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NOTE

A Counterexample on Implicit Variational Inequalities

Department of Applied Mathematics, Korea Maritime University, Pusan, 606-791, Korea

Submitted by E. Stanley Lee

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Our aim in this note is to give a counterexample to show that some existence theorems on implicit variational inequalities recently due to Fu are false.

In a recent paper, Fu [1] considered the following implicit variational problem, denoted by I.V.P.: Let X, Y be topological vector spaces; let C and D be nonempty subsets of X and Y, respectively. Given multivalued mappings $E: C \to 2^C$ and $F: C \to 2^D$, real functions $f: C \times C \times D \to R$ and $g: C \times C \to R$ such that for any $x \in C$, $y \in F(x)$, $f(x, x, y) \geq 0$. Find a vector $v \in C$ such that $v \in E(v)$ and $u \in F(v)$ such that

$$g(v, v) \leq f(v, w, u) + g(v, w) \qquad \text{for all } w \in E(v).$$

He proved that under some suitable conditions the I. V. P. has solutions. To be more specific, he derived the following existence theorem which plays a crucial role in his paper, and discussed many applications (see [1, Theorems 6−12]) of the result.

THEOREM 1. *Let X, Y be Hausdorff locally convex spaces, C be a nonempty compact convex subset of X, and D be a nonempty closed convex subset of Y. Let $E: C \to 2^C$ be upper semicontinuous with nonempty closed*

* Supported in part by GARC-KOSEF in 1996−1997. E-mail address: kum@hanara.kmaritime.ac.kr.

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convex values and $F: C \to 2^D$ *be a mapping with nonempty values. Suppose that* $f: C \times C \times D \to R$ *satisfies the following conditions*:

(i) *for each* $x \in C$ and $y \in F(x)$, $f(x, x, y) \geq 0$;

(ii) *for any fixed* $x \in C$ and $y \in D$, *the function* $f(x, u, y)$ of u *is convex*;

(iii) *for any fixed* $u \in C$, *the function* $\sup_{y \in F(x)} f(x, u, y)$ of x *is* u.s.c.

Then there exists an $\bar{x} \in C$ *such that* $\bar{x} \in E(\bar{x})$ and $\sup_{y \in F(\bar{x})} f(\bar{x}, u, y) \geq 0$ *for all* $u \in E(\bar{x})$.

Our aim in this note is to show that the above theorem is false. We also would like to point out that a number of results in [1] (including some arguments to derive well-known theorems from those false facts) must be corrected. To this end, we give a counterexample.

EXAMPLE. Let $X = Y$ be R^n the n-dimensional Euclidean space and let $C = D$ be B_n the closed unit ball of R^n. Define two multifunctions E and $F: B_n \to 2^{B_n}$ to be

$$E(x) = \begin{cases} B_n & \text{if } x = \mathbf{0} \text{ the origin,} \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

and

$$F(x) = (1, 0, \ldots, 0) \qquad \text{for all } x \in B_n.$$

For $(x, u, y) \in C \times C \times D$, we define $f(x, u, y) = \langle u - x, y \rangle$ where $\langle , \rangle$ is the inner product on R^n. Then E is clearly u.s.c. with nonempty compact convex values and it is easily checked that all the conditions of Theorem 1 are satisfied. In this case, the only fixed point $\bar{x}$ of E is the origin $\mathbf{0}$, and $E(\mathbf{0}) = B_n$. However, $\sup_{y \in F(\mathbf{0})} f(\mathbf{0}, u, y) = \langle u, (1, 0, \ldots, 0) \rangle = u_1$ for all $u \in E(\mathbf{0}) = B_n$ where $u = (u_1, \ldots, u_n)$. Hence $\sup_{y \in F(\mathbf{0})} f(\mathbf{0}, u, y) < 0$ for all $u \in E(\mathbf{0}) = B_n$ with $u_1 < 0$. This shows that the conclusion of Theorem 1 is not true.

Remarks. (1) Taking $g(x, u) = 0$ for all $(x, u) \in C \times C$, we see that our example also serves as a counterexample for Fu [1, Theorems 2 and 3].

(2) Taking $\phi(x) = 0$ for all $x \in C$, we know that our example is also a counterexample for Fu [1, Theorems 9 and 10].

(3) Our example shows that even if F is continuous with nonempty compact convex values in the hypotheses of Fu [1, Theorems 2–3 and 9–10], the conclusions still do not hold.

(4) Our example tells us that just the assumption that E is u.s.c. is not sufficient to ensure some existence results in [1] as we have seen. There needs to be more additional conditions such as the continuity of E to get the results. Note that E in the above example is u.s.c. but not l.s.c.

(5) The above example can be extended in a separable Hilbert space.

Note added in proof. After the acceptance of this paper the author realized that Dr. P. Cubiotti had already found a counterexample to Theorem 1 in the one-dimensional Euclidean space R (see *Comment. Math. Univ. Carolinae* **37** (1996), 415–418).

REFERENCES

1. J. Fu, Implicit variational inequalities for multivalued mappings, *J. Math. Anal. Appl.* **189** (1995), 801–814.