Minimal Distance Upper Bounds for the Perturbation of Least Squares Problems in Hilbert Spaces

J. Ding
Department of Mathematics, The University of Southern Mississippi
Hattiesburg, MS 39406-5045, U.S.A.

(Received December 2000; accepted January 2001)

Communicated by R. Srivastav

Abstract—Let $X$ and $Y$ be Hilbert spaces, and let $T : X \to Y$ be a bounded linear operator with closed range. In this paper, we present an optimal perturbation result on the least squares solutions to the operator equation $Tx = y$ under the most general condition. © 2002 Elsevier Science Ltd.

Keywords—Hilbert space, Least squares solution, Generalized inverses, Perturbation.

1. INTRODUCTION

Let $X$ and $Y$ be two complex Hilbert spaces, let $T : X \to Y$ be a bounded linear operator with closed range, and let $y \in Y$ be fixed. We shall present a new perturbation result for the least squares problem (LSP)

$$
\|Tx - y\| = \min_{z \in X} \|Tz - y\|.
$$

The LSP is widely used in various areas of computational and applied mathematics [1,2], and so its perturbation analysis is important in error estimates for computing least squares solutions. Recently, some results for various special perturbations have been obtained for the generalized inverse of $T$ and the related minimal norm solution of the LSP [3–5]. They generalize well-known results for matrices under rank-preserving perturbations. It is well known that the generalized inverse (and so the minimal norm solution of the LSP) is not continuous with respect to the operator norm (see [6]), so any reasonable upper bound on the perturbation of minimal norm solution of the LSP must restrict the perturbation to be of special kind, such as the perturbation of Type I or Type II in [4]. In [7], some perturbation results concerning the upper semicontinuity of the generalized inverse of bounded linear operators of a Banach space have been obtained, and a classic perturbation result on invertible operators, which is based on the Neumann Lemma in functional analysis [2], was extended to arbitrary ones with an additional

---

Research was partially supported by a China Bridge Fellowship through the National Natural Science Foundation of China and China Bridges International (North America).
condition. This condition was removed in [8] in which an expression of the unperturbed solution in the perturbation result is also obtained.

It was asked in [8] whether the perturbation result there for the consistent operator equation can be generalized to more general least squares problems for linear operators of Banach spaces. In this paper, we answer the question in the context of Hilbert spaces. Our result directly extends that of [9] from the matrix case to the general Hilbert space one, and the proof follows the same idea as in the finite-dimensional case. Specifically, we shall give an upper bound to the minimal distance of the perturbed solution to the affine set of all least squares solutions of the unperturbed problem, and this bound is optimal since it is with regard to the minimal distance from the perturbation. Moreover, we have actually obtained an explicit expression of the unperturbed solution. We are able to obtain the upper bound for the minimal distance for Hilbert spaces since, unlike the more general case of Banach spaces, solutions to the LSP can be expressed in terms of generalized inverses of linear operators. Since many problems in differential equations and numerical analysis are in the context of infinite-dimensional Hilbert spaces, this optimal perturbation result, which has not appeared in the literature for general Hilbert spaces, is expected to find applications in applied fields.

In the next section, we give the main result, and in Section 3, we apply the main result to least squares problems with equality constraints.

2. THE MAIN RESULT

Let $B(X,Y)$ be the Banach space of all bounded linear operators $T : X \to Y$ with norm $\|T\| = \sup \{\|Tx\| : \|x\| = 1\}$, and let $B_c(X,Y)$ be the subspace of all $T \in B(X,Y)$ such that $R(T)$ is closed in $Y$. We use the standard notation in functional analysis. Let $T \in B_c(X,Y)$. The bounded linear operator $T^\dagger : Y \to X$ defined by $T^\dagger x = x$ for $x \in N(T)^\perp$, and $T^\dagger y = 0$ for $y \in R(T)^\perp$ is called the generalized inverse of $T$. It is well known that the vector $x_{LS} \equiv T^\dagger y$ is not only a solution to the LSP (1), but also the unique minimal norm solution of (1) among all the solutions. See [1] for more details about $T^\dagger$.

The following two lemmas will be needed in the paper. We omit the proof to Lemma 2.2 (the so-called Neumann Lemma) since it is standard, and the proof of Lemma 2.3 is referred to in [2].

**Lemma 2.1.** Let $F$ be the solution set of (1) and $p \in X$ be any vector in $X$. If $x$ is the orthogonal projection of $p$ onto $F$, then

$$ p - x = T^\dagger (Tp - y). $$

**Proof.** Since $p - x \in N(T)^\perp$ and $Tx - y \in R(T)^\perp$,

$$ p - x = T^\dagger T(p - x) = T^\dagger (Tp - y). $$

**Lemma 2.2.** Let $E \in B(X,X)$ be such that $\|E\| < 1$. Then $I + E$ is invertible and $(I + E)^{-1} \in B(X,X)$. Furthermore,

$$ \|(I + E)^{-1}\| \leq \frac{1}{1 - \|E\|}. $$

**Lemma 2.3.** Let $T, \tilde{T} \in B_c(X,Y)$. Then

$$ T^\dagger - \tilde{T}^\dagger = T^\dagger \delta T \tilde{T}^\dagger - T^\dagger (T^\dagger)^* (\delta T)^* \left( I - \tilde{T} \tilde{T}^\dagger \right) - (I - T^\dagger T) (\delta T)^* \left( \tilde{T}^\dagger \right)^* \tilde{T}^\dagger. $$

Now we consider the general LSP (1). Suppose that $T \in B_c(X,Y)$ and (1) is perturbed to

$$ \|\tilde{T} \tilde{x} - \tilde{y}\| = \min_{\tilde{z} \in X} \|\tilde{T} \tilde{z} - \tilde{y}\|, $$

with $\tilde{T} = T + \delta T$ and $\tilde{y} = y + \delta y$. In this paper, we assume that the perturbation is such that the perturbed operator $\tilde{T} \in B_c(X,Y)$, and so $\tilde{T}^\dagger$ is also well defined and the least squares problems (5) always has a solution. Let $F$ and $\tilde{F}$ be the solution set of (1) and (5), respectively, and let $\kappa = \|T\|\|T^\dagger\|$ be the condition number of $T$. 


THEOREM 2.1. Suppose that $\|T^t\delta T\| < 1$ and $\|TT^t\| < 1$. Then for any $\tilde{x} \in \tilde{F}$, there is $x \in F$ such that

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leq \frac{\kappa}{1 - \|T^t\delta T\|} \left[ \frac{\|T^t\| \|Tx - y\| \|\delta T\| + 2\|\delta y\| + 2\|\delta T\|}{\|T\| \|x\|} \right].$$

(6)

PROOF. Let $x$ be the orthogonal projection of $\tilde{x}$ onto $F$. Let $\tilde{\rho} = T^t(\tilde{x} - \tilde{y})$ be the residual of $\tilde{x}$. Then, since $\tilde{x}$ is a least squares solution of (5),

$$\tilde{T}^t \tilde{\rho} = 0$$

(7)

and

$$\|\tilde{\rho}\| = \|T^t(\tilde{x} - \tilde{y})\| \leq \|Tx - \tilde{y}\| \leq \|Tx - y\| + \|\delta y - T\delta x\|.$$  

(8)

Thus, by (7),

$$T^t \tilde{\rho} = (T^t - \tilde{T}^t) \tilde{\rho}.$$  

(9)

On the other hand, from Lemma 2.1, we have

$$\tilde{x} - x = T^t(T\tilde{x} - y) = T^t(\tilde{\rho} + \delta y - \delta T\tilde{x}) = T^t[\tilde{\rho} + \delta y - \delta T(\tilde{x} - x) - \delta T\tilde{x}],$$

which implies that

$$(I + T^t\delta T) (\tilde{x} - x) = T^t(\tilde{\rho} + \delta y - \delta T\tilde{x}) = T^t(\tilde{\rho} + \delta y - \delta T\tilde{x}) + (T^t - \tilde{T}^t)(\tilde{\rho} + \delta y - \delta T\tilde{x}).$$

(10)

Now, Lemma 2.3 and (7) give that

$$(T^t - \tilde{T}^t) \tilde{\rho} = -T^t(\tilde{T}^t)^*(\delta T)^*\tilde{\rho} = -T^t(\delta TT^t)^*\tilde{\rho},$$

from which, it follows that

$$\left\| (T^t - \tilde{T}^t) \tilde{\rho} \right\| \leq \|T^t\| \|\delta TT^t\| \|\tilde{\rho}\|.$$  

(11)

Therefore, by (8), (10), (11), and Lemma 2.2,

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leq \frac{\|T^t\|}{1 - \|T^t\delta T\|} \frac{\|\delta TT^t\| \|\tilde{\rho}\| + \|\delta y - \delta T\tilde{x}\|}{\|x\|} \leq \frac{\|T^t\|}{1 - \|T^t\delta T\|} \frac{\|\delta TT^t\| (\|Tx - y\| + \|\delta y - \delta T\tilde{x}\|) + \|\delta y - \delta T\tilde{x}\|}{\|x\|} \leq \frac{\kappa}{1 - \|T^t\delta T\|} \left[ \frac{\|\delta TT^t\| \|Tx - y\| + \|\delta y\| (\|\delta TT^t\| + 1)}{\|x\|} + \frac{\|\delta T\| (\|\delta TT^t\| + 1)}{\|T\|} \right].$$

COROLLARY 2.1. If, in addition, $y \in R(T)$, then

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leq \frac{\kappa}{1 - \|T^t\delta T\|} \left[ \frac{\|\delta TT^t\| \|Tx - y\|}{\|x\|} + \frac{\|\delta T\|}{\|T\|} \right].$$

(12)

REMARK 2.1. The difference between (12) and the main result (6) of [8] is that here the perturbed linear operator equation may not be consistent, while the perturbation in Theorem 3.1 of [8] keeps the equation consistent.
Remark 2.2. If $T$ is invertible and $T^{-1} \in B(Y, X)$, then the LSP (1) is reduced to a consistent operator equation and of course the assumption of Corollary 2.1 is automatically satisfied. Thus, (12) is valid in this special case. However, since the perturbed LSP (5) is also reduced to a consistent operator equation, the constant factor $2$ in (12) can be removed as Corollary 3.1 of [8] shows in the more general case of Banach spaces.

Remark 2.3. In general, even a stronger condition $\|T^\dagger\|\|\delta T\| < 1$ does not imply that $\tilde{T} \in B_c(X, Y)$, so the perturbed problem (5) may not have a solution. Some rather strong conditions (see, e.g., [3, 7]) imply that $\tilde{T} \in B_c(X, Y)$. It will be interesting to see under what general conditions a small perturbation will guarantee the closeness of the range of the perturbed operator, and so the well definedness of its generalized inverse.

One disadvantage of Theorem 2.1 is that the bound in (6) is dependent on $x$ which is unknown in advance and depends on $\tilde{x}$. However, using the fact that $\|Tx - y\| = \|Tx_{LS} - y\|$ and $\|x\| \geq \|x_{LS}\|$, we immediately have the following bound which is independent of $x$.

Theorem 2.2. Suppose that $\|T^\dagger\|\|\delta T\| < 1$ and $\|\delta TT^\dagger\| < 1$. Then for any $\tilde{x} \in \tilde{F}$, there is $x \in F$ such that

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leq \frac{\kappa}{1 - \|T^\dagger\|\|\delta T\|} \left[ \frac{\|T^\dagger\|\|\delta T\| + 2\|\delta y\|}{\|T\|\|x_{LS}\|} + 2\|\delta T\| \right],$$

where $r = Tx_{LS} - y$ is the residual of the minimal norm least squares solution $x_{LS}$ of (1).

3. LEAST SQUARES PROBLEMS WITH EQUALITY CONSTRAINTS

Now we apply the main result in the previous section to the least squares problem with equality constraints (LSE)

$$\min \|Tx - g\|, \quad \text{subject to} \quad \|Sx - h\| = \min_{u \in X} \|Su - h\|,$$

where $T, S \in B_c(X, Y)$ and $g, h \in Y$. The next proposition, which is easy to prove, shows that the LSE can be reduced to a usual LSP.

Proposition 3.1. The LSE (14) is equivalent to the LSP

$$\min_{z \in X} \|Az - f\|,$$

where

$$A \equiv T(I - S^\dagger S), \quad f \equiv g - TS^\dagger h,$$

in the sense that $z$ is a solution to (15) if and only if $x = z + S^\dagger(h - Sz)$ is a solution to (14).

Remark 3.1. By (2), $x$ is exactly the orthogonal projection of $z$ onto the feasible set of the LSE (14). Also it is easy to see that

$$Az - f = Tx - g.$$

Let the LSE (14) be perturbed to

$$\min \|\tilde{T}\tilde{x} - \tilde{g}\|, \quad \text{subject to} \quad \|\tilde{S}\tilde{x} - \tilde{h}\| = \min_{\tilde{z} \in \tilde{X}} \|\tilde{S}\tilde{z} - \tilde{h}\|,$$

where $\tilde{T} = T + \delta T, \tilde{S} = S + \delta S, \tilde{g} = g + \delta g$, and $\tilde{h} = h + \delta h$. Then the corresponding LSP (15) is perturbed to

$$\min_{\tilde{z} \in \tilde{X}} \|\tilde{A}\tilde{z} - \tilde{f}\|,$$

with $\tilde{A} = \tilde{T}(I - \tilde{S}^\dagger\tilde{S})$ and $\tilde{f} = \tilde{g} - \tilde{T}\tilde{S}^\dagger\tilde{h}$. 
Write $\tilde{A} = A + \delta A$ and $\tilde{f} = f + \delta f$. Then one easily obtains
\begin{align}
\|\delta A\| & \leq \|\delta T\| + \|T\| \left\| \tilde{S}^\dagger S - S^\dagger S \right\|, \\
\|\delta f\| & \leq \|\delta g\| + \|\delta T\| \left\| \tilde{S}^\dagger h \right\| + \|T\| \left\| \tilde{S}^\dagger h - S^\dagger h \right\|. 
\end{align}

Using Proposition 3.1 and Theorem 2.1, and noting (16), we immediately have the following perturbation result for the LSE (14).

**Theorem 3.1.** Suppose that $\|\delta T\|$ and $\|\tilde{S}^\dagger S - S^\dagger S\|$ are small enough. Then for any solution $\tilde{x}$ to (17) of the form $\tilde{x} = \tilde{S}^\dagger h + (I - \tilde{S}^\dagger S)\tilde{z}$, there is a solution $x$ to (14) of the form $x = S^\dagger h + (I - S^\dagger S)z$ such that
\begin{align}
\frac{\|\tilde{x} - x\|}{\|x\|} & \leq \frac{\|\tilde{S}^\dagger h - S^\dagger h\|}{\|x\|} + \frac{\|\tilde{z} - z\|}{\|x\|} + \left\| \tilde{S}^\dagger S - S^\dagger S \right\| \frac{\|z\|}{\|x\|}, 
\end{align}

where
\begin{align}
\|\tilde{z} - z\| & \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger \delta A\|} \left[ \|A^\dagger\| \|\delta A\| \|Tx - g\| + 2(\|\delta A\| \|z\| + \|\delta f\|) \right].
\end{align}

**Remark 3.2.** From the results in [3–5], we can obtain some upper bounds for $\|\tilde{S}^\dagger - S^\dagger\|$ and $\|\tilde{S}^\dagger S - S^\dagger S\|$ in terms of $\|\delta S\|$ under some additional conditions.

**References**