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On some contributions of John Horváth to the theory of distributions

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Dedicated to Professor J. Horváth on the occasion of his 80th birthday

Abstract

Our main task is a presentation of J. Horváth's results concerning

- singular and hypersingular integral operators,
- the analytic continuation of distribution-valued meromorphic functions, and
- a general definition of the convolution of distributions.

At some instances minor supplements to his results are given.

Keywords: Singular integral operators; Distributions; Convolution of distributions

Introduction

This contribution in honour of Professor John Horváth's 80th birthday attempts to throw light on the development of his mission for a general use of L. Schwartz's distribution theory in Analysis, in particular concerning the Theory of (Linear) Partial Differential Equations and in Harmonic Analysis. Considering modern textbooks on these subjects one gets the impression that the use of (some parts of) distribution theory is by now common sense, e.g., [1,14,51,53,72,86–88,96,97]. This development began with [50] and has its roots in

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Schwartz's monograph [75]. To quote an example: in 1940, H. Weyl proved that any weak solution u of $\Delta u = 0$ is $\mathcal{C}^\infty$. Schwartz's generalization [75, p. 143, Théorème XII] reads as follows: "Si D est un opérateur différentiel (matriciel) à coefficients indéfiniment dérivables, ayant un noyau élémentaire à gauche très régulier (respectivement analytiquement très régulier) $E_{x,\xi}$, il est hypo-elliptique (respectivement analytique hypo-elliptique). Si alors T converge vers 0 dans $\mathcal{D}'$, et si DT converge vers 0 dans $\mathcal{E}$ (en particulier si T est solution de l'équation homogène $DT = 0$), alors T converge vers 0 dans $\mathcal{E}$." (Note that this theorem implies classical theorems of Weierstraß, Harnack and Täcklind relating to solutions to $(\partial_x + \mathrm{i}\partial_y)u = 0$, $\Delta u = 0$, $(\partial_t - \Delta)u = 0$, converging locally uniformly.)

Horváth's commitment towards a universal use of distribution theory proceeded concurrently in two directions.

The first one investigated the nature of distributions and the spaces to which they belong, and, more generally, the theory of locally convex topological vector spaces. Around 1954 L. Schwartz encouraged him to write an "introductory textbook" [32, p. ix] which appeared in 1966 as "Topological Vector Spaces and Distributions, vol. I". The first 3 chapters treat Banach spaces and—in the spirit of N. Bourbaki [2,3]—general topology, locally convex topological vector spaces and duality. Later, in lecture series, Professor Horváth above all made the (modified) content of Chapter 4 on distributions [33,38,41,45–48] widely known.

The second direction is characterized by the examination of the following questions:

(1) What is the nature of singular integrals, of their symbols and their composition?
(2) What are hypersingular integrals?
(3) How should convolution be defined in general?

In the following pages I will be concerned with shedding light on Professor Horváth's clarifications of these questions.

Let me add a few further remarks. L. Schwartz's own interests focussed on the first direction (cf. L. Gårding in his article "The impact of distributions in analysis" [18, Chapter 12, pp. 77–88]: "... and one gets the impression that the author's heart is with the linear topological spaces rather than with the problems of analysis." (p. 79)), which is demonstrated impressively in his monumental "Théorie des distributions à valeurs vectorielles" [76], but also in: "Espaces de fonctions différentiables à valeurs vectorielles" [77].

Nevertheless, besides the above quoted theorem [75, p. 143], Schwartz contributed in a manifold and lasting manner to the theory of partial differential equations, which is testified, e.g., by his article on convolution equations [78]. The latter is expanded to detail in [16] and is also taken as a basis by S. Gindikin, L.R. Volevich in [23,24]. Also his presentation of the regularity theory for elliptic systems in his Bogotá-Lectures [79] had model character, cf. [71, pp. 201–204]; [98, pp. 120–139]. The notes of the Bogotá-Lectures were edited by Professor J. Horváth, who invited Schwartz to Bogotá in 1956.

How much distribution theory is needed in analysis? This question is not easy to answer. The usual wide-spread belief is that tempered distributions (Schwartz's "distributions sphériques" $\mathcal{S}'$), Sobolev spaces H^s and the kernel theorem are irredeemable [51,53,72,84, 86,88,96]. Nevertheless, there are some presentations which do not use distributions, e.g., [15] or [5].

Since this is an article in honour of Professor J. Horváth, I take the liberty to quote also some manuscripts which unfortunately are unpublished until now.

All notations in the sequel are those of [75]. In particular, the Fourier transform is defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i}\xi x}\varphi(x)\,\mathrm{d}x, \quad \varphi \in \mathcal{S}\big(\mathbb{R}^n_x\big),$$

Y denotes the Heaviside function, $\check{\varphi}$ is the function $x \mapsto \varphi(-x)$, $*$ means convolution and χ_A the characteristic function of the set A. Constantly we use "les nouveaux espaces des distributions, les $\mathcal{D}'_{L^p}$" of L. Schwartz [75, p. 199] which are connected with the Bessel potential spaces

$$L^p_s = \big\{T \in \mathcal{S}'/\mathcal{F}^{-1}\big(\big(1+|x|^2\big)^{s/2}\mathcal{F}T\big) = \big(\mathcal{F}^{-1}\big(1+|x|^2\big)^{s/2}\big) * T \in L^p\big\}$$

$$\text{by} \quad \mathcal{D}'_{L^p} = \bigcup_{s\in\mathbb{R}} L^p_s = \bigcup_{k=0}^{\infty} L^p_{-k}$$

([87, p. 135]; [6]; [81, exposé 5]). A systematic study of certain other spaces used in distribution theory is given by J. Horváth in [43,44].

1. Elucidation of the very nature of singular integrals, their symbols and their composition

1.1. One-dimensional Hilbert transform and distribution theory

In 1924, 1927, Horváth's mentor Marcel Riesz proved that the Hilbert transform $H : L^p(\mathbb{R}^1) \to L^p(\mathbb{R}^1)$, $f \mapsto Hf$, is well-defined for $1 < p < \infty$. Herein, Hf means a function defined (almost everywhere) by the following limit of integrals [58,59]:

$$Hf(x) = \lim_{\varepsilon\searrow 0} \int_{|y|\geqslant\varepsilon} f(x-y)\frac{\mathrm{d}y}{y}.$$

L. Schwartz [75, (II,2;29), p. 42] introduces the principal value distribution $\mathrm{vp}(1/x)$ either as the distributional derivative of the locally integrable function $\log|x|$, i.e.,

$$\mathrm{vp}\,\frac{1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}\log|x|,$$

or, equivalently, as the distributional limit

$$\mathrm{vp}\,\frac{1}{x} = \lim_{\varepsilon\searrow 0}\frac{Y(|x|-\varepsilon)}{x}.$$

Then he states: "Pour $f \in L^p$ $(1 < p < \infty)$, on démontre [59] que $\mathrm{vp}(1/x) * f$ s'ecrit v.p. $\int_{-\infty}^{\infty} f(t)/(x-t)\,\mathrm{d}t$, cette intégrale étant convergente pour presque toutes les valeurs de x. On peut généraliser immédiatement les propriétés classiques de cette intégrale, en montrant que la convolution avec $\mathrm{vp}(1/x)$ est une opération linéaire continue de $\mathcal{D}'_{L^p}$ dans $\mathcal{D}'_{L^q}$ $(1 < p < \infty,\ p \leqslant q$, ou $1 = p < q)$" [75, p. 259].

Hence, Schwartz identifies the Hilbert transform Hf of f with the convolution of two single entities, $\mathrm{vp}(1/x)$ and f. The limit of integrals in the original definition is shifted to the definition of $\mathrm{vp}(1/x)$.

It was this point of view on singular integrals which was adopted by J. Horváth [27–30].

The identification of Hf with $\mathrm{vp}(1/x) * f$ also depends on the *definition of convolution*. Within the framework of Schwartz's "Théorie des Distributions" [75] the convolution is defined in Théorème XXVI, p. 203: "Si $S \in \mathcal{D}'_{L^p}$, $T \in \mathcal{D}'_{L^q}$, $1/p + 1/q \geqslant 1$, on peut donner un sens au produit de convolution $S * T$; alors $S * T \in \mathcal{D}'_{L^r}$, $1/r = 1/p + 1/q - 1$; l'application bilinéaire $(S, T) \mapsto S * T$ de $\mathcal{D}'_{L^p} \times \mathcal{D}'_{L^q}$ dans $\mathcal{D}'_{L^r}$ est continue."

Note that this $\mathcal{D}'_{L^p}$-convolution theorem does not imply the above quoted mapping property [75, p. 259] of the Hilbert transform,

$$\mathrm{vp}\,\frac{1}{x} * f \in \mathcal{D}'_{L^p} \quad \text{for } f \in \mathcal{D}'_{L^p},\ p > 1. \tag{1}$$

The reason is that $\mathrm{vp}(1/x) \notin \mathcal{D}'_{L^1}$, i.e., $\mathrm{vp}(1/x)$ is not an integrable distribution (otherwise its Fourier transform would be a continuous function—but

$$\mathcal{F}\left(\mathrm{vp}\,\frac{1}{x}\right) = -\mathrm{i}\pi\,\mathrm{sign}\,\xi$$

[75, (VII,7;19), p. 259].

To prove the mapping property (1), we use a description of distributions in $\mathcal{D}'_{L^p}$, the consistency of the convolution product $\mathrm{vp}(1/x) * f \in \mathcal{D}'_{L^r}$ of $\mathrm{vp}(1/x) \in \mathcal{D}'_{L^q}$, $q > 1$, and $f \in L^p \subset \mathcal{D}'_{L^p}$ with $Hf \in L^p \subset \mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^r}$, and an associativity property of the $\mathcal{D}'_{L^p}$-convolution, formulated by J. Horváth:

Proposition 1 [28, p. 65]. *$S \in \mathcal{D}'_{L^p}$, $T \in \mathcal{D}'_{L^q}$, $U \in \mathcal{D}'_{L^r}$, $1/p + 1/q + 1/r \geqslant 2$. Then $S * (T * U) = (S * T) * U$.*

Proof of (1). For $f \in \mathcal{D}'_{L^p}$ there exist $g_\alpha \in L^p$, $|\alpha| \leqslant m$, such that $f = \sum_{|\alpha| \leqslant m} \partial^\alpha g_\alpha$ [75, Théorèmè XXV, 1°, p. 201]. M. Riesz's result implies $\mathrm{vp}(1/x) * g_\alpha \in L^p$ and hence

$$\sum_{|\alpha| \leqslant m} \partial^\alpha \left(\mathrm{vp}\,\frac{1}{x} * g_\alpha\right) \in \mathcal{D}'_{L^p}.$$

By taking $U = \partial^\alpha \delta \in \mathcal{D}'_{L^r}$ in Proposition 1 we conclude

$$\sum_{|\alpha| \leqslant m} \mathrm{vp}\,\frac{1}{x} * \left(\partial^\alpha g_\alpha\right) \in \mathcal{D}'_{L^p}$$

and, finally, $\mathrm{vp}(1/x) * f \in \mathcal{D}'_{L^p}$. $\square$

1.2. The reciprocity relation

Before turning to n-dimensional analogues let me point out four issues in the one-dimensional case which we often encounter in the early papers on singular integrals.

(a) *The connexion with the conjugate Poisson kernel.*

By approximating $\log|x|$ by the smooth function $\frac{1}{2}\log(x^2+\varepsilon^2)$ and by use of the continuity of the distributional derivation we get

$$\operatorname{vp}\frac{1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(\log|x|\big) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{\varepsilon\searrow 0}\frac{1}{2}\log\big(x^2+\varepsilon^2\big)\right) = \frac{1}{2}\lim_{\varepsilon\searrow 0}\frac{\mathrm{d}}{\mathrm{d}x}\log\big(x^2+\varepsilon^2\big)$$
$$= \lim_{\varepsilon\searrow 0}\frac{x}{x^2+\varepsilon^2} = \frac{1}{2}\lim_{\varepsilon\searrow 0}\left(\frac{1}{x+\mathrm{i}\varepsilon}+\frac{1}{x-\mathrm{i}\varepsilon}\right).$$

Note that the relation

$$\lim_{\varepsilon\searrow 0}\left(\frac{Y(|x|-\varepsilon)}{x} - \frac{x}{x^2+\varepsilon^2}\right) = 0 \tag{2}$$

is the distributionally interpreted version of Proposition 2.2 in [26, p. 19] (in $\mathbb{R}^1$) and of Lemma 1.2 in [89, p. 218].

(b) *The connexion with M. Riesz's elliptic kernels.*

These are defined as

$$R_\lambda = \frac{\Gamma\left(\frac{n-\lambda}{2}\right)}{\pi^{n/2}2^\lambda\Gamma\left(\frac{\lambda}{2}\right)}|x|^{\lambda-n}, \quad \lambda\in\mathbb{C},\ n\in\mathbb{N}$$

([60, (5), (6), p. 3]; [62, (1), p. 16 and (7), p. 19]; [26, p. 28]; [31, (1), p. 433]; [34, p. 154]; [35, (5), p. XII.5]; [39, p. 180]; [40, p. 8-04]; [41, p. 22]; [42, p. 434]; [45, p. 456]; [46, p. 28]; [47, p. 10]).

If $\lambda = n = 1$, $R_1 = \operatorname{Pf}_{\lambda=1}R_\lambda = -\frac{1}{\pi}(\log|x|+C)$, and hence

$$-\frac{1}{\pi}\operatorname{vp}\frac{1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}R_1 = \frac{\mathrm{d}\delta}{\mathrm{d}x} * R_1 = -\frac{1}{\pi}\lim_{\varepsilon\searrow 0}|x|^{\varepsilon-1}\operatorname{sign}x. \tag{3}$$

Note that the representation (3) of $\operatorname{vp}(1/x)$ is different from (2), since it comes from the different approximation

$$\lim_{\varepsilon\searrow 0}\frac{|x|^\varepsilon-1}{\varepsilon} = \log|x|$$

of the logarithm. An interpretation of (3) is: the odd singular integral kernel $\operatorname{vp}(1/x)$ is the convolution of the odd distribution $\mathrm{d}\delta/\mathrm{d}x$ with the weakly singular, even kernel $\log|x|$.

(c) *The composition of the Hilbert transform with itself.*

It was well known before the arrival of distribution theory that the Hilbert transform can be iterated to give

$$H(Hf) = -\pi^2 f. \tag{4}$$

A classical problem was to determine spaces of functions f for which the identity (4) holds. Due to $Hf = \operatorname{vp}(1/x) * f$ the equality (4) is equivalent to

$$\left(-\frac{1}{\pi}\operatorname{vp}\frac{1}{x}\right) * \left(-\frac{1}{\pi}\operatorname{vp}\frac{1}{x}\right) = -\delta \tag{5}$$

which is an immediate consequence of the Fourier transform of $\mathrm{vp}(1/x)$ and the exchange theorem $\mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T$ for $S, T \in \mathcal{D}'_{L^2}$ in [75, p. 270, remarque].

Putting $N_0 = \frac{1}{\pi}\mathrm{vp}(1/x)$ [31, (6), p. 434] and observing

$$R_{-2k} = (-1)^k \frac{\mathrm{d}^{2k}\delta}{\mathrm{d}x^{2k}}, \quad k \in \mathbb{N}_0$$

[39, p. 180], we derive the equations

$$N_0 * N_0 = -R_0 \tag{6}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} R_1\right) * \left(\frac{\mathrm{d}}{\mathrm{d}x} R_1\right) = -\delta = -R_0 = R_1 * \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} R_1\right) = -R_1 * R_{-1}.$$

Note that the last two equations are equivalent to (5), (6) upon shifting the differentiation (employing Proposition 6 below) and a subsequent use of the composition formula of the elliptic Riesz kernels $R_\alpha * R_\beta = R_{\alpha+\beta}$, $\mathrm{Re}(\alpha + \beta) < n$.

(d) *The symbol of the singular integral operator H.*

Due to

$$\mathcal{F}(Hf) = \mathcal{F}\left(\mathrm{vp}\,\frac{1}{x} * f\right) = -\mathrm{i}\pi\,\mathrm{sign}\,\xi\,\mathcal{F}f, \quad f \in \mathcal{D}'_{L^2},$$

we obtain

$$Hf = \mathcal{F}^{-1}(-\mathrm{i}\pi\,\mathrm{sign}\,\xi\,\mathcal{F}f).$$

Hence, the symbol of H, i.e., $-\mathrm{i}\pi\,\mathrm{sign}\,\xi$, is the Fourier transform of the kernel of H. Compare [82, p. 311]: "The mysterious nature of the symbol (i.e., in Giraud, 1935) was not eliminated until sixteen years later in the work of Calderon and Zygmund [8,9,11], of Horváth [26,30,31], and of Kohn. There it enters rather naturally as the Fourier transform of the convolution part of the kernel ...".

1.3. $N_0 * N_0 = -\delta$

Above all, distribution theory was created and developed to be applied to problems in $\mathbb{R}^n$, $n > 1$. In [11, p. 901], cf. also [85, A4-08, Théorème], the objective of Calderón and Zygmund is to represent an elliptic differential operator P, homogeneous of degree m, as

$$P = H\Lambda^m,$$

where H is a singular integral operator and Λ a square root of the Laplacean. To define Λ the (M.) Riesz kernel N_0 was introduced by J. Horváth as the vector-valued principal-value distribution defined by

$$N_0 = -\nabla R_1 = c_n \mathrm{vp}\left(\frac{x}{|x|^{n+1}}\right) \in \left(\mathcal{D}'_{L^r}\right)^n, \quad r > 1,\ c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}$$

([8, Theorem 2, p. 129]; [26, p. 28]; [31, p. 434]; [34, (5.1), p. 155]; [35, (3), p. XII.3]; [87, (5), p. 57]; [89, p. 224]; [15, p. 225]; [14, p. 77]; [74, (1.24), p. 280]).

Inspired by the relation

$$\nabla Z_1 * \nabla Z_1 = \delta \tag{7}$$

([61, pp. 157, 158]; [62, pp. 5, 6]) for the gradients (with respect to the Lorentz scalar product on $\mathbb{R}^n$) of the hyperbolic M. Riesz kernels

$$Z_\lambda = \frac{s_+^{\lambda-n}}{\pi^{n/2-1}2^{\lambda-1}\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(1+\frac{\lambda-n}{2}\right)}$$

(where $s_+(x) := (x_1^2 - x_2^2 - \cdots - x_n^2)^{1/2}$ if $x_1 \geqslant 0,\ x_1^2 \geqslant x_2^2 + \cdots + x_n^2$ and 0 elsewhere ([61, p. 156, (2), (3)]; [62, p. 4,(2), (3)]; [75, (II, 3;31), p. 50]; [35, XII. 9]; [36, (0.2), p. 49]; [46, p. 2]; [47, p. 8])) Horváth showed the relation (6) in $\mathbb{R}^n$, i.e.,

$$N_0 * N_0 = -\delta,$$

in [26, pp. 24, 27], compare also [89, p. 224, (2.9)].

A *distributional proof* is immediate: use the above quoted exchange theorem for $\mathcal{D}'_{L^2}$—[75, p. 270, remarque]—and $\mathcal{F}N_0 = -\mathrm{i}\xi/|\xi|$ ([31, (10), p. 434]; [15, p. 225]; [14, (4.8), p. 76]). A second proof alluded to in [31, p. 434], uses the associativity of the convolution (Proposition 1) and the composition formula for the elliptic Riesz kernels:

$$N_0 * N_0 = (-\nabla R_1) * (-\nabla R_1) = \Delta(R_1 * R_1) = -R_{-2} * R_2 = -\delta.$$

(This proof works only if $n \geqslant 3$, since $R_1 * R_1 = R_2$ does not hold if $n \leqslant 2$.) Note that the analogous *proof of* (7) is considerably easier since the composition formula for the hyperbolic Riesz kernels, i.e.,

$$Z_\lambda * Z_\nu = Z_{\lambda+\nu} \tag{8}$$

holds for all $\lambda, \nu \in \mathbb{C}$ (due to the fact that the support of Z_λ lies in the forward light cone $x_1 \geqslant 0,\ x_1^2 \geqslant x_2^2 + \cdots + x_n^2$ [75, (VI,5;19), p. 177] and hence

$$\nabla Z_1 * \nabla Z_1 = \Box(Z_1 * Z_1) = Z_{-2} * Z_2 = Z_0 = \delta$$

by (18) Proposition in [37, p. 192].

Defining

$$\Lambda = \mathrm{i}\nabla N_0 \quad (\text{i.e., } \nabla N_0 = \operatorname{div} N_0)$$

[11, Definition 1, p. 908], we obtain, in fact,

$$\Lambda * \Lambda = (\mathrm{i}\nabla N_0) * (\mathrm{i}\nabla N_0) = -\Delta\delta * (N_0 * N_0) = \Delta\delta.$$

Or, equivalently, $\nabla N_0 = -\Delta R_1 = R_{-1}$, i.e., $\Lambda = \mathrm{i}R_{-1}$ and hence $\Lambda * \Lambda = -R_{-2} = \Delta\delta$. Thus Λ is a (convolution-)square root of $\Delta\delta$ ([31, p. 435]; [11, (24), p. 909]).

An immediate consequence is the "*representation of a function by its derivatives*" $\delta = R_1 * (N_0 * \nabla\delta)$ [87, p. 125]:

$$R_1 * (N_0 * \nabla\delta) = R_{+1} * R_{-1} = R_0 = \delta \tag{9}$$

due to the composition formula of the elliptic Riesz kernels. To prove (9), Stein proceeds by directly manipulating definite integrals in $\mathbb{R}^n$, "but avoids the use of the rather deep theory of the Riesz transforms" [87, p. 125].

Two points remained open:

(i) $\mathrm{vp}(1/x)$ and N_0 are defined only in an ad hoc manner:

$$\text{for } \varphi \in \mathcal{D}, \quad \langle \varphi, N_0 \rangle = c_n \int_{\mathbb{R}^n} \frac{x}{|x|^{n+1}} \varphi(x)\,\mathrm{d}x$$

where the integral has to be understood as a Cauchy principal value. Hence the question arises: Is there a systematic way to define distributions T_λ depending on a parameter $\lambda \in \mathbb{C}$ if for certain λ the distribution T_λ belongs to L^1_{loc}? An answer will be given in Section 2.

(ii) How to define convolution? Whereas we could interpret Eq. (6) $N_0 * N_0 = -\delta$ as a convolution equation in $\mathcal{D}'_{L^p}$, $p > 1$, such an interpretation is not possible in the case of the composition formula of the elliptic Riesz kernels

$$R_\alpha * R_\beta = R_{\alpha+\beta}$$

valid if $\mathrm{Re}(\alpha + \beta) < n$, unless the additional assumptions $\mathrm{Re}\,\alpha < n/2$, $\mathrm{Re}\,\beta < n/2$ ensure $R_\alpha, R_\beta \in \mathcal{D}'_{L^2}$. In fact the convolvability of R_α and R_β is guaranteed under the sole and essential condition $\mathrm{Re}(\alpha + \beta) < n$ if a concept of convolution of two distributions S, T is used which is more general than those treated in [75] or [32], e.g.,

$$(S, T) \in \mathcal{E}' \times \mathcal{D}', \qquad (S, T) \in \mathcal{D}'_{+\Gamma} \times \mathcal{D}'_{+\Gamma},$$
$$(S, T) \in \mathcal{D}'_{L^p} \times \mathcal{D}'_{L^q}, \qquad (S, T) \in \mathcal{O}'_C \times \mathcal{S}'.$$

It will be presented in Section 3.

2. Analytic continuation of distribution-valued functions—hypersingular integrals

2.1. Marcel Riesz's distributions

In [60, (1), p. 2, (5), p. 3] and [61, (2), p. 156], M. Riesz defined three operators $f \mapsto I_\alpha f$, $f \mapsto K_\alpha f$, $f \mapsto J_\lambda f$ (cf. also [62]) by

$$I_\alpha f(x) = \frac{Y(x)}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad \alpha > 0,\ f : \mathbb{R}_+ \to \mathbb{C},$$

$$K_\alpha f(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} |x-\xi|^{\alpha-n} f(\xi)\,\mathrm{d}\xi, \quad \alpha > 0,\ f : \mathbb{R}^n \to \mathbb{C},$$

$$J_\lambda f(x) = \frac{1}{\pi^{n/2-1} 2^{\lambda-1} \Gamma\left(\frac{\lambda}{2}\right) \Gamma\left(1 + \frac{\lambda-n}{2}\right)} \int_{\mathbb{R}^n} s_+(x-\xi)^{\lambda-n} f(\xi)\,\mathrm{d}\xi, \quad \lambda > n-2,$$

depending holomorphically on $\alpha, \lambda \in \mathbb{C}$. He was able to perform the analytic continuation with respect to α, λ and to prove the following:

(i) the identity relations

$$I_0 f = f, \qquad K_0 f = f, \qquad J_0 f = f, \tag{10}$$

(ii) the composition formulae

$$I_\alpha(I_\beta f) = I_{\alpha+\beta} f, \qquad K_\alpha(K_\beta f) = K_{\alpha+\beta} f, \qquad J_\lambda(J_\nu f) = J_{\lambda+\nu} f, \tag{11}$$

and the differentiation formulae

$$\frac{\mathrm{d}}{\mathrm{d}x}(I_\alpha f) = I_{\alpha-1} f, \qquad \Delta(K_\alpha f) = -K_{\alpha-2} f, \qquad \Box(J_\lambda f) = J_{\lambda-2} f. \tag{12}$$

L. Schwartz substituted the operators I_α, K_α, J_λ by the distributions Y_α, R_α, Z_λ defined by

$$\langle \varphi, Y_\alpha \rangle = \big(I_\alpha \check{\varphi}\big)(0), \quad \langle \varphi, R_\alpha \rangle = \big(K_\alpha \check{\varphi}\big)(0), \quad \langle \varphi, Z_\lambda \rangle = \big(J_\lambda \check{\varphi}\big)(0), \quad \varphi \in \mathcal{D}\big(\mathbb{R}^n\big).$$

For $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\lambda > n-2$ the distributions Y_α, R_α, Z_α are locally integrable functions for which (at least formally)

$$Y_\alpha * f = I_\alpha f, \qquad R_\alpha * f = K_\alpha f, \qquad Z_\lambda * f = J_\lambda f.$$

L. Schwartz observed that the functions

$$\mathbb{C} \to \mathcal{D}'_+, \quad \alpha \mapsto Y_\alpha, \qquad \text{and} \qquad \mathbb{C} \to \mathcal{D}'_{+\Gamma}, \quad \lambda \mapsto Z_\lambda,$$

are weakly holomorphic on $\mathbb{C}$, i.e., entire [75, pp. 43, 49].

The values in the points of $-\mathbb{N}_0$ respectively $-2\mathbb{N}_0$ are given by

$$Y_{-l} = \delta^{(l)}, \quad l \in \mathbb{N}_0, \qquad Z_{-2k} = \Box^k \delta, \quad k \in \mathbb{N}_0.$$

Thus the identity relations (10) are consequences. The spaces $\mathcal{D}'_+$ and $\mathcal{D}'_{+\Gamma}$ are convolution algebras, hence $Y_\alpha * Y_\beta$ and $Z_\lambda * Z_\nu$ exist for all $\alpha, \beta, \lambda, \nu \in \mathbb{C}$. The composition formulae (11) yield

$$Y_\alpha * Y_\beta = Y_{\alpha+\beta}, \qquad Z_\lambda * Z_\nu = Z_{\lambda+\nu},$$

which also imply the differentiation formulae (12).

Let me comment on two shortcomings in the procedure outlined above and thereby answer the following questions:

(a) Are there *explicit expressions* for Y_α, R_α, Z_λ in the regions $\operatorname{Re}\alpha \leqslant 0$, $\operatorname{Re}\lambda \leqslant n-2$?
(b) How to define convolvability and how to obtain composition formulae for the *elliptic Riesz kernels* R_α?

We treat question (a) in 2.2 whereas question (b) will be addressed in more detail in Section 3.

2.2. *Meromorphic continuation of the function $\alpha \mapsto Y_\alpha$*

Let us concentrate on question (a). As we indicated in Section 1, L. Schwartz could define the distribution $\mathrm{vp}(1/x)$ ad hoc by setting

$$\mathrm{vp}\,\frac{1}{x} := \lim_{\varepsilon \searrow 0} \frac{Y(|x|-\varepsilon)}{x},$$

since this limit exists in $\mathcal{D}'(\mathbb{R})$. (Applied to test-functions φ this means that the integral

$$\left\langle \varphi, \mathrm{vp}\,\frac{1}{x} \right\rangle = \lim_{\varepsilon \searrow 0} \int\limits_{|x| \geqslant \varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \int\limits_{\mathbb{R}^1} \frac{\varphi(x)}{x}\,\mathrm{d}x$$

exists as a Cauchy principal value.)

In the simplest case of the one-dimensional *Riemann–Liouville distributions*

$$Y_\alpha = \frac{1}{\Gamma(\alpha)} x_+^{\alpha-1}, \quad \operatorname{Re}\alpha > 0,$$

this is not possible since $\lim_{\varepsilon \searrow 0} x^{\alpha-1} Y(x-\varepsilon)$ does not exist in $\mathcal{D}'(\mathbb{R})$ if $\operatorname{Re}\alpha \leqslant 0$.

Hence, let me proceed differently.

For $\operatorname{Re}\alpha > 0$ we have

$$Y_\alpha = \frac{\mathrm{d}}{\mathrm{d}x} Y_{\alpha+1}.$$

Hence, Y_α is defined by this relation for all $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \leqslant 0$. In particular, we have

$$Y_0 = \frac{\mathrm{d}}{\mathrm{d}x} Y = \delta, \qquad Y_{-l} = \delta^{(l)} \quad \text{for } l \in \mathbb{N}_0.$$

In the strip $-1 < \operatorname{Re}\alpha < 0$ we obtain

$$\begin{aligned} Y_\alpha &= \frac{\mathrm{d}}{\mathrm{d}x} Y_{\alpha+1} = \lim_{\varepsilon \searrow 0} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x_+^\alpha}{\Gamma(\alpha+1)} Y(x-\varepsilon) \right) \\ &= \lim_{\varepsilon \searrow 0} \left[\frac{x_+^{\alpha-1}}{\Gamma(\alpha)} Y(x-\varepsilon) - \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \delta_\varepsilon \right] = \lim_{\varepsilon \searrow 0} \frac{1}{\Gamma(\alpha)} \left[x_+^{\alpha-1} Y(x-\varepsilon) - \frac{\varepsilon^\alpha}{\alpha} \delta \right]. \end{aligned}$$

Applied to test functions this expression is identified with Hadamard's finite part $\frac{1}{\Gamma(\alpha)} \mathrm{Pf} \int_0^\infty x^{\alpha-1} \varphi(x)\,\mathrm{d}x$, i.e.,

$$\langle \varphi, Y_\alpha \rangle = \frac{1}{\Gamma(\alpha)} \lim_{\varepsilon \searrow 0} \left(\int\limits_\varepsilon^\infty x^{\alpha-1} \varphi(x)\,\mathrm{d}x - \frac{\varepsilon^\alpha}{\alpha} \varphi(0) \right) \tag{13}$$

for $-1 < \operatorname{Re}\alpha \leqslant 0$, $\alpha \neq 0$.

By iteration, explicit representations of Y_α for all $\alpha \in \mathbb{C}$ are derived easily ([75, p. 42, (II, 2;26)]; [25, (4), (5), p. 25]; [36, (2.2.5) Ejemplo, pp. 85–88]).

The derivation of a representation of Y_α sketched above reveals that Hadamard's finite parts are special approximations "by cutting off" the power function $x_+^{\alpha-1}$. The resulting distributions are called "*pseudo-fonctions monômes*" by L. Schwartz; J. Horváth calls the distributions Y_α "*Riemann–Liouville distributions*" [36, (2.2.8.1), p. 90].

There is another way to represent the analytic continuation of Y_α in the region $\operatorname{Re}\alpha < 0$—relying essentially on M. Riesz's method of analytic continuation [61, p. 154]. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $-1 < \operatorname{Re}\alpha \leqslant 0$, $\alpha \neq 0$. Integrating by parts we get

$$\begin{aligned}
\langle \varphi, Y_\alpha \rangle &= -\langle \varphi', Y_{\alpha+1} \rangle = -\frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha \varphi'(x)\,\mathrm{d}x \\
&= \frac{-1}{\Gamma(\alpha+1)} \left(\int_0^1 x^\alpha \varphi'(x)\,\mathrm{d}x + \int_1^\infty x^\alpha \varphi'(x)\,\mathrm{d}x \right) \\
&= \frac{-1}{\Gamma(\alpha+1)} \Bigg[x^\alpha \big(\varphi(x) - \varphi(0)\big)\big|_0^1 \\
&\quad - \alpha \int_0^1 x^{\alpha-1} \big(\varphi(x) - \varphi(0)\big)\,\mathrm{d}x + x^\alpha \varphi(x)\big|_1^\infty - \alpha \int_1^\infty x^{\alpha-1} \varphi(x)\,\mathrm{d}x \Bigg] \\
&= \frac{1}{\Gamma(\alpha)} \left[\int_0^1 x^{\alpha-1} \big(\varphi(x) - \varphi(0)\big)\,\mathrm{d}x + \int_1^\infty x^{\alpha-1} \varphi(x)\,\mathrm{d}x + \frac{\varphi(0)}{\alpha} \right]. \qquad (14)
\end{aligned}$$

The functions

$$\alpha \mapsto \frac{\varphi(0)}{\alpha}, \qquad \alpha \mapsto \int_0^1 x^{\alpha-1} \big(\varphi(x) - \varphi(0)\big)\,\mathrm{d}x, \qquad \alpha \mapsto \int_1^\infty x^{\alpha-1} \varphi(x)\,\mathrm{d}x$$

and $1/\Gamma(\alpha)$ are holomorphic in $-1 < \operatorname{Re}\alpha < 0$. Thus, formula (14) represents $\langle \varphi, Y_\alpha \rangle$ differently from (13). Obviously (14) can be derived directly from (13) by breaking up the integral, subtracting $\varphi(0)$ and observing that

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^1 x^{\alpha-1} \big[\varphi(x) - \varphi(0)\big]\,\mathrm{d}x = \int_0^1 x^{\alpha-1} \big(\varphi(x) - \varphi(0)\big)\,\mathrm{d}x \quad \text{if } -1 < \operatorname{Re}\alpha.$$

This is the reason why Schwartz wrote: "La partie finie d'une intégrale apparaît ainsi comme le prolongement analytique d'une intégrale ordinaire" [75, p. 39].

For Schwartz the concept of Hadamard's finite part f is related to definite integrals defining the values of distributions applied to test functions: Pf appears as a precautionary measure in treating expressions like

$$\int_0^\infty x^{-3/2} \varphi(x)\,\mathrm{d}x.$$

Furthermore, he discovered that

$$\operatorname{Pf} \int_0^\infty x^{-3/2} \varphi(x)\,\mathrm{d}x = F\left(-\frac{3}{2}\right)$$

where $F(\lambda)$ is the analytic continuation of $\lambda \mapsto \int_0^\infty x^\lambda \varphi(x)\,\mathrm{d}x$, thereby revealing the connexion with M. Riesz's method of analytic continuation.

2.3. Analytic continuation of distribution-valued holomorphic functions

Generalizing an informal discussion of the analytic continuation of holomorphic functions with values in $\mathcal{D}'$ in [22, Chapter I, Appendix A 2.3 (pp. 149–151)], around 1970 J. Dieudonné and J. Horváth considered holomorphic functions $\Lambda \to E$, $\Lambda \subset \mathbb{C}$, E a locally convex, quasi-complete, Hausdorff topological vector space over $\mathbb{C}$ ([13, XVII. 9, pp. 260–268]; [34–36,39,46,49]). Holomorphy is understood as complex differentiability. Specializing some results of Grothendieck [20], J. Horváth proved the following two theorems. The first one states conditions ensuring that weakly holomorphic functions are holomorphic:

Theorem 1 ([34, p. 147]; [36, (1.1.4), p. 57]). *Let F be a quasi-complete, barrelled, locally convex Hausdorff space and F' its dual. If $f : \Lambda \to F'$ is such that for every $y \in F$ the numerical function $\Lambda \to \mathbb{C}$, $\lambda \mapsto \langle y, f(\lambda)\rangle$, is holomorphic, then f is holomorphic if we equip F' with the strong topology $\beta(F', F)$.*

The next theorem allows to define distributions by analytic continuation and is a generalization of the Appendix 2, n° 3 in [22, Chapter I].

Theorem 2 ([49]; [34, p. 147]; [36, (1.3.1), p. 68]). *Let F be a quasi-complete, barrelled, locally convex Hausdorff space and F' its dual equipped with the strong topology $\beta(F', F)$. Let Λ and Λ_1 be two domains in $\mathbb{C}$ such that $\Lambda \subset \Lambda_1$, and $f : \lambda \to F'$ a holomorphic function. Assume that for every $y \in F$ the holomorphic numerical function*

$$\lambda \mapsto \big\langle y, f(\lambda)\big\rangle = g(\lambda, y)$$

has an analytic continuation into Λ_1. Then there exists a holomorphic function $f_1 : \Lambda_1 \to F'$ such that $f_1(\lambda) = f(\lambda)$ for $\lambda \in \Lambda$ and $\langle y, f_1(\lambda)\rangle = g(\lambda, y)$ for $\lambda \in \Lambda_1$ and $y \in F$.

Note that the conditions concerning F' are satisfied in particular if F' is a reflexive locally convex Hausdorff space.

In [34] and [36] J. Horváth applied Theorems 1 and 2 to the analytic continuation of distribution-valued holomorphic functions and showed thereby that the fundamental operations between distributions are preserved:

Theorem 3 ([34, pp. 149, 150]; [36, (2.1.1)–(2.1.11)]). *Let Λ be a domain in $\mathbb{C}$ and Ω an open subset in $\mathbb{R}^n$ and $T : \Lambda \to F'$ a holomorphic function taking its values in one of the spaces $F' = \mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$, $\mathcal{S}'$, $\mathcal{O}'_C$. If Λ_1 is a domain in $\mathbb{C}$ containing Λ and if for every $\varphi \in F$ the numerical function $\lambda \mapsto \langle \varphi, T(\lambda)\rangle$ has an analytic continuation to Λ_1, then T can be extended to a holomorphic function defined in Λ_1 with values in F', and the following properties hold*:

(1) *If A is a closed subset of Ω such that* $\operatorname{supp} T(\lambda) \subset A$ *for $\lambda \in \Lambda$, then* $\operatorname{supp} T(\lambda) \subset A$ *for $\lambda \in \Lambda_1$.*

(2) *If* $S(\lambda) = \partial^\alpha T(\lambda)$ *for* $\lambda \in \Lambda$, $\alpha \in \mathbb{N}_0^n$, *then* $\lambda \mapsto S(\lambda)$ *can also be extended to a holomorphic function in* Λ_1 *and* $S(\lambda) = \partial^\alpha T(\lambda)$ *for* $\lambda \in \Lambda_1$.

(3) *If* $S(\lambda)$ *is the Fourier transform of* $T(\lambda) \in \mathcal{S}'$ *for* $\lambda \in \Lambda$, *then* $\lambda \mapsto S(\lambda)$ *has an analytic continuation into* Λ_1 *and* $S(\lambda) = \mathcal{F}(T(\lambda))$ *for* $\lambda \in \Lambda_1$.

(4) *If* $\lambda \mapsto T(\lambda)$ *is holomorphic in* Λ *with values in* $\mathcal{D}'(\Omega)$ (*respectively in* $\mathcal{S}'$) *and if* $\lambda \mapsto \alpha(\lambda)$ *is holomorphic in* Λ *with values in* $\mathcal{E}'(\Omega)$ (*respectively in* $\mathcal{O}_M$), *then* $\lambda \mapsto S(\lambda) = \alpha(\lambda)T(\lambda)$ *is holomorphic in* Λ *with values in* $\mathcal{D}'(\Omega)$ (*respectively in* $\mathcal{S}'$). *Furthermore if both* $\lambda \mapsto T(\lambda)$ *and* $\lambda \mapsto \alpha(\lambda)$ *have analytic continuations into* Λ_1, *then* $\lambda \mapsto S(\lambda)$ *has an analytic continuation into* Λ_1 *and* $S(\lambda) = \alpha(\lambda)T(\lambda)$ *for* $\lambda \in \Lambda_1$.

(5) *Let* $\lambda \mapsto S(\lambda)$ *be holomorphic in* Λ *with values in* $\mathcal{D}'$ *and* $\lambda \mapsto T(\lambda)$ *holomorphic in* Λ *with values in* $\mathcal{D}'$. *Assume that there exist two closed subsets* A *and* B *of* $\mathbb{R}^n$ *such that* $\operatorname{supp} S(\lambda) \subset A$ *and* $\operatorname{supp} T(\lambda) \subset B$ *for* $\lambda \in \Lambda$ *and that* A *and* B *verify the condition* (Σ) *for every compact subset* K *in* $\mathbb{R}^n$, *the set* $(A \times B) \cap K^\Delta$ *is compact in* $\mathbb{R}^{2n}$, $K^\Delta = \{(x, y) \in \mathbb{R}^{2n} / x + y \in K\}$.
Then $\lambda \mapsto R(\lambda) = S(\lambda) * T(\lambda)$ *is holomorphic in* Λ *with values in* $\mathcal{D}'$. *Furthermore if both* $\lambda \mapsto S(\lambda)$ *and* $\lambda \mapsto T(\lambda)$ *have analytic continuations into* Λ_1, *then* $\lambda \mapsto R(\lambda)$ *has an analytic continuation into* Λ_1, *and* $R(\lambda) = S(\lambda) * T(\lambda)$ *for* $\lambda \in \Lambda_1$.

Let me refer to *three applications* of Theorem 3(5):

(a) Schwartz's derivation of the *fundamental solution*

$$\frac{(-1)^{k+1}\lambda^{\frac{n}{4}-\frac{k}{2}}}{2^{\frac{n}{2}+k}\pi^{\frac{n}{2}-1}(k-1)!}|x|^{k-\frac{n}{2}}N_{\frac{n}{2}-k}\big(\sqrt{\lambda}\,|x|\big)$$

of the *iterated Helmholtz operator* $(\Delta + \lambda)^k$, $\lambda > 0$, $k \in \mathbb{N}$, by analytic continuation with respect to $\lambda \in \mathbb{C}\backslash(-\mathrm{i}\mathbb{R}_+)$ of the unique tempered fundamental solution

$$\frac{(-1)^{k}\lambda^{\frac{n}{4}-\frac{k}{2}}}{2^{\frac{n}{2}+k-1}\pi^{\frac{n}{2}}(k-1)!}|x|^{k-\frac{n}{2}}K_{\frac{n}{2}-k}\big(\sqrt{\lambda}\,|x|\big)$$

of the *iterated metaharmonic operator* $(\Delta - \lambda)^k$, $\lambda > 0$, $k \in \mathbb{N}$ [75, pp. 286, 287].

(b) The derivation of the *fundamental solution*

$$\frac{1}{2\pi}\frac{Y(t-|x|)}{\sqrt{t^2-|x|^2}} = Z_2$$

of the *two-dimensional wave operator* $\partial_t^2 - \Delta$ by analytic continuation of the fundamental solution $-1/(4\pi|x|)$ of Δ_3 (with respect to a suitably introduced parameter). This derivation renders precise a procedure which was used around 1900 without any mathematical rigour. Obviously this example has only pedagogical character since the result is contained in the composition formula for M. Riesz's hyperbolic kernels, i.e.,

$$Z_{-2} * Z_2 = Z_0, \qquad \Box Z_2 = \big(\partial_t^2 - \Delta\big)Z_2 = \delta.$$

(c) A more demanding application of the method of analytic continuation is the derivation of the *fundamental solution of homogeneous, hyperbolic operators of fourth degree* of the form $\sum_{j,k=1}^3 c_{jk}\partial_j^2\partial_k^2$ from that of elliptic operators of this type [94,95].

2.4. Finite parts as distributions

Using the theory of analytic continuation of holomorphic distribution-valued functions, the sign "Pf" can be dispensed with, when dealing with the analytic continuation into a particular value λ_0. Hence, e.g., $\langle\varphi, x_+^{-3/2}\rangle = \lim_{\lambda\to -3/2}\langle\varphi, x_+^{\lambda}\rangle$ in contrast to Schwartz's symbolism. L. Schwartz uses the sign "Pf" also in a second meaning, e.g., for $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\left\langle\varphi, \operatorname{Pf}|x|^{-n-2k}\right\rangle = \lim_{z\to 0}\left(\left\langle\varphi, \operatorname{Pf}|x|^{-n-2k-z}\right\rangle - \frac{A}{z}\right) \tag{15}$$

[75, (II,3;7), p. 45]. On the right-hand side, we write $\langle\varphi, |x|^{-n-2k-z}\rangle$, since the distribution-valued function $z \mapsto |x|^{-n-2k-z}$ is holomorphic in $|z| < 2$, $z \neq 0$.

It is this second meaning which led to the new concept of the finite part:

Definition 1 (*Finite part of distribution-valued functions* ([13, XVII. 9, pp. 261, 262]; [36, (2.2.1), p. 84]; [46, pp. 23, 24])). Let Λ be a domain in $\mathbb{C}$, Ω an open subset of $\mathbb{R}^n$ and $T : \Lambda \to F'$, $\lambda \mapsto T(\lambda)$ a meromorphic function with values in one of the spaces $F' = \mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$, $\mathcal{O}'_C$, $\mathcal{S}'$. If $\lambda_0 \in \Lambda$ is a pole of order m of T, then the finite part $\operatorname{Pf}_{\lambda=\lambda_0} T(\lambda)$ is the value of the regular part of the Laurent series of T at λ_0. Hence, if

$$T(\lambda) = \frac{S_m}{(\lambda-\lambda_0)^m} + \frac{S_{m-1}}{(\lambda-\lambda_0)^{m-1}} + \cdots + \frac{S_1}{\lambda-\lambda_0} + S(\lambda), \quad S_j \in F',$$

and $S : \Lambda \to F'$ the regular part of the Laurent series of T at λ_0, then

$$\operatorname*{Pf}_{\lambda=\lambda_0} T(\lambda) = S(\lambda_0) = \lim_{\lambda\to\lambda_0}\left(T(\lambda) - \sum_{j=0}^{m} \frac{S_j}{(\lambda-\lambda_0)^j}\right).$$

This definition is the abstract version of Schwartz's equation (15): Pf is not the finite value of a definite integral but a distribution.

Example. Equation (14) yields for $\operatorname{Re}\alpha > 0$

$$\left\langle\varphi, x_+^{\alpha-1}\right\rangle = \int_0^1 x^{\alpha-1}\big(\varphi(x)-\varphi(0)\big)\,\mathrm{d}x + \int_1^\infty x^{\alpha-1}\varphi(x)\,\mathrm{d}x + \frac{\varphi(0)}{\alpha}.$$

The first term possesses an analytic continuation into the strip $-1 < \operatorname{Re}\alpha \leqslant 0$, the second one is an entire function of α and the third one has a simple pole at $\alpha = 0$. Therefore, by definition,

$$\left\langle\varphi, \operatorname*{Pf}_{\alpha=0} x_+^{\alpha-1}\right\rangle = \int_0^1 \frac{\varphi(x)-\varphi(0)}{x}\,\mathrm{d}x + \int_1^\infty \varphi(x)\frac{\mathrm{d}x}{x}.$$

This was known long before 1970 ([25, (5), p. 25]; [22, (3), p. 48]) but the precise definition of Pf was given in 1970 by Dieudonné and divulged by J. Horváth [34,36]. Later on the definition was accepted as the natural one [70, § 15, p. 39].

To avoid the treatment of finite parts, M. Riesz divided $x_+^{\alpha-1}$ by $\Gamma(\alpha)$. Then the residue of $\alpha \mapsto x_+^{\alpha-1}$ at 0, $\mathrm{Res}_{\alpha=0} x_+^{\alpha-1} = \delta$, disappears since $\lim_{\alpha\to 0} \varphi(0)/\Gamma(\alpha) = 0$. This is the reason for $\alpha \mapsto x_+^{\alpha-1}$ to be meromorphic in $\mathbb{C}\backslash(-\mathbb{N}_0)$. However, the Riemann–Liouville distribution $Y_\alpha = x_+^{\alpha-1}/\Gamma(\alpha)$ is entire.

The behaviour of the finite part and the residue with respect to some of the fundamental operations is studied in the following

Theorem 4 [36, Proposiciónes (2.2.6), p. 88, (2.2.9), p. 91 and (2.2.11), p. 92]. *Let Λ be a domain in $\mathbb{C}$, Ω an open subset of $\mathbb{R}^n$ and $T : \Lambda \to F'$ a meromorphic distribution-valued function with values in $F' = \mathcal{D}'(\Omega)$ or $\mathcal{S}'$ and with a simple pole at $\lambda_0 \in \Lambda$.*

(1) *Assume that there exists a closed subset A of Ω such that* $\operatorname{supp} T(\lambda) \subset A$ *for* $\Lambda\backslash\{\lambda_0\}$. *Then,* $\operatorname{supp} \mathrm{Pf}_{\lambda=\lambda_0} T(\lambda) \subset A$.
(2) $\partial^\alpha(\mathrm{Pf}_{\lambda=\lambda_0} T(\lambda)) = \mathrm{Pf}_{\lambda=\lambda_0}(\partial^\alpha T(\lambda))$ *for* $\alpha \in \mathbb{N}_0^n$.
(3) $\mathcal{F}(\mathrm{Pf}_{\lambda=\lambda_0} T(\lambda)) = \mathrm{Pf}_{\lambda=\lambda_0}(\mathcal{F}T(\lambda))$.
(4) $\partial^\alpha(\mathrm{Res}_{\lambda=\lambda_0} T(\lambda)) = \mathrm{Res}_{\lambda=\lambda_0}(\partial^\alpha T(\lambda))$ *for* $\alpha \in \mathbb{N}_0^n$.
(5) $\mathcal{F}(\mathrm{Res}_{\lambda=\lambda_0} T(\lambda)) = \mathrm{Res}_{\lambda=\lambda_0} \mathcal{F}T(\lambda)$.

Concerning the multiplication of distribution-valued meromorphic functions with meromorphic multipliers let us state.

Proposition 2. *Let Λ be a domain in $\mathbb{C}, \Omega$ an open subset of $\mathbb{R}^n$ and $T : \Lambda \to F'$ a distribution-valued meromorphic function with values in $F' = \mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$, $\mathcal{S}'$, $\mathcal{O}'_C$ and with a simple pole at $\lambda \in \Lambda$. Then the following holds*:

(i) $\mathrm{Pf}_{\lambda=\lambda_0}\alpha(\lambda)T(\lambda) = \alpha(\lambda_0)\mathrm{Pf}_{\lambda=\lambda_0} T(\lambda) + \partial_\lambda\alpha(\lambda_0)\,\mathrm{Res}_{\lambda=\lambda_0} T(\lambda)$ *if $\alpha : \Lambda \to E$ is holomorphic in $\lambda_0 \in \Lambda$ with values in $E = \mathcal{E}(\Omega)$ or $\mathcal{O}_M$* [42, Lemma 1, p. 431].
(ii) *Let $\alpha : \Lambda \to E$ be a meromorphic function with values in $E = \mathcal{E}(\Omega)$ or $\mathcal{O}_M$ and with a simple pole at $\lambda_0 \in \Lambda$. Then*

$$\operatorname*{Res}_{\lambda=\lambda_0} \alpha(\lambda)T(\lambda) = \operatorname*{Res}_{\lambda=\lambda_0} \alpha(\lambda) \operatorname*{Pf}_{\lambda=\lambda_0} T(\lambda) + \operatorname*{Pf}_{\lambda=\lambda_0} \alpha(\lambda) \operatorname*{Res}_{\lambda=\lambda_0} T(\lambda)$$

[68, p. 679], *and*

$$\operatorname*{Pf}_{\lambda=\lambda_0} \alpha(\lambda)T(\lambda) = \operatorname*{Res}_{\lambda=\lambda_0} \alpha(\lambda) \operatorname*{Pf}_{\lambda=\lambda_0} \partial_\lambda T(\lambda) + \operatorname*{Pf}_{\lambda=\lambda_0} \partial_\lambda\alpha(\lambda) \operatorname*{Res}_{\lambda=\lambda_0} T(\lambda) + \operatorname*{Pf}_{\lambda=\lambda_0} \alpha(\lambda) \operatorname*{Pf}_{\lambda=\lambda_0} T(\lambda)$$

[92, p. 478].

Remarks.

(1) L. Schwartz proves [75, p. 42, (II,2;27), (II,2;28)]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x_+^\alpha\right) = \alpha x_+^{\alpha-1} \quad \text{if } \alpha \in \mathbb{C}\backslash(-\mathbb{N}_0) \quad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{Pf}x_+^{-k}\right) = \mathrm{Pf}\left(-kx_+^{-k-1}\right) + (-1)^k \frac{1}{k!}\delta^{(k)} \quad \text{if } k \in \mathbb{N}_0$$

which seemingly contradicts Theorem 4(2).

At several instances ([34, p. 151, Remark]; [36, (2.2.10.1), p. 92]; [45, p. 455]) J. Horváth ravelled out this "contradiction": one has to interpret $\mathrm{Pf}(-kx_+^{-k-1})$ as $\mathrm{Pf}_{\lambda=-k}(-kx_+^{\lambda-1})$ and to use Proposition 2(i).

(2) The equation on p. 151 in [34] (2nd line from below) is a special case of Proposition 2(i).

(3) Defining the Bessel function of order $\lambda \in \mathbb{C}$ by

$$J_\lambda(x_+) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\lambda+k+1)} \left(\frac{x_+}{2}\right)^{2k+\lambda}$$

for $\mathrm{Re}\,\lambda > 0$, we obtain by analytic continuation

$$J_{-m}(x_+) = 2^m \sum_{0 \leqslant 2j < m} \frac{(-1)^{m-j-1}}{2^{2j} j!(m-2j-1)!} \delta^{(m-2j-1)} + (-1)^m J_m(x_+)$$

[52, (1.12), p. 188].

(4) As an application of the second equation in Proposition 2(ii) in [92, Satz 7, p. 478], the convolution $|x|^{-n} * |x|^{-n}$ is computed as follows:

$$|x|^{-n} * |x|^{-n} = \frac{4\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} |x|^{-n} \log|x| + \frac{\pi^n}{\Gamma\left(\frac{n}{2}\right)^2} \left(\psi'\left(\frac{n}{2}\right) - \frac{\pi^2}{6}\right)\delta$$

($|x|^{-n}$ means $\mathrm{Pf}_{\lambda=-n}|x|^\lambda$ and $|x|^{-n}\log|x|$ means $\mathrm{Pf}_{\lambda=-n}|x|^\lambda \log|x|$).

2.5. *Meromorphic continuation of Euclidean pseudofunctions*

The main purpose of this section is to define rigorously distributions like $|x|^\lambda$, R_α, N_λ. Since these distributions are singular at the origin, and homogeneous outside the origin, J. Horváth calls them Euclidean pseudofunctions and his task is the analytic continuation of Euclidean pseudofunctions. His main result is

Theorem 5 ([34, Theorem 3.1, p. 152]; [35, XII. 3-4]; [39, Theorem 1, p. 173]; [40, 8-03, 8-04]; [46, pp. 24–26]). *Let $k \in L^1(\mathbb{S}_{n-1})$ and the moments of order $\alpha \in \mathbb{N}_0^n$ be defined by*

$$M_\alpha = \int_{\mathbb{S}_{n-1}} \omega^\alpha k(\omega)\,\mathrm{d}s(\omega),$$

wherein $\mathrm{d}s$ *means the surface measure on the $(n-1)$-dimensional unit sphere $\mathbb{S}_{n-1}$. For $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > -n$, the distribution K_λ is defined as the $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-function $k(x/|x|)\cdot|x|^\lambda$. Let m be a positive integer or $+\infty$. Then the distribution K_λ is a holomorphic function of λ with values in $\mathcal{S}'^m$ in the half-plane $\mathrm{Re}\,\lambda > -n$. It has an analytic continuation as a holomorphic function with values in $\mathcal{S}'^m$ into the half-plane $\mathrm{Re}\,\lambda > -n-m$, with the*

exception of the points $\lambda = -n - j$ $(j \in \mathbb{N}_0,\ 0 \leqslant j \leqslant m)$ *which are simple poles* (*if* $M_\alpha \neq 0$ *for* $|\alpha| = j$) *with residue*

$$(-1)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} M_\alpha \partial^\alpha \delta.$$

Remarks.

(1) Earlier the analytic continuation of K_λ was performed in [25, §2, pp. 35–44] and in [22, Chapter III, 3.2, pp. 297–303; 3.3, pp. 303–309; 3.4, pp. 309–312].
(2) A similar theorem is in [70, Theorem 15.1, p. 39]: $k \in \mathcal{C}^\infty(\mathbb{S}_{n-1})$).
(3) A special case of Theorem 5 is the analytic continuation of the locally integrable function $x \mapsto x_+^{\alpha-1}$, $\operatorname{Re}\alpha > 0$ into the half-plane $-1 < \operatorname{Re}\alpha$, $\alpha \neq 0$ which was given in Eq. (14); compare also the example following Definition 1.
(4) Usually the function k depends also on λ.

Example 1 (*The elliptic M. Riesz kernels* R_α ([31, p. 433]; [34, p. 154]; [39, pp. 179–181]; [40, 8-04, 8-05]; [41, p. 22]; [45, pp. 456–457]; [46, pp. 26–28])). For $\operatorname{Re}\alpha > 0$, R_α is defined by the locally integrable function

$$\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha-n}.$$

For $\alpha \in \mathbb{C}$, $\alpha \neq n + 2k$ $(k \in \mathbb{N}_0)$, R_α is the analytic continuation of the distribution-valued function $\alpha \mapsto R_\alpha$. For $\alpha = n + 2k$, $k \in \mathbb{N}_0$, R_α is the finite part of the meromorphic distribution-valued function R_α. Explicit expressions at particular values $\alpha \in \mathbb{C}$ are:

$$R_{-2k} = (-\Delta)^k \delta, \quad k \in \mathbb{N}_0,$$
$$R_{n+2k} = \frac{(-1)^k |x|^{2k}}{\pi^{n/2} 2^{n+2k} \Gamma\left(\frac{n}{2} + k\right) k!} \left(2 \log \frac{2}{|x|} + \psi(k+1) + \psi\left(k + \frac{n}{2}\right)\right).$$

(Note the slightly different constant in formula (2) [31, p. 433].)

Example 2 (*The higher Hilbert–Riesz kernels* N_λ). They were introduced in [31, p. 433], and more closely investigated in [34, pp. 155, 156]; [39, pp. 181–184]; [40, p. 8-05] and [41, p. 22]. For $\operatorname{Re}\lambda > 0$ N_λ is defined by the locally integrable function

$$\frac{\Gamma\left(\frac{n-\lambda+1}{2}\right)}{2^\lambda \pi^{n/2} \Gamma\left(\frac{\lambda+1}{2}\right)} \frac{x}{|x|} |x|^{\lambda-n}.$$

For $\lambda \in \mathbb{C}$, $\lambda \neq n+2k+1$, $k \in \mathbb{N}_0$, N_λ is the analytic continuation of the distribution-valued function $\lambda \mapsto N_\lambda$. For $\lambda = n + 2k + 1$, $k \in \mathbb{N}_0$, N_λ is the finite part of the meromorphic distribution-valued function N_λ. We get $N_\lambda = -\nabla R_{\lambda+1}$, and therefore $N_{-2k-1} = (-1)^{k+1} \nabla(\Delta^k \delta)$, $k \in \mathbb{N}_0$.

3. General convolution

3.1. The definition

J. Horváth recognized that the "support-restricting" conditions to ensure convolvability of two distributions (i.e., condition (Σ) in Theorem 3(5) ([75, p. 170]; [32, p. 383])) are not appropriate to convolve kernels of elliptic singular and hypersingular integrals. Thus, in his earlier papers [28,30,31] he used the $\mathcal{D}'_{L^p} \times \mathcal{D}'_{L^q}$-convolution. But even the framework of $\mathcal{D}'_{L^p} \times \mathcal{D}'_{L^q}$ does not yield the validity of the composition formula of the elliptic M. Riesz kernels $R_\alpha * R_\beta = R_{\alpha+\beta}$ under the sole condition $\mathrm{Re}(\alpha+\beta) < n$ as we remarked in 1.3.

Hence he looked for a general definition which is as symmetric as possible. His starting point is the very symmetric definition of the convolution of two Radon measures given by Bourbaki ([4, Chapitre VIII, § 1, n° 1, Définition 1, pp. 120, 121]; [12, 14.5, pp. 246–248]):

Two *Radon measures* $\mu, \nu \in \mathcal{K}' = \mathcal{D}'^0 = \mathcal{M}$ are said to be convolvable

$$\text{iff } \forall \varphi \in \mathcal{K}(\mathbb{R}^n): \quad \varphi^\Delta(\mu \otimes \nu) \in \mathcal{M}^1(\mathbb{R}^{2n}).$$

Herein, $\mathcal{K}$ denotes the space of continuous functions with compact support, $\mathcal{K}'$ its dual, the space of Radon measures. $\mathcal{M}^1$, the dual of $\mathcal{C}_0$, is the space of integrable measures which usually are called bounded measures. φ^Δ means the function $\varphi^\Delta(x,y) := \varphi(x+y)$ if $\varphi \in \mathcal{K}(\mathbb{R}^n)$. The convolution $\mu * \nu$ is then defined by

$$\mathcal{K}(\mathbb{R}^n) \to \mathbb{C}, \quad \varphi \mapsto \underset{\mathcal{K}}{\langle} \varphi, \mu * \nu \underset{\mathcal{M}}{\rangle} = \int_{\mathbb{R}^{2n}} \varphi^\Delta(\mu \otimes \nu).$$

Interpreting the integral as application to the test function 1 we could also define:

$$\underset{\mathcal{K}}{\langle} \varphi, \mu * \nu \underset{\mathcal{M}}{\rangle} = \underset{\mathcal{BC}_\beta}{\langle} 1, \varphi^\Delta(\mu \otimes \nu) \underset{\mathcal{M}^1}{\rangle},$$

wherein $\mathcal{BC}_\beta$ is the space of bounded and continuous functions on $\mathbb{R}^n$ equipped with the strict topology (whose dual is also $\mathcal{M}^1$).

The analogues of the spaces

$$\mathcal{K} \subset \mathcal{C}_0 \subset \mathcal{BC}_\beta, \qquad \mathcal{M}^1 \subset \mathcal{M},$$

in the distributional setting are:

$$\mathcal{D} \subset \dot{\mathcal{B}} \subset \mathcal{B}_c, \qquad \mathcal{D}'_{L^1} \subset \mathcal{D}'.$$

L. Schwartz defined $\mathcal{B}_c$ as the space

$$\mathcal{D}_{L^\infty} = \bigcap_{k=0}^{\infty} L_k^\infty = \bigcap_{k=0}^{\infty} W^{k,\infty},$$

equipped with the finest locally convex topology which induces on the bounded subsets of $\mathcal{D}_{L^\infty}$ the same topology as $\mathcal{E}$ ([75, p. 203]; [77, p. 100]).

Definition 2 (*General convolution* ([37, (1), (2) Définition, p. 185]; [38, p. 36]; [39, pp. 184, 185]; [40, p. 8-08]; [41, pp. 15–17]; [45, p. 453]; [46, p. 6]; [48, pp. 81, 82])). Let S and T be distributions on $\mathbb{R}^n$. S and T fulfill *condition* (Γ)

$$\text{iff } \forall \varphi \in \mathcal{D}(\mathbb{R}^n): \quad \varphi^\Delta(S \otimes T) \in \mathcal{D}'_{L^1}(\mathbb{R}^{2n}).$$

If S and T fulfill condition (Γ) the convolution $S * T$ is defined by

$$\underset{\mathcal{D}}{\langle}\, \varphi, S * T \,\underset{\mathcal{D}'}{\rangle} = \underset{\mathcal{B}_c}{\langle}\, 1, \varphi^{\Delta}(S \otimes T) \,\underset{\mathcal{D}'_{L^1}}{\rangle} \qquad \text{for } \varphi \in \mathcal{D}\big(\mathbb{R}^n\big).$$

Remark. Note that condition (Σ) in Theorem 3(5), concerning $\operatorname{supp} S$ and $\operatorname{supp} T$, is equivalent with:

$$\forall \varphi \in \mathcal{D}\big(\mathbb{R}^n\big): \quad \varphi^{\Delta}(S \otimes T) \in \mathcal{E}'\big(\mathbb{R}^{2n}\big).$$

Due to $\mathcal{E}' \subset \mathcal{D}'_{L^1}$ condition (Σ) implies (Γ). Therefore,

$$\underset{\mathcal{D}}{\langle}\, \varphi, S * T \,\underset{\mathcal{D}'}{\rangle} = \underset{\mathcal{E}}{\langle}\, 1, \varphi^{\Delta}(S \otimes T) \,\underset{\mathcal{E}'}{\rangle}.$$

Immediately after the completion of his paper on the symmetric definition of the convolution [37], J. Horváth noticed that his new definition had in fact been discovered by L. Schwartz 20 years earlier [80, exposé n° 22, p. 2] (cf. [83, Definition 1, p. 22]).

L. Schwartz presented yet another definition of the convolution of 2 distributions:

Definition 3 ([80, exposé n° 22, p. 1]; [83, Definition 2, p. 23]). Let $S, T \in \mathcal{D}'$. Then the convolution $S * T$ of S and T is defined by

$$\underset{\mathcal{D}}{\langle}\, \varphi, S * T \,\underset{\mathcal{D}'}{\rangle} = \underset{\mathcal{B}_c}{\langle}\, 1, \big(\varphi * \check{S}\big)T \,\underset{\mathcal{D}'_{L^1}}{\rangle}$$

if the condition

$$\forall \varphi \in \mathcal{D}: \quad \big(\varphi * \check{S}\big)T \in \mathcal{D}'_{L^1}\big(\mathbb{R}^n\big)$$

is satisfied.

The *equivalence of Definitions* 2 *and* 3 and also with a variant of Definition 3 with the roles of S and T reversed, was shown by R. Shiraishi in [83, Theorem 2, p. 24] and [73, pp. 194, 195].

Let me add a little *remark concerning Definition* 2.

L. Schwartz writes [80, exposé n° 22, p. 2]: "Cette intégrale (i.e., $\langle_{\mathcal{B}_c} 1, \varphi^{\Delta}(S \otimes T)\rangle_{\mathcal{D}'_{L^1}}$) définit une forme linéaire continue sur $\mathcal{D}$, en vertu du théorème de Banach–Steinhaus, puisqu'elle est limite d'intégrales

$$\underset{\mathcal{B}_c}{\langle}\, 1, \alpha(x)\beta(y)\varphi^{\Delta}(S \otimes T) \,\underset{\mathcal{D}'_{L^1}}{\rangle}$$

(ou α (respectivement β) sont à support compact et convergent vers 1 au sens de $\mathcal{E}$ en restant bornées dans $\mathcal{B}$) définissant des formes linéaires continues."

This means that convolvability in the sense of Definitions 2 and 3 is equivalent with convolvability in the sense of

Definition 4 [91, p. 62]. $S, T \in \mathcal{D}'(\mathbb{R}^n)$ are said to be convolvable iff for every sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^{2n})$ tending to 1 in $\mathcal{E}$ and bounded in $\mathcal{B} = \mathcal{D}_{L^\infty}$ and for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ the limit $\lim_{k\to\infty}\langle \eta_k, \varphi^{\Delta}(S \otimes T)\rangle$ exists.

A further proof of the equivalence of Definitions 2 and 4 was given in [17].

3.2. Convolution and analytic continuation

The general definition of convolution (Definitions 2, 3) allows a slight generalization of J. Horváth's statements on the analyticity and on the analytic continuation of the convolution of two distribution-valued functions in Theorem 3(5):

Proposition 3. *Let Λ be a domain in $\mathbb{C}$ and S, T distribution-valued holomorphic functions $S, T : \Lambda \to \mathcal{D}'$ fulfilling the following condition (Γ_Λ) $\forall \varphi \in \mathcal{D}$: the mapping $\Lambda \to \mathcal{D}'_{L^1}$, $\lambda \mapsto (\varphi * \check{S}(\lambda))T(\lambda)$ is weakly continuous.*

(i) *Then, the distribution-valued convolution mapping*

$$R : \Lambda \to \mathcal{D}', \quad \lambda \mapsto S(\lambda) * T(\lambda),$$

is holomorphic.

(ii) *If Λ_1 is a domain in $\mathbb{C}$ containing Λ and if S, T have analytic continuations defined in Λ_1 fulfilling the condition (Γ_{Λ_1}) where Λ_1 replaces Λ, then R has an analytic continuation into Λ_1 and $R(\lambda) = S(\lambda) * T(\lambda)$ for $\lambda \in \Lambda_1$.*

Proof. Let us consider

$$\begin{tikzcd} \Lambda \arrow[r, "R"] \arrow[dr] & \mathcal{D}'_{L^1} \arrow[d, hook, "j"] & \lambda \arrow[r, mapsto] & \big(\varphi * \check{S}(\lambda)\big)T(\lambda) \\ & \mathcal{D}' & & \end{tikzcd}$$

$j \circ R$ is holomorphic due to Theorem 3(4) and due to the holomorphy of $\Lambda \to \mathcal{E}$, $\lambda \mapsto \varphi * \check{S}(\lambda)$. The holomorphy of R is implied by [36, (1.2.6) Corollaire, p. 64].

(ii) By the first part, the function $\Lambda_1 \to \mathcal{D}'$, $\lambda \mapsto S(\lambda) * T(\lambda)$ is holomorphic. Due to the representation $\langle \varphi, R(\lambda)\rangle = \langle 1, (\varphi * \check{S}(\lambda))T(\lambda)\rangle$ and Theorem 3(4) R has an analytic continuation into Λ_1. $\square$

Remarks.

(1) I conjecture that condition (Γ_Λ) is equivalent with the more symmetrical one: $\forall \varphi \in \mathcal{D}$: the mapping $\Lambda \to \mathcal{D}'_{L^1}(\mathbb{R}^{2n})$, $\lambda \mapsto \varphi^\Delta(S(\lambda) \otimes T(\lambda))$ is weakly continuous.

(2) Perhaps condition (Γ_Λ) is a consequence of the sole convolvability condition: $\forall \lambda \in \Lambda$, $\forall \varphi \in \mathcal{D}$: $(\varphi * \check{S}(\lambda))T(\lambda) \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$?

(3) Proposition (2.1.11) and Remark (2.1.12) in [36] are immediate since for $S \in \mathcal{E}'$ or $\mathcal{O}'_C$ and $T \in \mathcal{D}'$ or $\mathcal{S}'$, $\varphi \in \mathcal{D}$, the mappings $\Lambda \to \mathcal{E}'$, $\lambda \mapsto (\varphi * \check{S}(\lambda))T(\lambda)$ and $\Lambda \to \mathcal{O}'_C$, $\lambda \mapsto (\varphi * \check{S}(\lambda))T(\lambda)$ are holomorphic and hence also $\Lambda \to \mathbb{C}$, $\lambda \mapsto \langle 1, (\varphi * \check{S}(\lambda))T(\lambda)\rangle = \langle \varphi, S(\lambda) * T(\lambda)\rangle$.

For the sake of completeness let us supplement the analyticity properties of distribution-valued functions with values in $\mathcal{D}'_{L^p}$ and $\mathcal{D}'_{L^q}$:

Proposition 4. *Let Λ be a domain in $\mathbb{C}$ and S, T distribution-valued holomorphic functions $S : \Lambda \to \mathcal{D}'_{L^p}$, $T : \Lambda \to \mathcal{D}'_{L^q}$ with $1/p + 1/q \geqslant 1$. Then*

(i) *the distribution-valued convolution mapping*

$$R : \Lambda \to \mathcal{D}'_{L^r}, \quad \lambda \mapsto S(\lambda) * T(\lambda), \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q},$$

is holomorphic.

(ii) *If Λ_1 is a domain in $\mathbb{C}$ containing Λ and if S, T have analytic continuations defined in Λ_1, then R has an analytic continuation into Λ_1 and $R(\lambda) = S(\lambda) * T(\lambda)$ for $\lambda \in \Lambda_1$.*

Proof. Condition (Γ_Λ) of Proposition 3 is satisfied since $\Lambda \to \mathcal{D}_{L^p}$, $\lambda \mapsto \varphi * \check{S}(\lambda)$ and hence $\Lambda \to \mathcal{D}'_{L^1}$, $\lambda \mapsto (\varphi * \check{S}(\lambda))T(\lambda)$ are continuous by Théorème XXVI in [75, p. 203]. □

3.3. *Associativity of the convolution*

Before we turn to the announced applications of the general convolution let me quote J. Horváth's results on the *associativity of the convolution*.

Definition 5 ([37, p. 190]; [38, p. 38]; [40, pp. 8–10]; [41, p. 18]; [46, p. 6]; [48, p. 83]). If φ is a function defined on $\mathbb{R}^n$ denote by $\varphi^{h\Delta}$ the function $(x_1, \ldots, x_h) \mapsto \varphi(x_1 + \cdots + x_h)$ on $\mathbb{R}^{hn}$. The finite sequence $(T_1, \ldots, T_h)$ of distributions on $\mathbb{R}^n$ is *convolvable* if $\varphi^{h\Delta}(T_1 \otimes \cdots \otimes T_h) \in \mathcal{D}'_{L^1}(\mathbb{R}^{h\cdot n})$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$. One then defines $*_{j=1}^h T_j = T_1 * \cdots * T_h$ by

$$\underset{\mathcal{D}}{\langle} \varphi, T_1 * \cdots * T_h \underset{\mathcal{D}'}{\rangle} = \underset{\mathcal{B}_c}{\langle} 1, \varphi^{h\Delta}(T_1 \otimes \cdots \otimes T_h) \underset{\mathcal{D}'_{L^1}}{\rangle}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Proposition 5 ([37, (15) Proposition, p. 191]; [83, Lemma 1, p. 28]; [73, p. 195]; [48, Proposition 8, p. 84]). *If the sequence (R, S, T) of distributions is convolvable and $R \neq 0$, then S and T are convolvable, R and $S * T$ are convolvable and $R * (S * T) = R * S * T$.*

Remark. Proposition 1 on the associativity of the convolution of distributions in $\mathcal{D}'_{L^p}$, $\mathcal{D}'_{L^q}$, $\mathcal{D}'_{L^r}$ is an immediate consequence.

At this occasion let us give an application of Proposition 1 to the *solution to an advanced problem in the American Mathematical Monthly* [19]: show that for $a_j > 0$

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_h \frac{\sin(a_1 x_1)}{x_1} \cdots \frac{\sin(a_h x_h)}{x_h} \cdot \frac{\sin a_0(x_1 + \cdots + x_h)}{x_1 + \cdots + x_h} \\ &= \pi^h \min(a_0, a_1, \ldots, a_h). \end{aligned}$$

Putting $T_j = \sin(a_j x)/x \in \mathcal{D}_{L^p} \subset L^p \subset \mathcal{D}'_{L^p}$ for $1 < p \leqslant \infty$ the convolution $T_1 * \cdots * T_h$ is defined and belongs to $\mathcal{D}'_{L^r}$, $r > 1$. Viewing $T_0 \in \mathcal{D}_{L^s}$, $1/r + 1/s = 1$, as a test function and taking into account that

$$T_0^{h\Delta}(T_1 \otimes \cdots \otimes T_h) \in L^1\big(\mathbb{R}^h\big),$$

we obtain

$$\begin{aligned}\underset{\mathcal{D}_{L^s}}{\langle}\ T_0, T_1 * \cdots * T_h\ \underset{\mathcal{D}'_{L^r}}{\rangle} &= \underset{\mathcal{B}_c}{\big\langle}\ 1, T_0^{h\Delta}(T_1 \otimes \cdots \otimes T_h)\ \underset{\mathcal{D}'_{L^1}}{\big\rangle} \\ &= \underset{L^\infty}{\big\langle}\ 1, T_0^{h\Delta}(T_1 \otimes \cdots \otimes T_h)\ \underset{L^1}{\big\rangle} = I.\end{aligned}$$

Hence, we have identified the required integral with a convolution applied to a test function. This leads to the representation $I = (T_1 * \cdots * T_h * T_0)(0)$.

To evaluate the convolution we use the Exchange Theorem [75, p. 270]: it yields $\mathcal{F}(T_0 * T_1 * \cdots * T_h) = (\mathcal{F}T_0)(\mathcal{F}T_1)\cdots(\mathcal{F}T_h)$.

Due to $\mathcal{F}(T_j) = \pi\chi_{(-a_j/2\pi, a_j/2\pi)}$, where $\chi_{(-a,a)}$ is the characteristic function of $(-a, a)$, we arrive at $\mathcal{F}(T_0 * T_1 * \cdots * T_h) = \pi^{h+1}\chi_{(-a_{h+1}/2\pi, a_{h+1}/2\pi)}$ with $a_{h+1} = \min(a_0, a_1, \ldots, a_h)$. By Fourier inversion, $T_0 * T_1 * \cdots * T_h = \pi^h T_{h+1}$, and $I = \pi^h T_{h+1}(0) = \pi^h a_{h+1}$.

Remarks.

(1) The proof above yields

$$\frac{\sin(a_0 x)}{x} * \frac{\sin(a_1 x)}{x} * \cdots * \frac{\sin(a_h x)}{x} = \pi^h \frac{\sin(a_{h+1} x)}{x}$$

with $a_{h+1} = \min(a_0, a_1, \ldots, a_h)$, $a_j > 0$.

(2) An immediate consequence of the convolution-product representation and (2) is the formula

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_h \frac{\sin(a_1 x_1)}{x_1} \cdots \frac{\sin(a_h x_h)}{x_h} \cdot f(x_1 + \cdots + x_h) \\ &\quad = \pi^{h-1} \int_{-\infty}^{\infty} \frac{\sin(a_{h+1} x)}{x} f(x)\,\mathrm{d}x\end{aligned}$$

with $a_{h+1} = \min(a_1, \ldots, a_h)$, $a_j > 0$ [19, p. 85].

(3) Thomas Delmer's formula [19, p. 86] is a special case of

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_h \frac{a_1}{\pi(x_1^2 + a_1^2)} \cdots \frac{a_h}{\pi(x_h^2 + a_h^2)} f(x_1 + \cdots + x_h) \\ &\quad = \int_{-\infty}^{\infty} \frac{a_0 f(x)}{\pi(x^2 + a_0^2)}\,\mathrm{d}x \quad \text{with } a_0 = a_1 + \cdots + a_h,\ a_j > 0.\end{aligned}$$

The analogue of this formula concerning the convolution of Poisson kernels in $\mathbb{R}^{n+1}$ is straightforward.

(4) Obviously it is not necessary to start a "distributional machinery" to compute simple integrals as I. In fact it can be done in 3 lines:

$$\begin{aligned} I &= \operatorname{Re}\int_0^{a_0} \mathrm{d}t \left(\int_{-\infty}^{\infty} \frac{\sin(a_1x_1)}{x_1} \mathrm{e}^{\mathrm{i}tx_1}\,\mathrm{d}x_1 \right) \cdots \left(\int_{-\infty}^{\infty} \frac{\sin(a_hx_h)}{x_h} \mathrm{e}^{\mathrm{i}tx_h}\,\mathrm{d}x_h \right) \\ &= \int_0^{a_0} \mathrm{d}t \left(\int_{-\infty}^{\infty} \frac{\sin(a_1x_1)}{x_1} \cos(tx_1)\,\mathrm{d}x_1 \right) \cdots \left(\int_{-\infty}^{\infty} \frac{\sin(a_hx_h)}{x_h} \cos(tx_h)\,\mathrm{d}x_h \right) \\ &= \pi^h \int_0^{a_0} \chi_{(-a_1,a_1)}(t) \cdots \chi_{(-a_h,a_h)}(t)\,\mathrm{d}t. \end{aligned}$$

(5) An advantage of the procedure outlined above is the possibility to treat more general situations. A generalization of I in $\mathbb{R}^n$ was published in [69, Remark 3, p. 613]:

$$\begin{aligned} &|x|^{-n/2} J_{n/2}\big(a_1|x|\big) * |x|^{-n/2} J_{n/2}\big(a_2|x|\big) * \cdots * |x|^{-n/2} J_{n/2}\big(a_h|x|\big) \\ &\quad = (2\pi)^{n(h-1)/2} \left(\frac{a_0}{a_1 \cdots a_h |x|} \right)^{n/2} J_{n/2}\big(a_0|x|\big) \end{aligned}$$

with $a_0 = \min(a_1, \ldots, a_h)$, $a_j > 0$.

3.4. *Composition formulae*

Originally M. Riesz proved the composition formula for the elliptic potentials, i.e., $K_\alpha(K_\beta f) = K_{\alpha+\beta} f$ (in the notation of Section 2, "for functions such that all occurring integrals converge absolutely") under the assumptions $\alpha > 0$, $\beta > 0$, $\alpha + \beta < n$ ([60, (7), p. 3]; [62, (11), p. 20]; [57, (3), p. 124]).

J. Horváth defined the corresponding distributions R_α for all $\alpha \in \mathbb{C}$ by analytic continuation and taking the finite part of a distribution-valued meromorphic function (Example 1 in Section 2.5). In [31, (3), p. 433], he stated that $R_\alpha * R_\beta = R_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}(\alpha + \beta) < n$. He remarked that a proof is immediate by the use of L. Schwartz's exchange theorem [75, remarque, p. 270]. But this is true only if $\operatorname{Re}\alpha < n/2$ and $\operatorname{Re}\beta < n/2$ since $\operatorname{Re}\alpha < n/2 \Leftrightarrow R_\alpha \in \mathcal{D}'_{L^2}$.

The case $\operatorname{Re}\alpha \geqslant n/2$, $\operatorname{Re}(\alpha + \beta) < n$ was settled later. The result is

Theorem 6 ([65, Satz 6, p. 31]; [67, p. 12-05 and p. 12-09]; [41, p. 23]). *R_α and R_β ($\alpha, \beta \in \mathbb{C}$) are convolvable if and only if one of the following conditions is fulfilled*:

$$\alpha = -2k \quad \textit{or} \quad \beta = -2k\ (k \in \mathbb{N}_0) \quad \textit{or} \quad \operatorname{Re}(\alpha + \beta) < n.$$

In these cases, $R_\alpha * R_\beta = R_{\alpha+\beta}$.

There are 3 *proofs* in the references quoted above.

The following *generalization* was found by P. Wagner:

Theorem 7 [93, Satz 6, p. 415, Satz 7, p. 416]. *Let P be a homogeneous polynomial of degree m in n variables with $P(\xi) > 0$ for $\xi \neq 0$, such that P cannot be expressed as a power of another polynomial. Denoting by $T_\lambda = \mathcal{F}^{-1}(P^\lambda)$ the "convolution group" of P the following assertions are equivalent*:

(i) *T_λ and T_ν are convolvable.*
(ii) *$\lambda \in \mathbb{N}_0$ or $\nu \in \mathbb{N}_0$ or* $\operatorname{Re}(\lambda + \nu) > -n/m$.

In this case, $T_\lambda * T_\nu = T_{\lambda+\nu}$.

Remark. The elliptic M. Riesz's kernels are defined for the polynomial $P(\xi) = 4\pi^2|\xi|^2$ by

$$R_\alpha = \mathcal{F}^{-1}\big((2\pi|\xi|)^{-\alpha}\big).$$

In [31] J. Horváth also defined the *gradients of the elliptic M. Riesz distributions* (which we called *higher Hilbert–Riesz kernels* in Example 2 of Section 2.5), $N_\lambda = -\nabla R_{\lambda+1}$. He presented their composition (i.e., convolution) formula $N_\lambda * N_\nu = -R_{\lambda+\nu}$ [31, (9), p. 434] and sketched two proofs: a first one relying on the exchange theorem for the Fourier transform mentioned above; it works only for $\lambda, \nu \in \mathbb{C}$ with $\operatorname{Re}\lambda < n/2$ and $\operatorname{Re}\nu < n/2$. A second one which shifts the differentiation in the definition of N_λ, i.e.,

$$N_\lambda * N_\nu = (\nabla R_{\lambda+1}) * (\nabla R_{\nu+1}) = \Delta_2(R_{\lambda+1} * R_{\nu+1}) = -R_{-2} * R_{\lambda+\nu+2} = -R_{\lambda+\nu}.$$

But the differentiation rule

$$P(\partial)(S * T) = \big(P(\partial)S\big) * T = S * \big(P(\partial)T\big)$$

for linear partial differential operators [37, (18) Proposition, p. 192] supposes the convolvability of S and T, which is equivalent with $\operatorname{Re}(\lambda + \nu) < n - 2$ by Theorem 6 in the case $S = R_\lambda$, $T = R_\nu$. However, a slightly more refined argument saves the idea. We only have to use

Proposition 6 ([66, Proposition 1, p. 534]; [40, Proposition, p. 8–11]; [41, Proposition, p. 19]; [48, Theorem, p. 84]). *Let S and T be two distributions on $\mathbb{R}^n$ such that* $(1 \leqslant j \leqslant n)$

(i) *S and $\partial_j T$ are convolvable,*
(ii) *$\partial_j S$ and T are convolvable,*
(iii) *for every $\varphi \in \mathcal{D}$: $(\varphi * \check{S})T \in \dot{\mathcal{B}}'$, wherein $\dot{\mathcal{B}}'$ denotes the closure of $\mathcal{E}'$ in $\mathcal{D}'_{L^\infty}$* [75, p. 200].

Then $(\partial_j S) * T = S * (\partial_j T)$.

Remarks.

(1) As in 3.2 I conjecture that condition (iii) is equivalent to the more symmetrical one: for every $\varphi \in \mathcal{D}$: $\varphi^\Delta(S \otimes T) \in \dot{\mathcal{B}}'$.

(2) The application of Proposition 6 completes the second proof of Theorem II. 1(2) in [7, p. 257], if $n > 1$, i.e.,

$$N_0 * B = -(\nabla R_1) * B = -R_1 * (\nabla B) = 0,$$

if $B \in L^p,\ 1 < p < \infty$ and $\operatorname{div} B = 0$.

Proposition 5 yields now

Theorem 8 ([66, Proposition 2, p. 535]; [67, Proposition, pp. 12-05 and 12-10]; [41, p. 23]). *N_λ, N_ν $(\lambda, \nu \in \mathbb{C})$ are convolvable if and only if one of the following conditions is satisfied*:

$$\lambda = -2k - 1 \quad \textit{or} \quad \nu = -2k - 1\ (k \in \mathbb{N}_0) \quad \textit{or} \quad \operatorname{Re}(\lambda + \nu) < n.$$

In these cases, $N_\lambda * N_\nu = -R_{\lambda+\nu}$. The particular case

$$N_0 * N_0 = -\delta \tag{16}$$

is the reciprocity formula for the Hilbert–M. Riesz transform.

Remarks.

(1) The importance of Eq. (16) (i.e., (5), (6) in Section 1.2) is underlined in [10, p. 299]; [64, (viii), p. 417], and in [63, Theorem 5, p. 103]. These references attribute it to J. Horváth. Later on (16) became common knowledge, everybody used it, e.g., [81, p. 4-07]; [89, (2.9), p. 224]; [15, (3.13), p. 225]; [14, (4.9), p. 76]; [90, (2.110), p. 46].

(2) In Example 2, Section 2.5 we mentioned the relation $N_\lambda = -\nabla R_{\lambda+1}$, which is equivalent to $N_\lambda = N_{-1} * R_{\lambda+1}$ for $\lambda \in \mathbb{C}$. Under the convolvability conditions stated above we obtain, more generally,

$$N_{\lambda+\nu} = R_\lambda * N_\nu.$$

3.5. *Continuity of singular integral operators of the second generation*

Let us return to J. Horváth's starting point concerning singular integral operators: his view to consider them as convolution operators in Schwartz's spaces $\mathcal{D}'_{L^p}$ [28–30]. Let me recall and prove his main result under the more general assumptions stated in [14, Proposition 5.5, p. 94; Corollary 5.8, p. 97] for singular integral operators defined by convolution kernels (not necessarily homogeneous).

Theorem 9 ([30, Theorem 2, p. 55]; [31, p. 437]). *Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ be such that*

(i) $\lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} K(x)\,\mathrm{d}x$ *exists*;

(ii) $\left|\int_{a<|x|<b} K(x)\,\mathrm{d}x\right| \leqslant A,\ 0 < a < b$;

(iii) $\int_{a<|x|<2a} |K(x)|\,\mathrm{d}x \leqslant B,\ a > 0$;

(iv) $\int_{|x| \geqslant 2|y|} |K(x-y) - K(x)|\,\mathrm{d}x \leqslant C,\ y \in \mathbb{R}^n$.

Then

(i) *The distributional limit* $\lim_{\varepsilon\searrow 0}\chi_{\mathbb{R}^n\setminus B_\varepsilon}\cdot K =: K \in \mathcal{D}'_{L^q}$, $1 < q < \infty$, $B_\varepsilon = \{x \in \mathbb{R}^n / |x| \leqslant \varepsilon\}$, *exists.*
(ii) *The convolution mapping* $K*: \mathcal{D}'_{L^p} \to \mathcal{D}'_{L^p}$, $T \mapsto K * T$, $1 < p < \infty$, *is well-defined and continuous.*

Proof. (i) By Proposition 5.7 in [14, p. 96], $K \in \mathcal{S}'$ and by Corollary 5.8, p. 97, $K * \varphi \in \mathcal{D}_{L^q}$ for $\varphi \in \mathcal{D}_{L^q}$, $1 < q < \infty$. The closed graph theorem for Fréchet spaces implies the continuity of the mapping $\mathcal{D}_{L^q} \to \mathcal{D}_{L^q}$, $\varphi \mapsto K * \varphi$, and, hence, $K : \mathcal{D}_{L^q} \to \mathbb{C}$, $\varphi \mapsto (K * \check{\varphi})(0)$ is continuous, i.e., $K \in \mathcal{D}'_{L^q}$.

(ii) To prove that $K*$ is well-defined we generalize the reasoning for the Hilbert transform in Section 1.1: For $T \in \mathcal{D}'_{L^p}$ there exist $g_\alpha \in L^p$, $|\alpha| \leqslant m$, such that $T = \sum_{|\alpha|\leqslant m} \partial^\alpha g_\alpha$. Thus, by Proposition 1,

$$K * \left(\sum_{|\alpha|\leqslant m} \partial^\alpha g_\alpha\right) = \sum_{|\alpha|\leqslant m} \partial^\alpha (K * g_\alpha).$$

Since $K * g_\alpha \in L^p$ by Corollary 5.8, p. 97 [14], $K * T \in \mathcal{D}'_{L^p}$.

The continuity of $T \mapsto K * T$, $\mathcal{D}'_{L^p} \to \mathcal{D}'_{L^p}$, follows from the closed graph theorem, either in the form of Theorem B in [21, p. 17] (note that $\mathcal{D}'_{L^p}$ is an inductive limit of Banach spaces in the sense of the definition in [21, p. 13], cf. [81, p. 5-01], or in the form of Theorem 4 in [32, p. 301] ($\mathcal{D}'_{L^p}$ is reflexive by [75, p. 200], and hence barrelled by the corollary to Proposition 6 in [32, p. 229]; $\mathcal{D}'_{L^p}$ is also a Pták space by Proposition 6 in [32, p. 300]. □

For the spaces $\mathcal{D}'_{L^1}$, $\mathcal{D}'_{L^\infty}$, the same difficulties arise as in the classical theory (i.e., $T \mapsto K * T$, $\mathcal{D}'_{L^1} \to \mathcal{D}'_{L^r}$, $r > 1$ but not for $r = 1$). Hence let us define spaces $\mathcal{D}'_{H^1}$ and $\mathcal{D}'_{BMO}$ since "H^1 is a natural substitute for L^1, with *BMO* playing a similar role with respect to L^∞" [88, p. 139]. Recall that the Hardy space H^1 can be defined as [88, 6.4, p. 179]

$$H^1 = \left\{f \in L^1\left(\mathbb{R}^n\right) / N_0 * f \in L^1\left(\mathbb{R}^n\right)^n\right\}$$

and the space *BMO* of functions of bounded mean oscillation (modulo constants) as its dual, $BMO = (H^1)'$ [14, Theorem 6.15, p. 129].

Definition 6. Let us define the following three spaces:

$$\mathcal{D}_{H^1} = \left\{\varphi \in \mathcal{E}\left(\mathbb{R}^n\right) / \forall \alpha \in \mathbb{N}_0^n\colon\ \partial^\alpha \varphi \in H^1\right\},$$

equipped with the locally convex topology generated by the countable family of seminorms $\varphi \mapsto \|\partial^\alpha \varphi\|_{H^1}$, $\alpha \in \mathbb{N}_0^n$;

$$\mathcal{D}'_{H^1} = \left\{T \in \mathcal{D}'_{L^1}\left(\mathbb{R}^n\right) / \exists m \in \mathbb{N}_0,\ \exists g_\alpha \in H^1\colon\ T = \sum_{|\alpha|\leqslant m} \partial^\alpha g_\alpha\right\},$$

equipped with the topology of the inductive limit of the Banach spaces $\sum_{|\alpha|\leqslant m} \partial^\alpha H^1$, $m = 0, 1, 2, \ldots$;

$$\mathcal{D}'_{BMO} = (\mathcal{D}_{H^1})',$$

equipped with the strong topology $\beta(\mathcal{D}'_{BMO}, \mathcal{D}_{H^1})$.

Remark. The space $\mathcal{D}'_{BMO}$ contains the space BMO^{-1} of [55, Theorem 1, p. 24] and of [56, Proposition 16.1, p. 160].

One easily proves

Lemma.

(1) *$\mathcal{D}_{H^1} = \mathcal{D}_{L^1} \cap H^1$ and the topology of $\mathcal{D}_{H^1}$ is identical to the projective topology of $\mathcal{D}_{L^1}$ and H^1. $\mathcal{D}_{H^1}$ is a Fréchet space.*
(2) *$\mathcal{D}'_{H^1} = \{T \in \mathcal{D}'_{L^1}/N_0 * T \in (\mathcal{D}'_{L^1})^n\}$ and also $\mathcal{D}'_{H^1} = (\dot{\mathcal{B}}_{VMO})'$ wherein $\dot{\mathcal{B}}_{VMO}$ is the space of $\mathcal{C}^\infty$-functions φ such that $\partial^\alpha\varphi \in VMO$, $\alpha \in \mathbb{N}_0^n$, equipped with the projective topology with respect to the maps $\partial^\alpha : \dot{\mathcal{B}}_{VMO} \to VMO$, $\alpha \in \mathbb{N}_0^n$, VMO being the Banach space of functions with vanishing mean oscillation. $\mathcal{D}'_{H^1}$ is a complete $(\mathcal{DF})$ space.*
(3) *$\mathcal{D}'_{BMO}$ can be identified with a subspace of $\mathcal{D}'/_{\mathbb{C}}$. In fact, $\mathcal{D}'_{BMO} = \mathcal{D}'_{L^\infty}/_{\mathbb{C}} + BMO$. $\mathcal{D}'_{BMO}$ is a complete $(\mathcal{DF})$ space.*

Proposition 7. *Let $K \in \mathcal{C}(\mathbb{R}^n\backslash\{0\})$ fulfill the conditions*

(i) $\lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} K(x)\,\mathrm{d}x$ *exists*;
(ii) $\mathcal{F}K \in L^\infty$;
(iii) $|K(x)| \leqslant A|x|^{-n}$ $(x \neq 0)$;
(iv) $|K(x-y) - K(x)| \leqslant C\frac{|y|^\varepsilon}{|x|^{n+\varepsilon}}$, $0 < \varepsilon < 1$, $|x| \geqslant 2|y|$ $(x \neq 0)$.

Then the convolution mappings

$$K*: \mathcal{D}_{H^1} \to \mathcal{D}_{H^1}, \quad \varphi \mapsto K * \varphi, \quad \text{and} \quad K*: \mathcal{D}'_{H^1} \to \mathcal{D}'_{H^1}, \quad T \mapsto K * T,$$

are well-defined and continuous. The transpose

$${}^t(K*): \mathcal{D}'_{BMO} \to \mathcal{D}'_{BMO}$$

is well-defined and continuous.

Proof. (i) $\varphi \in \mathcal{D}_{H^1}$ implies $K * \varphi \in \mathcal{D}_{H^1}$ since $\partial^\alpha(K * \varphi) = K * \partial^\alpha\varphi$ belongs to H^1 for $\alpha \in \mathbb{N}_0^n$ due to Theorem 4 in [88, p. 115]. The continuity follows from the closed graph theorem for Fréchet spaces.

(ii) If $T \in \mathcal{D}'_{H^1}$, there exist $m \in \mathbb{N}_0$ and $g_\alpha \in H^1$ such that $T = \sum_{|\alpha|\leqslant m} \partial^\alpha g_\alpha$ and hence $K * T = \sum_{|\alpha|\leqslant m} \partial^\alpha(K * g_\alpha)$. By Theorem 4 in [88, p. 115], $K * g_\alpha \in H^1$ and thus $K * T \in \mathcal{D}'_{H^1}$.

The continuity follows from Theorem B in [21, p. 17].

(iii) The assertion is an immediate consequence of the definition of the space $\mathcal{D}'_{BMO}$, of the continuity of the mapping $K*:\mathcal{D}_{H^1} \to \mathcal{D}_{H^1}$, $\varphi \mapsto K * \varphi$, and the corollary in [32, p. 256]. □

Remark. Considering $N_0*:\mathcal{D}'_{BMO} \to (\mathcal{D}'_{BMO})^n$ as a singular integral operator we obtain

$$A \log|x| + \mathbb{C} = R_n + \mathbb{C} = {}^t(N_0*)(N_n) \quad \text{in } \mathcal{D}'_{BMO}$$

and hence $\log|x| \in BMO$ (compare [54, p. 231]). Note that by Proposition 6, N_0 and N_n are not convolvable.

In dimension 1, the above equation reads

$$-\frac{1}{\pi}\log|x| + \mathbb{C} = {}^t\left(\frac{1}{\pi}\mathrm{vp}\,\frac{1}{x}*\right)\left(\frac{1}{2}\operatorname{sign} x\right).$$

(By translations, the well-known Hilbert transform of $\chi_{(a,b)}$ is therefrom immediately derived (compare [14, p. 120]).)

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