; see the proof of Lemma 1.2 in Mitra and Odell [16] in this regard.

LEMMA 2.1. *The following statements are equivalent:*

$$A \overset{-}{<} B, \tag{11a}$$

$$B = A \oplus (B - A), \tag{11b}$$

$$P(A, B - A) = 0. \tag{11c}$$

LEMMA 2.2. *When the matrix A is of index* 1, *the condition* (10) *is equivalent to the following condition*:

$$BA = A^2 = AB. \tag{12}$$

Proof. Since $A^{\#}A^2 = AA^{\#}A = A^2A^{\#} = A$, clearly (10) $\Rightarrow$ (12). That (12) $\Rightarrow$ (10) is a simple consequence of (5). ■

Thus the condition (12) could replace (10) in the definition of the sharp order. It is shown by Drazin [3] that in a finite semigroup a necessary and sufficient condition for $a^2 = ab = ba$ to define a partial order is that the semigroup is quasiseparative.

LEMMA 2.3.

(a) $A \eqslantless B$ *if and only if* $(B - A) \eqslantless B$.

(b) $A \overset{*}{<} B$ *if and only if* $(B - A) \overset{*}{<} B$.

(c) $A \overset{\#}{<} B$ *if and only if* $(B - A) \overset{\#}{<} B$.

Proof. (a), (b), and (c) follow respectively from Lemma 2.1, (6), and (12).

Note that (12) $\Rightarrow$ $B^2 = A^2 + (B - A)^2$ and $A \overset{\#}{<} B \Rightarrow A \eqslantless B$. Hence, in view of (11), if A and B are of index 1, so is $B - A$. ■

THEOREM 2.1.

(a) $A \eqslantless B$ *if and only if*

$$\{B^-\} \subset \{A^-\}. \tag{13}$$

(b) $A \overset{*}{<} B$ *if and only if*

$$\{B_m^-\} \subset \{A_m^-\}, \qquad \{B_l^-\} \subset \{A_l^-\}. \tag{14}$$

(c) $A \overset{\#}{<} B$ *if and only if*

$$\{B_{\text{com}}^-\} \subset \{A_{\text{com}}\}. \tag{15}$$

Proof. (a) and (b) are proved in [15].

For the "only if" part of (c), clearly $A \overset{\#}{<} B \Rightarrow A \eqslantless B \Rightarrow \{B^-\} \subset \{A^-\} \Rightarrow B = A \oplus (B - A)$. Let B_{com}^- be an arbitrary commuting g-inverse of B; then $BB_{\text{com}}^- = BB^\#$ is the unique projector projecting onto $\mathscr{M}(B)$ along $\mathscr{N}(B)$. Since $AA^\#$ is the projector onto $\mathscr{M}(A)$ along $\mathscr{N}(A)$ and $(B - A)(B - A)^\#$ is the projector onto $\mathscr{M}(B - A)$ along $\mathscr{N}(B - A)$, (12) implies that $AA^\# + (B - A)(B - A)^\#$ is the projector onto $\mathscr{M}(B)$ along $\mathscr{N}(A) \cap \mathscr{N}(B - A) = \mathscr{N}(B)$. Hence $BB_{\text{com}}^- = AB_{\text{com}}^- + (B - A)B_{\text{com}}^- = AA^\# + (B - A)(B - A)^\# = A^\# A + (B - A)^\#(B - A) = B_{\text{com}}^- A + B_{\text{com}}^-(B - A)$. Since $\mathscr{M}(A)$ and $\mathscr{M}(B - A)$ are virtually disjoint, and so are $\mathscr{M}(A')$ and $\mathscr{M}[(B - A)']$, this implies

$$AB_{\text{com}}^- = AA^\# = A^\# A = B_{\text{com}}^- A.$$

Thus B^-_{com}, which belongs to $\{B^-\}$ and hence to $\{A^-\}$, is indeed a commuting g-inverse of A, and hence $\{B^-_{\text{com}}\} \subset \{A^-_{\text{com}}\}$.

For the "if" part, assume now that (15) holds. Then $B^{\#} \in \{A^-_{\text{com}}\}$. We now show that (15) $\Rightarrow$ (13). To avoid triviality assume that B is singular. Observe that $\{B^-_{\text{com}}\} = \{B^{\#} + (I - B^{\#}B)U(I - BB^{\#}),\ U \text{ arbitrary}\}$. Then $\{B^-_{\text{com}}\} \subset \{A^-\} \Rightarrow A(I - B^{\#}B)U(I - BB^{\#})A = 0$, which in turn implies that $A(I - B^{\#}B)$ or $(I - BB^{\#})A$ is a null matrix. Without loss of generality let $(I - BB^{\#})A$ be null and $A(I - B^{\#}B)$ be nonnull. Choose and fix U such that

$$A(I - B^{\#}B)U(I - BB^{\#}) \neq 0.$$

For this choice of U, $B^{\#} + (I - B^{\#}B)U(I - BB^{\#}) \in \{B^-_{\text{com}}\} \subset \{A^-\}$ but $\notin \{A^-_{\text{com}}\}$, since $[B^{\#} + (I - B^{\#}B)U(I - BB^{\#})]A = B^{\#}A = AB^{\#} \neq A[B^{\#} + (I - B^{\#}B)U(I - BB^{\#})]$, which contradicts our assumption that

$$\{B^-_{\text{com}}\} \subset \{A^-_{\text{com}}\}.$$

Hence $A(I - B^{\#}B)$ is also null, which implies

$$\{B^-\} = \{B^{\#} + (I - B^{\#}B)U + V(I - BB^{\#}),\ U, V \text{ arbitrary}\} \subset \{A^-\};$$

therefore

$$P(A, B - A) = 0.$$

Here we have used Theorem 2.1(a) and Lemma 2.1 respectively. Then

$$\begin{aligned} P(A, B - A) &= AB^-(B - A) = AB^-_{\text{com}}(B - A) \\ &= B^-_{\text{com}}A(B - A) = 0 \\ \Rightarrow \quad AB^-_{\text{com}}A(B - A) &= A(B - A) = 0. \end{aligned}$$

Similarly

$$\begin{aligned} & P(B - A, A) = P(A, B - A) = 0 \\ \Rightarrow \quad & (B - A)B^-_{\text{com}}A = (B - A)AB^-_{\text{com}} = 0 \\ \Rightarrow \quad & (B - A)AB^-_{\text{com}}A = (B - A)A = 0 \quad \Rightarrow \quad A \overset{\#}{<} B \end{aligned}$$

This completes the proof. ■

In our next theorem we give conditions which specify when the minus and the star order coincide.

THEOREM 2.2. *The following statements are equivalent:*

(a) $A \overset{*}{<} B$;

(b) $A \overset{-}{<} B$, *and for some* $G_a \in \{A_m^-\}$

$$G_a + (B-A)^+ \in \{B^-\}; \tag{16}$$

(c) $A \overset{-}{<} B$, *and for some* $G_a \in \{A_l^-\}$

$$G_a + (B-A)^+ \in \{B^-\}. \tag{17}$$

Proof. We shall first establish the equivalence of (a) and (b). That (a) ⇒ (b) follows straightforwardly from (9), (6), (11), and the explicit expression for the Moore-Penrose inverse

$$A^+ = A^*(A^*AA^*)^- A^*$$

given in [13, p. 111]. Note that if (a) holds,

$$B[A^+ + (B-A)^+] = AA^+ + (B-A)(B-A)^+,$$

$$[A^+ + (B-A)^+]B = A^+A + (B-A)^+(B-A)$$

are both hermitian, and further

$$B[A^+ + (B-A)^+]B = AA^+A + (B-A)(B-A)^+(B-A) = B.$$

Then

$$\text{Rank}\, B \leqslant \text{Rank}[A^+ + (B-A)^+] \leqslant \text{Rank}\, A^+ + \text{Rank}(B-A)^+$$

$$= \text{Rank}\, A + \text{Rank}(B-A) = \text{Rank}\, B,$$

which in turn implies

$$A^+ + (B-A)^+ = B^+.$$

Another proof of the proposition that $A \overset{*}{<} B$ implies $A^{+} + (B-A)^{+} = B^{+}$ appears in Hartwig and Styan [10].

(b) $\Rightarrow$ (a): $A \overset{-}{<} B$ if and only if $\{B^{-}\} \subset \{A^{-}\}$ [Theorem 2.1(a)]. Hence

$$G_a + (B-A)^{+} \in \{B^{-}\} \subset \{A^{-}\}.$$

Since $G_a \in \{A^{-}\}$, this implies

$$A(B-A)^{+}A = 0.$$

In Lemma 2.1 we have noted that $A \overset{-}{<} B \Leftrightarrow P(B-A, A) = P(A, B-A) = 0$. But then

$$\begin{aligned}
P(B-A,A) = (B-A)B^{-}A = (B-A)&\left[G_a + (B-A)^{+}\right]A = 0 \\
\Leftrightarrow \quad & \left[(B-A)^{+}\right]^{*}\left[G_a + (B-A)^{+}\right]A = 0 \\
\Leftrightarrow \quad & \left[(B-A)^{+}\right]^{*}A^{*}G_a^{*} + \left[(B-A)^{+}\right]^{*}(B-A)^{+}A = 0. \\
\Rightarrow \quad & A^{*}\left[(B-A)^{+}\right]^{*}(B-A)^{+}A = 0 \\
\Leftrightarrow \quad & (B-A)^{+}A = 0 \\
\Leftrightarrow \quad & (B-A)^{*}A = (B-A)^{*}(B-A)(B-A)^{+}A = 0 \\
\Rightarrow \quad & (B-A)G_aA = 0 \\
\Leftrightarrow \quad & (B-A)A^{*} = (B-A)G_aAA^{*} = 0,
\end{aligned}$$

whence $(B-A)^{*}A = 0$, $(B-A)A^{*} = 0 \Rightarrow$ (a).

(a) $\Rightarrow$ (c): We have noted earlier in this proof that (a) implies $A \overset{-}{<} B$ and

$$A^{+} + (B-A)^{+} = B^{+},$$

which in turn implies (c)

(c) $\Rightarrow$ (a): $\{(B^{-})^{*}\} = \{(B^{*})^{-}\}$, $[(B-A)^{+}]^{*} = [(B-A)^{*}]^{+}$, and $\{(A_l^{-})^{*}\} = \{(A^{*})_m^{-}\}$; see [19, Theorem 3.2.4]. Further, $A \overset{-}{<} B \Leftrightarrow A^{*} \overset{-}{<} B^{*}$. Hence, in view of the equivalence of (a) and (b),

$$(c) \quad \Rightarrow \quad A^{*} \overset{*}{<} B^{*} \quad \Rightarrow \quad (a). \qquad \blacksquare$$

REMARK 1. Theorem 2.2 is not true if the Moore-Penrose inverse $(B-A)^{+}$ in conditions (b) and (c) is replaced either by $(B-A)_{mr}^{-}$ or by $(B-A)_{lr}^{-}$ or even by $(B-A)_{lm}^{-}$. Consider for example

$$B=\begin{pmatrix}1&0\\1&1\end{pmatrix},\qquad A=\begin{pmatrix}1&0\\1&0\end{pmatrix},\qquad B-A=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Clearly $A \overset{-}{<} B$. Observe that

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\in\{A_{mr}^{-}\},\qquad \begin{pmatrix}0&0\\-1&1\end{pmatrix}\in\{(B-A)_{mr}^{-}\}$$

and

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\-1&1\end{pmatrix}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}=B^{-1}.$$

However $(B-A)^{*}A\neq 0$. Hence $A \overset{*}{\nless} B$.

Similarly consider

$$B=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad A=\begin{pmatrix}1&0\\1&0\end{pmatrix},\qquad B-A=\begin{pmatrix}0&0\\-1&1\end{pmatrix}.$$

Clearly $A \overset{-}{<} B$,

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\in\{A_{mr}^{-}\},\qquad \begin{pmatrix}0&0\\0&1\end{pmatrix}\in\{(B-A)_{lr}^{-}\},$$

and

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=B^{-1}.$$

Further, $(B-A)A^{*}\neq 0 \;\Rightarrow\; A \overset{*}{\nless} B$. Finally,

$$\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}\in\{A_{lm}^{-}\},\qquad \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}\in\{(B-A)_{lm}^{-}\},$$

and the two matrices sum to

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}=B^{-1}.$$

REMARK 2. Theorem 2.2 exhibits the equivalence of (v), (vi), and (vii) with (i) in Theorem 2 of Hartwig and Styan [10]. Along with Remark 1, it shows perfectly the extent to which the equivalence depends on the Moore-Penrose inverses, an open problem raised in [10]. The proof of Theorem 2.2 is an algebraic proof of the equivalence of (vi) and (i), thus answering another question which is raised in [10].

REMARK 3. A result similar to Theorem 2.2 is not available for the sharp order. To see this, let S and T' be nonnull matrices of the same order with at least two rows and columns, such that

$$ST = 0, \qquad TS = 0.$$

The matrices

$$A = \begin{pmatrix} I & S \\ T & 0 \end{pmatrix}, \qquad B - A = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix} \tag{18}$$

add up to

$$B = \begin{pmatrix} I & S \\ T & -I \end{pmatrix},$$

which is an involution. A and $A - B$ are idempotent; hence $A^{\#} = A$, $(B - A)^{\#} = (B - A)$, and $A^{\#} + (B - A)^{\#} = B^{\#}$. Clearly $A \overset{-}{<} B$. However,

$$A(B - A) = \begin{pmatrix} 0 & -S \\ 0 & 0 \end{pmatrix} \neq 0 \quad \text{and} \quad A \overset{\#}{\not<} B.$$

THEOREM 2.3.

(a) *If* $A \overset{*}{<} B$ *then* $A \overset{-}{<} B$ *and* $BA^{+}B = A$. *Conversely,* $A \overset{-}{<} B$ *and* $\mathcal{M}(BA^{+}B) \subset \mathcal{M}(A)$, $M[(BA^{+}B)'] \subset \mathcal{M}(A')$ *imply*

$$A \overset{*}{<} B.$$

(b) *If* $A \overset{\#}{<} B$, *then* $A \overset{-}{<} B$ *and* $BA^{\#}B = A$. *Conversely,* $A \overset{-}{<} B$, B *of index* 1, *and* $\mathcal{M}(BA^{\#}B) \subset \mathcal{M}(A)$, $\mathcal{M}[(BA^{\#}B)'] = \mathcal{M}(A')$, *imply*

$$A \overset{\#}{<} B.$$

Proof. Part (a) was proved in Hartwig and Styan [10, Theorem 2]. For part (b), first, $A \overset{\#}{<} B \Rightarrow A \overset{-}{<} B$ follows from the definitions of the minus and the sharp orders. Next, $BA^{\#}B = A$ is a consequence of (10). Now assume $A \overset{-}{<} B$ and $\mathcal{M}(BA^{\#}B) \subset \mathcal{M}(A)$ hold. By Theorem 1(a), since $A \overset{-}{<} B$, we have $\{B^{-}\} \subset \{A^{-}\}$, so $V(I - BB^{-})A = 0$, $A(I - B^{-}B)U = 0$ for arbitrary U and V and arbitrary choice of B^{-}, whence $A = BB^{-}A = AB^{-}B$. Then

$$A \overset{-}{<} B \quad \Rightarrow \quad AB^{\#}A = A, \quad A = BB^{\#}A = AB^{\#}B \tag{19}$$

and

$$\begin{aligned} \mathcal{M}(BA^{\#}B) \subset \mathcal{M}(A) \quad &\Rightarrow \quad AA^{\#}BA^{\#}B = BA^{\#}B \\ &\Rightarrow \quad AB^{\#}(AA^{\#}BA^{\#}B) = AB^{\#}(BA^{\#}B) \\ &\Rightarrow \quad AA^{\#}BA^{\#}B = AA^{\#}B = BA^{\#}B \\ &\Rightarrow \quad AA^{\#}B(B^{\#}A) = BA^{\#}B(B^{\#}A) \\ &\Rightarrow \quad A = AA^{\#}A = BA^{\#}A \quad \Rightarrow \quad AA^{\#} = BA^{\#}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathcal{M}\left[(BA^{\#}B)'\right] \subset \mathcal{M}(A'), \quad A \overset{-}{<} B \\ &\Rightarrow \quad \mathcal{M}\left[B'(A')^{\#}B'\right] \subset \mathcal{M}(A'), \quad A' \overset{-}{<} B' \\ &\Rightarrow \quad A'(A')^{\#} = B'(A')^{\#} \Rightarrow A'(A^{\#})' = B'(A^{\#})' \\ &\Rightarrow \quad A^{\#}A = A^{\#}B. \end{aligned}$$

■

THEOREM 2.4.

(a) *If* $A \overset{*}{<} B$, *then* $A \overset{-}{<} B$ *and* $B^{+}AB^{+} = A^{+}$. *Conversely, if* $A \overset{-}{<} B$ *and* $\mathcal{M}(B^{+}AB^{+}) \subset \mathcal{M}(A^{+})$, $\mathcal{M}[(B^{+}AB^{+})'] \subset \mathcal{M}[(A')]$, *then*

$$A \overset{*}{<} B.$$

(b) *If* $A \overset{\#}{<} B$, *then* $A \lesssim B$ *and* $B^{\#}AB^{\#} = A^{\#}$. *Conversely, if* $A \lesssim B$ *and* $\mathcal{M}(B^{\#}AB^{\#}) \subset \mathcal{M}(A^{\#})$, $\mathcal{M}[(B^{\#}AB^{\#})'] \subset \mathcal{M}[(A^{\#})']$, *then*

$$A \overset{\#}{<} B.$$

Proof. The first part of (a) was proved in [10]. We next prove both parts of (b), and this will suggest a proof of the converse part of (a).

From (2), (3), and (12) it is seen that

$$\begin{aligned} A \overset{\#}{<} B \quad &\Rightarrow \quad (B-A)A^{\#} = A(B-A)^{\#} = A^{\#}(B-A) = (B-A)^{\#}A = 0 \\ &\Rightarrow \quad B\left[A^{\#} + (B-A)^{\#}\right] = AA^{\#} + (B-A)(B-A)^{\#} \\ &\Rightarrow \quad A^{\#} + (B-A)^{\#} \in \{B^{-}\}. \end{aligned} \tag{20}$$

Further, $\mathcal{M}[A^{\#} + (B-A)^{\#}] \subset \mathcal{M}(B)$, $\mathcal{M}[\{A^{\#} + (B-A)^{\#}\}'] \subset \mathcal{M}(B')$. Hence

$$A^{\#} + (B-A)^{\#} = B^{\#} \tag{21}$$

and $A \overset{\#}{<} B$ is seen to imply

$$A^{\#} \overset{\#}{<} B^{\#}. \tag{22}$$

Thus $B^{\#}AB^{\#} = A^{\#}AA^{\#} = A^{\#}$. We have already seen

$$A \overset{\#}{<} B \quad \Rightarrow \quad A \lesssim B.$$

Conversely,

$$\begin{aligned} \mathcal{M}(B^{\#}AB^{\#}) \subset \mathcal{M}(A^{\#}) \quad &\Rightarrow \quad A^{\#}AB^{\#}AB^{\#} = B^{\#}AB^{\#} \\ &\Rightarrow \quad A^{\#}AB^{\#}AB^{\#}A = B^{\#}AB^{\#}A \end{aligned}$$

which together with $A \lesssim B$ implies

$$A^{\#}A = B^{\#}A, \qquad \text{using (13)}.$$

Similarly,

$$\mathscr{M}\left[(B^{\#}AB^{\#})'\right] \subset \mathscr{M}\left[(A^{\#})'\right] \quad \Rightarrow \quad (A')^{\#}A' = (B')^{\#}A'$$

$$\Rightarrow \quad AA^{\#} = AB^{\#}.$$

Therefore $A^{\#} \overset{\#}{<} B^{\#}$, which implies $A \overset{\#}{<} B$, as required. ■

THEOREM 2.5

(a) *The following conditions are equivalent*:

(i) $A \overset{*}{<} B$;

(ii) $A \lesssim B$, *and* AB^* *and* B^*A *are hermitian*;

(iii) $A \lesssim B$, *and* AB^+ *and* B^+A *are hermitian*;

(iv) $A \lesssim B$, *and* A^+B *and* BA^+ *are hermitian.*

(b) *The following conditions are equivalent*:

(i) $A \overset{\#}{<} B$;

(ii) $A \lesssim B$, *A commutes with B, and B is of index* 1;

(iii) $A \lesssim B$, *and A commutes with* $B^{\#}$;

(iv) $A \lesssim B$, $A^{\#}$ *commutes with B, and B is of index* 1.

Proof. Part (a) was proved in [10, Theorem 2]. Since $A^{\#}$ and $B^{\#}$ are polynomials in A and B respectively [14, Theorem 5.3], conditions (ii), (iii), and (iv) of (b) are seen to be equivalent. It suffices to establish the equivalence of (i) and (iii). Clearly $A \overset{\#}{<} B \Rightarrow A \lesssim B$ and $A(B-A)=(B-A)A=0 \Rightarrow AB=BA \Rightarrow AB^{\#}=B^{\#}A$. Conversely $A \lesssim B$, $AB^{\#}=B^{\#}A \Rightarrow$

$$P(A, B-A) = AB^{-}(B-A) = AB^{\#}(B-A) = A(B-A)B^{\#} = 0,$$

since $BB^{\#} = B^{\#}B$, which in turn implies

$$A(B-A) = A(B-A)B^{\#}(B-A) = 0.$$

Similarly $P(B-A, A) = 0 \Rightarrow (B-A)A = 0$. Hence $A \overset{\#}{<} B$. We note that (12) $\Rightarrow B^2 = A^2 + (B-A)^2$. Hence when B is of index 1, so is A. ■

THEOREM 2.6. *The following conditions are equivalent:*

(i) $A \overset{*}{<} B$,

(ii) $A \lessapprox B,\ AA^* \overset{*}{<} BB^*$,

(iii) $A \lessapprox B,\ A^*A \overset{*}{<} B^*B$,

(iv) $A \lessapprox B,\ (AA^*)^n \overset{*}{<} (BB^*)^n$,

(v) $A \lessapprox B,\ (A^*A)^n \overset{*}{<} (B^*B)^n$,

(vi) $A \lessapprox B,\ (AA^*)^n A \overset{*}{<} (BB^*)^n A$,

(vii) $A \lessapprox B,\ (A^*A)^n A^* \overset{*}{<} (B^*B)^n B^*$,

where n is a positive integer.

Proof. Trivially condition (i) implies all the rest. We first establish the equivalence of (i) and (ii), for which it suffices to show that (ii) $\Rightarrow$ (i). Now $AA^* \overset{*}{<} BB^* \ \Rightarrow\ (BB^* - AA^*)AA^* = 0 \ \Rightarrow\ BB^*AA^* = AA^*BB^*$, which in turn implies that for some unitary U we have $U^*BB^*U = D_b^2$, $U^*AA^*U = D_a^2$, where D_a and D_b are diagonal matrices with nonnegative diagonal elements. Also, $AA^* \overset{*}{<} BB^* \ \Rightarrow\ D_a^2 \overset{*}{<} D_b^2$, which implies that for suitable permutation of the columns of U, if necessary,

$$D_a^2 = \begin{pmatrix} D_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_b^2 = \begin{pmatrix} D_1^2 & 0 & 0 \\ 0 & D_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

where D_1 is diagonal positive definite of order $r_1 \times r_1$, D_2 is diagonal positive definite of order $r_2 \times r_2$, $r_1 = \text{Rank } A$, and $r_1 + r_2 = \text{Rank } B$. Let the columns of U be partitioned as

$$U = (U_1 \ \vdots\ U_2 \ \vdots\ U_3)$$

corresponding to the partitioning used in (23). Hence

$$A = U_1 D_1 L_1^*, \qquad B = (U_1 \ \vdots\ U_2)\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix},$$

where

$$U_1^*U_1 = L_1^*L_1 = I_{r_1},$$

$$(U_1 \,\vdots\, U_2)^*(U_1 \,\vdots\, U_2) = (K_1 \,\vdots\, K_2)^*(K_1 \,\vdots\, K_2) = I_{r_1+r_2}.$$

Then

$$\begin{aligned}
A \lessapprox B \quad &\Rightarrow \quad \mathscr{M}(A^*) = \mathscr{M}(L_1) \subset \mathscr{M}(B^*) = \mathscr{M}(K_1 \,\vdots\, K_2)\\
&\Rightarrow \quad L_1 = K_1(I - \Lambda_1^*) - K_2\Lambda_2^* \qquad \text{for some matrices } \Lambda_1 \text{ and } \Lambda_2\\
&\Rightarrow \quad B - A = (U_1 \,\vdots\, U_2)\begin{pmatrix} D_1\Lambda_1 & D_1\Lambda_2 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix}\\
&\Rightarrow \quad \operatorname{Rank}(B - A) > r_2 \qquad [\text{unless } \Lambda_1 = 0] \text{ which contradicts (11)}\\
&\Rightarrow \quad \Lambda_1 = 0 \quad \Rightarrow \quad L_1 = K_1 - K_2\Lambda_2^*\\
&\Rightarrow \quad L_1^*L_1 = K_1^*K_1 + (L_1 - K_1)^*(L_1 - K_1)\\
&\Rightarrow \quad (L_1 - K_1) = 0 \qquad (\text{since} \quad L_1^*L_1 = K_1^*K_1 = I_{r_1})\\
&\Rightarrow \quad L_1 - K_1 = 0 \quad \Rightarrow \quad A = U_1D_1K_1^*, \quad B = U_1D_1K_1^* + U_2D_2K_2^*\\
&\Rightarrow \quad A \overset{*}{<} B.
\end{aligned}$$

Thus (i) and (ii) are equivalent.

We next show that (iv) $\Rightarrow$ (i). The hypothesis $(AA^*)^n \overset{*}{<} (BB^*)^n$ implies that $(AA^*)^n$ and $(BB^*)^n$ commute. Hence for some unitary U we have $U^*(AA^*)^nU = D_a^{2n}$, $U^*(BB^*)^nU = D_b^{2n}$, where D_a and D_b are diagonal matrices with nonnegative diagonal elements. Further, $(AA^*)^n \overset{*}{<} (BB^*)^n \Rightarrow D_a^{2n} \overset{*}{<} D_b^{2n} \Rightarrow D_a^2 \overset{*}{<} D_b^2 \Rightarrow AA^* = UD_a^2U^* \overset{*}{<} UD_b^2U^* = BB^*$. Hence (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Also, $(AA^*)^nA \overset{*}{<} (BB^*)^nB \Rightarrow (AA^*)^{2n+1} \overset{*}{<} (BB^*)^{2n+1}$. This shows (vi) $\Rightarrow$ (iv) $\Rightarrow$ (i). Next observe that (iii) $\Rightarrow$ $A^* \lessapprox B^*$, $A^*A \overset{*}{<} B^*B \Rightarrow A^* \overset{*}{<} B^*$ [using the equivalence of (i) and (ii)], which in turn implies (i). Similarly (v) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and all the equivalences in Theorem 2.6 are established. ■

REMARK 4. A result similar to Theorem 2.6 is not available for the sharp order. Consider matrices A and B as in Remark 3. Here $A^2 = A$ and $B^2 = I$. Hence $A^2(B^2 - A^2) = (B^2 - A^2)A^2 = 0 \Rightarrow A^2 \overset{\#}{<} B^2$. We have already noted in Remark 3 that $A \overset{\#}{<} B$ and $A \overset{\#}{\not<} B$. Nevertheless it is easy to check that $A \overset{\#}{<} B \Rightarrow A^n \overset{\#}{<} B^n$ for any positive integer n.

REMARK 5. Let A and B be complex matrices of order $m \times n$. Trivially $A \overset{*}{<} B \Rightarrow A^*A \overset{*}{<} B^*B$, $AA^* \overset{*}{<} BB^*$. Conversely $A^*A \overset{*}{<} B^*B$, $AA^* \overset{*}{<} BB^*$ $\Rightarrow AB^*(AB^*)^* = AA^*BB^*$, $A^*B(A^*B)^* = A^*AB^*B$. Nevertheless $A \overset{*}{<} B$ does not follow from the given premises, as the following counterexample shows. Let r and s be positive integers such that $r \geqslant 2$, $s \geqslant 1$, $r + s \leqslant \min(m, n)$, and L be a unitary matrix of order r, different from I_r. Also, let P and Q be unitary matrices of order m and n respectively. Define A and B as follows:

$$A = P\begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}Q, \qquad B = P\begin{pmatrix} I_r & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{pmatrix}Q.$$

Observe that $AA^* \overset{*}{<} BB^*$, $A^*A \overset{*}{<} B^*B$, but

$$A \overset{*}{\not<} B,$$

since $L \neq I$. It is shown by Hartwig and Styan [11, Theorem 4.2] that when A and B are idempotent,

$$A \overset{*}{<} B \quad \Leftrightarrow \quad A^*A \overset{*}{<} B^*B, \quad AA^* \overset{*}{<} BB^*.$$

The following theorem extends this result.

THEOREM 2.7. *$A \overset{*}{<} B$ is equivalent to $(AA^*)^nA \overset{*}{<} (BB^*)^nB$ for any positive integer n.*

Proof. The $\Rightarrow$ part is trivial. On the other hand $(AA^*)^nA \overset{*}{<} (BB^*)^nB$ $\Rightarrow (AA^*)^{2n+1} \overset{*}{<} (BB^*)^{2n+1}$ which in turn implies that $(AA^*)^{2n+1}$ and

$(BB^*)^{2n+1}$ commute. Hence for some unitary matrix P,

$$(AA^*)^{2n+1} = PD_\alpha P^*, \qquad (BB^*)^{2n+1} = PD_\beta P^*,$$

where D_α and D_β are diagonal matrices with nonnegative entries in the diagonal positions. Further, $(AA^*)^{2n+1} \overset{*}{<} (BB^*)^{2n+1} \Leftrightarrow D_\alpha \overset{*}{<} D_\beta$. Hence, redefining P if necessary, one can write

$$D_\alpha = \begin{pmatrix} D_1^{4n+2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_\beta = \begin{pmatrix} D_1^{4n+2} & 0 & 0 \\ 0 & D_2^{4n+2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where D_1 and D_2 are diagonal matrices with strictly positive diagonal entries. One can therefore write, with K and L unitary matrices,

$$A = PD_aK, \qquad B = PD_bL,$$

where

$$D_a = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_b = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now

$$(AA^*)^n A \overset{*}{<} (BB^*)^n B$$

$$\Rightarrow \quad PD_2^{2n+1}K \overset{*}{<} PD_b^{2n+1}L$$

$$\Rightarrow \quad \left(PD_b^{2n+1}L - PD_a^{2n+1}K\right)K^*D_a^{2n+1}P^* = 0$$

$$\Rightarrow \quad \left(D_b^+\right)^{2n}P^*\left(PD_b^{2n+1}L - PD_a^{2n+1}K\right)K^*D_a^{2n+1}P^*P\left(D_b^+\right)^{2n} = 0$$

$$\Rightarrow \quad (D_bL - D_aK)K^*D_a = 0 \quad \Rightarrow \quad (B-A)A^* = 0.$$

Similarly $(AA^*)^nA \overset{*}{<} (BB^*)^nB \Rightarrow (A^*A)^nA^* \overset{*}{<} (B^*]^nB^* \Rightarrow (B^* - A^*)A = 0$. Hence

$$A \overset{*}{<} B. \qquad \blacksquare$$

It was shown by Hartwig and Drazin [8] that complex matrices of order $m \times n$ constitute a lower semilattice under the $*$-order. The same cannot be said about the #-order. Let A and B be square matrices of order $n \times n$, both of index 1. Let $\underline{\mathscr{C}}$ be defined

$$\underline{\mathscr{C}} = \left\{ C : C \in \mathscr{F}^{n \times n},\ C \overset{\#}{<} A,\ C \overset{\#}{<} B \right\}.$$

If $\underline{\mathscr{C}}$ has a unique maximal element under the #-order, this element is called the sharp infimum of A and B and denoted by $A \overset{\#}{\wedge} B$. Given below are a pair of matrices A, B in $\mathscr{C}^{4 \times 4}$ for which $A \overset{\#}{\wedge} B$ does not exist. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 2 & 1 & -2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $A = B + K$, where

$$K = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & -2 \end{pmatrix}.$$

If the matrix C is dominated by the matrix B, then C is an idempotent matrix H of order 3×3 bordered by a null row and a null column as in B. Since $(B - C)C = C(B - C) = 0$, it is seen that $(A - C)C = C(A - C) = 0$ $\Rightarrow$ $KC = CK = 0$ $\Rightarrow$

$$\mathscr{M}(H) \subset \mathscr{M}(L), \qquad \mathscr{M}(H') \subset \mathscr{M}(R')$$

where

$$L = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 0 \end{pmatrix},$$

that is, $H = LZR$ for some matrix Z. But since $H = H^2 = LZRLZR$, we have

$$\text{Rank } H \leqslant \text{Rank } RL = \text{Rank} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 1$$

and

$$Z = Z(RL)Z.$$

To obtain a matrix C of maximum possible rank in $\underline{\mathscr{C}}$ one must therefore choose

$$Z \in \{(RL)_r^-\}.$$

A general solution to Z is given by

$$Z = \begin{pmatrix} 2ab & b \\ a & \frac{1}{2} \end{pmatrix},$$

where a and b are arbitrary complex numbers. This shows that a matrix C in $\underline{\mathscr{C}}$ of maximum possible rank, namely

$$C = \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & LZR \end{pmatrix}$$

is not unique, and $A \overset{\#}{\wedge} B$ does not exist. Note that here

$$B \overset{-}{<} A.$$

Hence $B = A \wedge B = 2P(A, B)$, where $A \wedge B$ is defined as in $A \overset{\#}{\wedge} B$ with minus order replacing the sharp order in the definition of $\underline{\mathscr{C}}$. The lattice properties of the $*$-order are also studied in Holladay [12], and those of the minus order in Mitra [15].

For a pair of complex matrices A and B of the same order, it was shown by Rao et al. [20] that

$$A^+ \in \{B^-\}, \quad B^+ \in \{A^-\} \quad \Rightarrow \quad A = B. \tag{24}$$

The same result is studied in an abstract algebraic setting in Hartwig [7]. The following example shows that a similar result is not true for the group inverse. Let S and T' be nonnull matrices of the same order with at least two columns so that $ST = 0$. Consider

$$A = \begin{pmatrix} I & S \\ T & TS \end{pmatrix}, \qquad B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly $A^2 = A = A^{\#}$, $B^2 = B = B^{\#}$, and

$$BAB = B \quad \Rightarrow \quad A = A^{\#} \in \{B^-\},$$

$$ABA = A \quad \Rightarrow \quad B = B^{\#} \in \{A^-\},$$

However, $A \neq B$.

The following characterization is however true.

THEOREM 2.8. *If* $A^{\#} \in \{B^-_{\text{com}}\}$ *and* $B^{\#} \in \{A^-\}$, *then*

$$A = B.$$

Proof. Note that $A^{\#} \in \{B^-\} \Rightarrow$ Rank $A^{\#}$ = Rank $A \geqslant$ Rank B. Similarly $B^{\#} \in \{A^-\} \Rightarrow$ Rank $B \geqslant$ Rank A. Hence $A^{\#} \in \{B^-_{\text{com}}\}$, $B^{\#} \in \{A^-\} \Rightarrow$ Rank A = Rank B. This implies $A^{\#}$ is a reflexive commuting inverse of B. Using a result of Englefield [4] and Erdelyi [5] quoted in Section 1 of this paper, we have $A^{\#} = B^{\#}$, which implies $A = B$ as required. ∎

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