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The Indivisibility of the Homogeneous K_n-Free Graphs

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We will prove that for each $n \geqslant 3$ the homogeneous K_n-free graph H_n is indivisible. That means that for every partition of H_n into two classes R and B there is an isomorphic copy of H_n in R or in B. This extends a result of Komjáth and Rödl [*Graphs Combin.* **2** (1986), 55–60] who have shown that H_3 is indivisible.

INTRODUCTION

Graphs are undirected and without loops. The graph A is a subgraph of the graph G if $V(A) \subset V(G)$ and $E(A) = (A \times A) \cap E(G)$. A countable graph H is *homogeneous* if for every two finite subgraphs A, B of H and every isomorphism $f: A \to B$ there is an automorphism of H which extends f. This definition follows Fraïssé [2, p. 313]. Lachlan and Woodrow [3] named those graphs "ultra-homogeneous." In [3] they characterised the homogeneous graphs.

We will rely on this classification of the homogeneous graphs and on the more general information in [2]. A graph H is *indivisible* if for every partition of the vertex set of H into two classes R and B there is an embedding $f: H \to H$ such that $f(H) \subset R$ or $f(H) \subset B$. (See [2, 0. 174].)

It is clear that the complement of a homogeneous graph is again homogeneous and that the complement of an indivisible graph is indivisible. Lachlan and Woodrow [3] proved that up to taking complements the only homogeneous graphs are disjoint copies of isomorphic complete graphs, the universal graph $\mathscr{U}$ and the K_n-free homogeneous graphs H_n $(n \geqslant 3)$. Of the graphs in the first group only K_ω and the graph which consists of ω copies of K_ω are indivisible. That the random graph $\mathscr{U}$ is indivisible can easily be seen, but also follows from [2; 10, 4.4, p. 288]. In [1] it is shown that H_3 is indivisible. In this paper we will now prove:

THEOREM. *Every homogenous K_n-free graph H_n $(n \geqslant 3)$ is indivisible.*

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1. Some Preliminary Observations

We assume familiarity with the description of H_n given in [4]. Familiarity with the proof that H_3 is indivisible as presented in [1] is not strictly necessary but probably helpful.

Lemma 1. *Let G be a finite K_n-free graph and $x \in G$. Then to each embedding $f: G - x \to H_n$ there are infinitely many extensions of f to an embedding of G into H_n.*

Proof. There is an embedding $g: G \to H_n$ [4, Theorem 2.3]. $f \circ g^{-1}$: $g(G) - g(x) \to f(G - x)$ is an isomorphism and can therefore be extended to an automorphism σ of H_n. $\sigma \circ g$ is then an extension of f. If G is a K_n-free graph with $V(G) = \{x_1, x_2, ..., x_n = x\}$, then the graph F with $V(F) = V(G) \cup \{y\}$, no edge from x to y, y is connected to x_i for $1 \leqslant i < n$ just in case x is connected to x_i, is also K_n-free. This observation implies that we can extend f in infinitely many ways. ▮

Lemma 2. *The K_n-free graph A is isomorphic to H_n if and only if:*

> *for every finite K_n-free graph G and $x \in G$ each embedding $f: G - x \to A$ can be extended to an embedding of G into A.*

Proof. Follows from Theorem 2.3 of [4]. ▮

Corollary 1. *If $X \subset V(H_n)$ is finite then $H - X$ is isomorphic to H_n.* ([4, Corollary 2.5].)

Corollary 2. *Let X be a finite subset of $V(H_n)$. Let $Y = \{y \in H_n - X;$ $(y, x) \notin E(H_n)$ for every $x \in X\}$. Then the subgraph induced by Y is isomorphic to H_n.*

Lemma 3. *Let G be a finite K_n-free graph and $X \subset V(H_n)$, such that G cannot be embedded into X. Then $H_n - X$ contains an isomorphic copy of H_n.*

Proof. This is Theorem 4.1 of [4]. ▮

2. Proof of the Theorem

We will prove the following:

Theorem. *The K_n-free homogeneous graph H_n is indivisible (for all natural numbers $n \geqslant 3$).*

Proof. We shall assume that H_n is defined on the set of natural numbers. We will use the following notation

$$X < Y \leftrightarrow \max X < \min Y \qquad (X, Y \subset \mathbb{N})$$

$$a < Y \leftrightarrow \{a\} < Y \qquad (a \in N, Y \subset \mathbb{N})$$

$$[\leftarrow a] = \{x \in \mathbb{N}: x \leqslant a\}$$

$$[\leftarrow a) = \{x \in \mathbb{N}: x < a\},$$

etc.

DEFINITION. For $a \in H_n$, $\Gamma(a) = \{x \in H_n: x < a \text{ and } xa \in E(H_n)\}$. The vertex a will be called a *strong vertex* if $\Gamma(a)$ is nonempty and contains the complete graph K_{n-2}. Otherwise a is a weak vertex.

LEMMA 4. *The set of strong vertices of H_n contains an isomorphic copy of H_n.*

Proof. The set of all weak vertices cannot contain K_{n-1}. Apply Lemma 3. ∎

DEFINITION. Let $a \in H_n$ be a strong vertex. We define

$$\gamma_1(a) = \min \Gamma(a).$$

$$\gamma_i(a) = \min\{x: \Gamma(a) \cap [\leftarrow x] \text{ contains } K_i\} \qquad \text{for} \quad 2 \leqslant i \leqslant n-2.$$

$$\gamma_{n-1}(a) = \min\{x: \Gamma(a) \subset [\leftarrow x)\}.$$

$$\Gamma_i(a) = \{x \in \Gamma(a): \gamma_i(a) \leqslant x < \gamma_{i+1}(a)\} \qquad \text{for} \quad 1 \leqslant i \leqslant n-2.$$

Note that $\gamma_{n-1}(a) \notin \Gamma(a)$. We could even have $\gamma_{n-1}(a) = a$.

In what follows we shall assume that the vertices of H_n are colored, in an arbitrary but fixed way, with two colors red and blue. We also introduce another (uncolored) copy H_n^* isomorphic to H_n. The vertices of H_n^* will be denoted by $a_1, a_2, ...$ and we assume that they are ordered such that $a_1 < a_2 < \cdots$. We do not assume that the graph and order isomorphisms between H_n and H_n^* coincide. Since the vertices of H_n^* are ordered, we can define strong and weak vertices, $\gamma_i(a_k)$, $\Gamma_i(a_k) \cdots$, etc. in H_n^*.

Statement. H_n contains a monochromatic copy of itself.

The statement will be proved by constructing a monotone map $\sigma: H_n^* \to H_n$ with the following properties:

1. The set $\{\sigma(a_k): a_k$ is a strong vertex of $H_n^*\}$ is monochromatic.
2. For every pair of natural numbers, i, j, $i<j$, we have

$\sigma(a_i)\,\sigma(a_j) \in E(H_n) \leftrightarrow a_j$ is a strong vertex of H_n^* and $a_i a_j \in E(H_n^*)$.

For every vertex $x \in H_n$ we define two formulas: the blue formula $\varphi(x)$ and the red formula $\psi(x)$. The blue formula $\varphi(x)$ is the following formula:

$$(\exists F_1\, \exists \alpha_1 : x = \min F_1 \wedge F_1 < \alpha_1 \wedge K_2 \not\subset F_1)$$
$$(\forall E_1\, \forall \beta_1 : \alpha_1 < E_1 < \beta_1 \wedge K_2 \not\subset F_1 \cup E_1)$$
$$(\exists F_2\, \exists \alpha_2 : \beta_1 < F_2 < \alpha_2 \wedge K_3 \not\subset F_1 \cup E_1 \cup F_2)$$
$$(\forall E_2\, \forall \beta_2 : \alpha_2 < E_2 < \beta_2 \wedge K_3 \not\subset F_1 \cup E_1 \cup F_2 \cup E_2)$$
$$\vdots$$
$$(\exists F_i\, \exists \alpha_i : \beta_{i-1} < F_i < \alpha_i \wedge K_{i+1} \not\subset F_1 \cup E_1 \cup \cdots \cup E_{i-1} \cup F_i)$$
$$(\forall E_i\, \forall \beta_i : \alpha_i < E_i < \beta_i \wedge K_{i+1} \not\subset F_1 \cup E_1 \cup \cdots \cup F_i \cup E_i)$$
$$\vdots$$
$$(\exists F_{n-2}\, \exists \alpha_{n-2} : \beta_{n-3} < F_{n-2} < \alpha_{n-2} \wedge K_{n-1} \not\subset F_1 \cup E_1 \cup \cdots \cup E_{n-3} \cup F_{n-2})$$
$$(\forall E_{n-2}\, \forall \beta_{n-2} : \alpha_{n-2} < E_{n-2} < \beta_{n-2} \wedge K_{n-1} \not\subset F_1 \cup E_1 \cup \cdots \cup F_{n-2} \cup E_{n-2}).$$

(The set $\{y \in H_n : y > \beta_{n-2}$ and $\Gamma(y) \cap [\leftarrow \beta_{n-2}] = F_1 \cup E_1 \cup \cdots \cup F_{n-2} \cup E_{n-2}\}$ contains infinitely many blue vertices.) $\varphi(x)$ is interpreted according to the following rules:

1. : means "such that."
2. $F_1, E_1, ..., F_{n-2}, E_{n-2}$ are restricted to be finite nonempty subsets of H_n and $\alpha_1, \beta_1, ..., \alpha_{n-2}, \beta_{n-2}$ are restricted to be elements of H_n.

The red formula $\psi(x)$ is obtained from $\varphi(x)$ by interchanging the quantifiers $\forall$, $\exists$ and replacing the word "blue" by "red." One can convince oneself that for every $x \in H_n$ at least one of $\varphi(x)$ and $\psi(x)$ is true.

DEFINITION. A vertex $x \in H_n$ is called a blue generator if $\varphi(x)$ is true. Otherwise x is called a red generator.

We shall divide the proof into two cases: The blue case where there are infinitely many blue generators in H_n and the Red Case where there are only finitely many blue generators.

The Blue Case

Here the formula φ will be used to generate certain subsets as follows. Suppose we have just chosen a vertex $x \in H_n$ which is a blue generator to be included in our construction. Then, since $\varphi(x)$ is true, there exist $F_1 \subset H_n$ and $\alpha_1 \in H_n$ such that $x = \min F_1$ and $F_1 < \alpha$. We say that F_1 and α_1 have been introduced and denote them respectively by $F_1(x)$ and $\alpha_1(x)$. We also say that the quantifier block

$$\varphi_1(x) \equiv (\forall E_1\, \forall \beta_1 : \alpha_1(x) < E_1 < \beta_1 \wedge K_2 \not\subset F_1(x) \cup E_1)$$

has been activated. If at a later stage some set $E_1 \subset H_n$ and an element $\beta_1 \in H_n$ were recognized to satisfy

$$\alpha_1(x) < E_1 < \beta_1 \qquad \text{and} \qquad K_2 \not\subset F_1(x) \cup E_1$$

then E_1, β_1 will be called an instance pair for $\varphi_1(x)$. Corresponding to E_1, β_1 there exist $F_2 \subset H_n$ and $\alpha_2 \in H_n$ such that

$$\beta_1 < F_2 < \alpha_2 \qquad \text{and} \qquad K_3 \not\subset F_1(x) \cup E_1 \cup F_2.$$

We shall denote these F_2 and α_2 by $F_2(x; E_1 : \beta_1)$ and $\alpha_2(x; E_1; \beta_1)$. In turn, the quantifier block

$$\varphi_2(x; E_1; \beta_1) \equiv (\forall E_2\, \forall \beta_2 : \alpha_2(x; E_1; \beta_1) < E_2 < \beta_2 \\ \wedge K_3 \not\subset F_1(x) \cup E_1 \cup F_2(x; E_1; \beta_1) \cup E_2)$$

will be activated.

In general, if the quantifier block

$$\begin{aligned} &\varphi_i(x; E_1, ..., E_{i-1}; \beta_1, ..., \beta_{i-1}) \\ &\quad \equiv (\forall E_i\, \forall \beta_i : \alpha_i(x; E_1, ..., E_{i-1}; \beta_1, ..., \beta_{i-1}) \\ &\quad < E_i < \beta_i \wedge K_{i+1} \not\subset F_1(x) \cup E_1 \cup \\ &\quad \cdots \cup F_i(x; E_1, ..., E_{i-1}; \beta_1, ..., \beta_{i-1}) \cup E_i) \end{aligned}$$

has been activated, then a pair E_i, β_i satisfying this quantifier block will be called an instance pair for it. This paper will induce the existence of

$$F_{i+1}(x; E_1, ..., E_i; \beta_1, ..., \beta_i) \qquad \text{and} \qquad \alpha_{i+1}(x; E_1, ..., E_i; \beta_1, ..., \beta_i)$$

which in turn will activate the quantifier block

$$\varphi_{i+1}(x; E_1, ..., E_i; \beta_1, ..., \beta_i).$$

Note that all the previous quantifier blocks

$$\varphi_1(x), \qquad \varphi_2(x; E_1; \beta_1), ..., \qquad \varphi_i(x; E_1, ...; \beta_1, ...)$$

are assumed to remain activated. In other words we can use them to generate sets as many times as we want. The sets $E_1, E_2, ...$ will be called initiators and the sets $F_1(x)$, $F_2(x; E_1, \beta_1)$, ... will be called the induced sets.

The general picture of our construction will be as follows. There will be an infinite sequence of blue generators $B_1, B_2, ...$ and another sequence consisting of $\sigma(a_1), \sigma(a_2), ...$ We shall use sets of the form $E_i = \{\sigma(a_{i_1}), ..., \sigma(a_{i_k})\}$ as initiators to generate sets of the form $F_j(B_l; E_1, ...; \beta_1, ...)$ as induced sets. The relative position of these elements and sets will be as follows. Above a generator B_k will be the set $F_1(B_k)$. This will be followed by $\sigma(a_k)$. Above $\sigma(a_k)$ there will be some induced sets of the form $F_i(B_l; E_{i_1}, ...; \beta_{i_1}, ...)$, $i \geqslant 2$ and $l \leqslant k$. Next will lie the generator B_{k+1}. The open interval $(\sigma(a_{k-1}), B_k)$ will be denoted by I_k. Except for the sets $F_1(B_j)$, each induced set will lie in an interval I_k for some k. All the induced sets will be so constructed to be disjoint and to contain no element B_j or $\sigma(a_k)$. In order to be able to do this we shall introduce the time bound T. At any stage of the construction the current value of T will be equal to some element in H_n larger than all the previously constructed sets and elements. Having introduced the next element or set into our construction, we set the current value of T to some element in H_n which is above the newly introduced element or set and so on.

Let us now describe our construction in full detail.

Choose a blue generator B_1 and introduce B_1 and $F_1(B_1)$. Set the current time bound $T = \alpha_1(B_1)$. Choose a vertex y such that

$$y > T \qquad \text{and} \qquad \Gamma(y) \cap [\leftarrow T] = \varnothing.$$

Put $\sigma(a_1) = y$ and set $T = \sigma(a_1) + 1$. Now the pair $(\{\sigma(a_1)\}, T)$ is an instance pair for the activated subformula $\varphi_1(B_1)$ so that $F_2(B_1; \{\sigma(a_1)\}; T)$ and $\alpha_2(B_1; \{\sigma(a_1)\}; T)$ exist. We introduce $F_2(B_1; \{\sigma(a_1)\}; T)$ and set $T = \alpha_2(B_1; \{\sigma(a_1)\}; T)$.

Next we choose a blue generator B_2 such that $B_2 > T$. We introduce B_2 and $F_1(B_2)$ and set $T = \alpha_1(B_2)$.

Let us now describe the construction after $\sigma(a_{k-1})$ has been introduced and the current time bound has been set to $T = \sigma(a_{k-1}) + 1$. Consider all pairs $(Y, \varphi_i(B_j))$ satisfying:

(i) $Y \subset \{\sigma(a_1), ..., \sigma(a_{k-1})\}$, $Y \neq \varnothing$.

(ii) $\phi_i(B_j)$ is an activated quantifier block belonging to B_j.

(iii) for any $T' \in H_n$, $T' \geqslant T$, the pair Y, T' is an instance pair for $\varphi_i(B_j)$.

Order all such pairs in an arbitrary way. Pick the first such pair Y, $\varphi_i(B_j)$. Since Y, T is an instance pair for $\varphi_i(B_j)$, we introduce $F_{i+1}(B_j; ..., Y; ..., T)$ and set $T = \alpha_{i+1}(B_j; ..., Y; ..., T)$. We repeat the same process one at a time for every such pair Y, $\varphi_i(B_j)$. Then we choose a blue generator $B_k > T$. We introduce $F_1(B_k)$ and set $T = \alpha_1(B_k)$.

The next step is to show how $\sigma(a_k)$ is constructed. If a_k is a weak vertex of H_n^*, then choose a vertex $y \in H_n$ satisfying

$$y > T \qquad \text{and} \qquad \Gamma(y) \cap [\leftarrow T] = \varnothing.$$

We put $\sigma(a_k) = y$ and set $T = \sigma(a_k) + 1$. Assume now that a_k is a strong vertex of H_n^*. Define $m_1, ..., m_{n-1}$ by

$$\gamma_i(a_k) = a_{m_i} \qquad (\text{in } H_n^*), \qquad i = 1, ..., n-1.$$

We also denote

$$D_i = \sigma(\Gamma_i(a_k)), \qquad i = 1, ..., n-2.$$

We claim that the subformula $\varphi_{n-2}(B_{m_1}; D_1, ..., D_{n-2}; T_1, ..., T_{n-2})$ has been activated where T_i is a value of the time bound satisfying $T_i \in I_{m_{i+1}}$ $(i = 1, ..., n-2)$. Assume for the moment that this is true. This means that $F_1(B_{m_1})$, $F_2(B_{m_1}; D_1; T_1)$, ...$F_i(B_{m_1}; D_1, ..., D_{i-1}; T_1, ..., T_{i-1})$, ... have been introduced. From the formula $\varphi(B_{m_1})$, we deduce that there are infinitely many blue vertices y such that

$$\begin{aligned} &y > T_{n-2} \qquad \text{and} \\ &\Gamma(y) \cap [\leftarrow T_{n-2}] = F_1(B_{m_1}) \cup D_1 \cup F_2(B_{m_2} \cdots) \\ &\qquad\qquad \cup \cdots \cup F_{n-2}(\cdots) \cup D_{n-2}, \end{aligned}$$

choose such a vertex y satisfying $y > T$ and put $\sigma(a_k) = y$. We also set $T = \sigma(a_k) + 1$.

Assume now that all $\sigma(a_l)$, $l < k$, have been constructed according to the previous description. We shall use induction on i to prove that all $\varphi_i(B_{m_1}, D_1, ..., D_i, T_1, ..., T_i)$ have been activated where $i \leqslant n-2$ and $T_j \in I_{m_{j+1}}$ $(j \leqslant i)$. First, if $a_i \in \Gamma_1(a_k)$ then there is no edge $a_i a_{m_1} \in H_n^*$. Therefore, according to the prescribed construction, $F_1(B_{m_1}) \cup D_1$ is an independent set. Thus $\varphi_2(B_{m_1}; D_1; T_1)$ was activated and $F_2(B_{m_1}; D_1; T_1)$ was introduced when $T = T_1$ where $T_1 \in I_{m_2}$.

Assume that for all $j \leqslant i$, the sets $F_j(B_{m_1}; D_1, ..., D_{j-1}; T_1, ..., T_{j-1})$ have been introduced and the formulas $\varphi_j(B_{m_1}; D_1, ..., D_{j-1}; T_1, ..., T_{j-1})$ have been activated. For simplicity we shall write $F_1, F_2, ...$ instead of $F_1(B_{m_1}), F_2(B_{m_1}; D_1; T_1), ...$. We also write $\varphi_1, \varphi_2, ...$ instead of $\varphi_1(B_{m_1})$,

$\varphi_2(B_{m_1}; D_1; T_1), \ldots$. We show that D_i, T_i is an instance pair for φ_i where $T_i \in I_{m_{i+1}}$. We have to prove that the set

$$X = F_1 \cup D_1 \cup \cdots \cup F_i \cup D_i$$

contains no K_{i+1}. Assume, on the contrary, that X contains vertices $x_1 < x_2 < \cdots < x_{i+1}$ which form a K_{i+1}. By the definition of F_i we have

$$x_{i+1} \notin F_1 \cup D_1 \cup \cdots \cup D_{i-1} \cup F_i$$

so that $x_{i+1} \in D_i$. We also observe that $\{x_1, \ldots, x_{i+1}\} \cap \bigcup_{j=1}^{i} F_i \neq \emptyset$ since $K_{i+1} \not\subset \bigcup_{j=1}^{i} \Gamma_j(a_k)$. Let h be maximum such that $x_h \in \bigcup_{j=1}^{i} F_j$, say $x_h \in F_p$, $p \leqslant i$. Write $x_l = \sigma(a_{q_l})$ for $h < l \leqslant i+1$. We observe that $h \leqslant p$ since $K_{p+1} \not\subset F_1 \cup D_1 \cup \cdots \cup F_p$. Now each x_l $(h < l)$ has an edge to F_p. By our construction, this implies that $\{\gamma_p(a_k)\} \cup \Gamma_1(a_k) \cup \cdots \cup \Gamma_{p-1}(a_k)$ for each $h < l$. This is a contradiction, however, since $D_1 \cup \cdots \cup \Gamma_1(a_k) \cup \cdots \cup \Gamma_{p-1}(a_k)$ contains a complete graph K_p which together with $a_{q_{h+1}}, \ldots, a_{q_{i+1}}$ will form a K_{i+1} in $\bigcup_{j=1}^{i} \Gamma_j(a_k)$. This shows that F_{i+1} has been introduced and φ_{i+1} has been activated or, in the case $i = n-2$, we can introduce $\sigma(a_k)$. This completes the proof in the blue case.

The Red Case

In this case, we can assume that each vertex in H_n is a red generator since deleting finitely many vertices of H_n we still have a copy isomorphic to H_n (in the graph-theoretic sense). The construction will go along similar lines as in the blue case except for some minor changes which we describe below. First of all we do not introduce special vertices to act as red generators since each vertex by itself is a red generator. We shall use the red formula to generate sets in the same way as we did in the blue case. The initiatiors, here the F_i's, will be again sets of the form $\{\sigma(a_{i_1}), \ldots, \sigma(a_{i_m})\}$. For each vertex of the form $\sigma(a_k)$ the subformula

$$\psi_1(\sigma(a_k)) \equiv (\forall F_1\ \forall a_1 : \sigma(a_k) = \min F_1 \wedge F_1 < \alpha_1 \wedge K_2 \not\subset F_1)$$

will be automatically activated. If the subformula

$$\begin{aligned}\psi_i(\sigma(a_k); F_1, \ldots, F_{i-1}; \alpha_1, \ldots, \alpha_{i-1}) \\ \equiv (\forall F_i\, \forall \alpha_i : \beta_{i-1}(\sigma(a_k), F_1, \ldots, F_{i-1}; \alpha_1, \ldots, \alpha_{i-1}) \\ < F_i < \alpha_i K_{i+1} \not\subset F_1 \cup E_1(\sigma(a_k); F_1) \\ \cup \cdots \cup E_{i-1}(\sigma(a_k); F_1, \ldots, F_{i-1}; \alpha_1, \ldots, \alpha_{i-1}))\end{aligned}$$

has been activated and F_i, α_i is an instance pair of it then we can introduce the new set $E_i(\sigma(a_k); F_1, ..., F_i; \alpha_1, ..., \alpha_i)$ and the next quantifier block will be activated.

The time bound T will act in the same way as in the blue case. Also the proof for the existence of $\sigma(a_k)$ for each k will be similar as in the blue case.

REFERENCES

1. P. KOMJÁTH AND V. RÖDL, Coloring of universal graphs, *Graphs Combin.* **2** (1986), 55–60.
2. R. FRAÏSSÉ, Theory of relations, *Stud. Logic Foundations Math.* **118** (1986).
3. A. H. LACHLAN AND R. E. WOODROW, Countable ultrahomogeneous undirected graphs, *Trans. Amer. Math. Soc.* **262** (1980), 51–94.
4. C. W. HENSON, A family of countable homogeneous graphs, *Pacific J. Math.* **38**, No. 1 (1971).