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Zero Polyhedral Cones

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Abstract—Properties of zero polyhedral cones are studied by making use of Fourier-Motzkin eliminations. Algorithms are presented for the characterization of zero polyhedral cones.

Keywords—Polyhedral cone, Fourier-Motzkin elimination, Hybrid systems.

1. INTRODUCTION

Polyhedral cones of the form $P = P(T) = \{x \mid x \in \mathcal{R}^n,\ Tx \geq 0\}$, for some $m \times n$ matrix T, are well-known objects in linear optimization [1] as feasible set of solutions. Here we are interested in the characterization of a zero polyhedral cone given an $m \times n$ matrix. The problem, an interesting mathematical problem by itself, is motivated by the studies of so-called hybrid systems [2]. In [2], the well-posedness of the solutions in the Caratheodary sense of piecewise linear systems was characterized by whether zero is the intersection of a number of lexicographic homogeneous linear inequalities. To check this, Imura and van der Schaft [2] proposed an algorithm based on linear programming to see whether the polyhedral cone determined by the first inequality of every lexicographic component is zero. If it is not, the original problem was reduced to lower-dimensional problem for iteration. It is noted, however, that the linear programming problem as formulated there in [2] is always degenerate.

Here we present elimination properties of zero polyhedral cones by making use of the Fourier-Motzkin transformation [3,4]. Two algorithms will be given for the characterization of zero polyhedral cones.

2. PROPERTIES OF ZERO POLYHEDRAL CONES

First, there is a necessary condition.

Lemma 1. *$P(T) = \{0\}$ only if* $\operatorname{rank} T = n$ *and* $m > n$.

Proof. If $\operatorname{rank} T = r < n$, then the nonzero solution of

$$Tx = 0$$

belongs to $P(T)$. So if $P(T) = \{0\}$, then $\operatorname{rank} T = n$.

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If $\operatorname{rank} T = n$, then $m \geq n$. Suppose $m = n$, then the nonzero solution to

$$\begin{aligned} T_1 x &= 1, \\ T_2 x &= 0, \\ &\vdots \\ T_n x &= 0, \end{aligned}$$

where T_i is the i^{th} row of T, belongs to $P(T)$. So necessarily, $m > n$ if $P(T) = \{0\}$.

From now on, we assume that $m > n$ and $\operatorname{rank} T = \left(T_1^\top, \dots, T_n^\top\right)^\top = n$, and denote

$$A = \begin{bmatrix} T_{n+1} \\ \vdots \\ T_m \end{bmatrix} \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix}^{-1}.$$

Define a coordinate transformation $z = (T_1, \dots, T_n)^\top x$, then $P(T)$ is equivalently defined by

$$P(T) = \{z \in \mathcal{R}^n \mid z \geq 0,\ Az \geq 0\}.$$

The cone defined by

$$\{x \in \mathcal{R}^n \mid x \geq 0,\ Ax \geq 0\}.$$

is called the positive cone of A and denoted by $P^+(A)$.

Immediately, we have the following equivalence.

LEMMA 2. *$P(T) = \{0\}$ if and only if $P^+(A) = \{0\}$.*

LEMMA 3. *$P(T) = \{0\}$ only if each column of A has at least one negative element.*

PROOF. If all elements of, say, the first column of A are greater or equal to zero, then it is easy to see that $(1, 0, \dots, 0)^\top$ belongs to $P^+(A)$. By Lemma 2, $P(T)$ contains a nonzero element. A contradiction.

Before we propose algorithms to check whether a polyhedral cone is zero, we develop an important property which is needed in the algorithms.

Suppose each column of the matrix $A \in \mathcal{R}^{m \times n}$ has at least one negative element. As usual, denote the i^{th} row of A by A_i. Denote

$$Q = \{k \mid a_{kn} < 0\}, \qquad P = \{k \mid a_{kn} > 0\}, \qquad Z = \{k \mid a_{kn} = 0\}.$$

For each $k \in Q$, define a matrix B^k, called a Fourier-Motzkin reduction of A, as

$$B^k = (b_{ij}) \in \mathcal{R}^{m \times (n-1)},$$

in which

$$b_{kj} = a_{kj},$$

for $j = 1, \dots, n-1$, and

$$b_{ij} = a_{ij} - \frac{a_{in}}{a_{kn}} a_{kj},$$

for $i \neq k$ and $j = 1, \dots, n-1$.

We also define matrix B, called a Fourier-Motzkin elimination of A, as

$$B = (b_{ij}) \in \mathcal{R}^{\tilde{m} \times (n-1)},$$

in which, denoting $|\cdot|$ for the size of the set $\cdot$,

$$\tilde{m} = |Z| + |P| + |Q \times P|.$$

For the first $|Z| + |P|$ rows,

$$b_{kj} = a_{kj'},$$

for $j = 1, \dots, n-1$, and some (one and only one) $j' \in Z + P$.

For the last $|P \times Q|$ rows,

$$b_{ij} = a_{i'j} - \frac{a_{i'n}}{a_{k'n}} a_{k'j},$$

for $j = 1, \dots, n-1$, and some (one and only one) index set $(i', k') \in P \times Q$.

THEOREM 1. *The following statements are equivalent.*

(1) $P^+(A) = \{0\}$.
(2) *For all* $k \in Q$, $P^+(B^k) = \{0\}$.
(3) $P^+(B) = \{0\}$.

PROOF. ((1) $\Rightarrow$ (2)) By contradiction. Without loss of generality, assume that $m \in Q$, and $P^+(B^m)$ contains a nonzero element $(\bar{x}_1, \dots, \bar{x}_{n-1})^\top$.

Then, we have, in particular,

$$a_{m1}\bar{x}_1 + \cdots + a_{mn-1}\bar{x}_{n-1} = b_{m1}\bar{x}_1 + \cdots + b_{mn-1}\bar{x}_{n-1} \geq 0.$$

Define

$$\bar{x}_n = \frac{a_{m1}\bar{x}_1 + \cdots + a_{mn-1}\bar{x}_{n-1}}{-a_{mn}}.$$

Since by assumption, a_{mn} is negative, we have $\bar{x}_n \geq 0$ and $A_m\bar{x} = 0$.

For this $\bar{x}$, and for $i = 1, \dots, m-1$,

$$\begin{aligned} A_i\bar{x} &= a_{i1}\bar{x}_1 + \cdots + a_{in-1}\bar{x}_{n-1} + a_{in}\bar{x}_n \\ &= \left(a_{i1} - \frac{a_{in}a_{m1}}{a_{mn}}\right)\bar{x}_1 + \cdots + \left(a_{in-1} - \frac{a_{in}a_{mn-1}}{a_{mn}}\right)\bar{x}_{n-1} \\ &= b_{i1}\bar{x}_1 + \cdots + b_{in-1}\bar{x}_{n-1} \\ &\geq 0, \end{aligned}$$

so $0 \neq \bar{x} \in P^+(A)$. A contradiction.

((2) $\Rightarrow$ (3)) By construction, each row of B is a row of one of the matrix B^k, so

$$P^+(B) \subset \bigcap_k P^+\left(B^k\right).$$

((3) $\Rightarrow$ (1)) Also by contradiction. Assume that $0 \neq \bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^\top \in P^+(A)$. Then, for all $k \in Q$,

$$a_{k1}\bar{x}_1 + \cdots + a_{kn-1}\bar{x}_{n-1} + a_{kn}\bar{x}_n \geq 0. \tag{1}$$

Define

$$\bar{\bar{x}}_n = \min_{k \in Q} \frac{a_{k1}\bar{x}_1 + \cdots + a_{kn-1}\bar{x}_{n-1}}{-a_{kn}}, \tag{2}$$

then by (1) and the fact that a_{kn}'s are negative, we have $\bar{\bar{x}}_n \geq 0$.

Define $\bar{\bar{x}} = (\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{\bar{x}}_n)^\top$, then by (1) and (2), it is verified that

$$A_k\bar{\bar{x}} \geq 0,$$

for $k \in Q$, and there is an index $\bar{k} \in Q$ such that

$$A_{\bar{k}}\bar{\bar{x}} = 0.$$

That is, $\bar{k} \in Q$ is chosen such that

$$a_{\bar{k}1}\bar{x}_1 + \cdots + a_{\bar{k}n-1}\bar{x}_{n-1} + a_{\bar{k}n}\bar{\bar{x}}_n = 0, \tag{3}$$

and for all other $k \in Q$,

$$\frac{a_{k1}\bar{x}_1 + \cdots + a_{kn-1}\bar{x}_{n-1}}{-a_{kn}} \geq \frac{a_{\bar{k}1}\bar{x}_1 + \cdots + a_{\bar{k}n-1}\bar{x}_{n-1}}{-a_{\bar{k}n}}. \tag{4}$$

We claim that $P^+(B)$ contains a nonzero element.

For the first $|Z|$ rows of B, obviously we have

$$b_{i1}\bar{\bar{x}}_1 + \cdots + b_{in-1}\bar{\bar{x}}_{n-1} \geq 0.$$

For each of the next $|Q|$ rows of B, there is a (unique) $j' \in Q$,

$$\begin{aligned} b_{i1}\bar{\bar{x}}_1 + \cdots + b_{in-1}\bar{\bar{x}}_{n-1} &= a_{j'1}\bar{x}_1 + \cdots + a_{j'n-1}\bar{x}_{n-1} \\ &\geq -a_{j'n}\bar{x}_n \\ &\geq 0, \end{aligned} \tag{5}$$

in which the last step is implied by the fact that $a_{j'n}$ is negative and $\bar{x}_n$ is nonnegative.

From (1) and (3), we know that $a_{kn}(\bar{x}_n - \bar{\bar{x}}_n) \geq 0$ or

$$\bar{\bar{x}}_n \geq \bar{x}_n. \tag{6}$$

Now for each of the last $|P \times Q|$ rows of B, since there is a (unique) index set $(i', k') \in P \times Q$,

$$\begin{aligned} b_{i1}\bar{\bar{x}}_1 + \cdots + b_{in-1}\bar{\bar{x}}_{n-1} &= \left(a_{i'1} - \frac{a_{i'n}}{a_{k'n}}a_{k'1}\right)\bar{x}_1 + \cdots + \left(a_{i'n-1} - \frac{a_{i'n}}{a_{k'n}}a_{k'n-1}\right)\bar{x}_{n-1} \\ &= a_{i'1}\bar{x}_1 + \cdots + a_{i'n-1}\bar{x}_{n-1} - \frac{a_{i'n}}{a_{k'n}}(a_{k'1}\bar{x}_1 + \cdots + a_{k'n-1}\bar{x}_{n-1}) \\ &\overset{(4)}{\geq} a_{i'1}\bar{x}_1 + \cdots + a_{i'n-1}\bar{x}_{n-1} - \frac{a_{i'n}}{a_{\bar{k}n}}(a_{\bar{k}1}\bar{x}_1 + \cdots + a_{\bar{k}n-1}\bar{x}_{n-1}) \\ &= a_{i'1}\bar{x}_1 + \cdots + a_{i'n-1}\bar{x}_{n-1} + a_{i'n}\bar{\bar{x}}_n \\ &\overset{(6)}{\geq} a_{i'1}\bar{x}_1 + \cdots + a_{i'n-1}\bar{x}_{n-1} + a_{i'n}\bar{x}_n \\ &\geq 0. \end{aligned} \tag{7}$$

Combining (5) and (7), we have $(\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_{n-1})^\top = (\bar{x}_1, \ldots, \bar{x}_{n-1})^\top \in P^+(B)$.

Finally, from (1), it is easy to see that $(\bar{x}_1, \ldots, \bar{x}_{n-1})^\top$ is not zero. A contradiction!

3. ALGORITHMS AND SPECIAL CASES

Now we are ready for the algorithm to check whether $P(T) = \{0\}$ for a matrix $T \in \mathcal{R}^{m \times n}$.

The first algorithm is based on (3) of Theorem 1.

ALGORITHM 1.

Step 1. Check: $n = \operatorname{rank} T$ and $m > n$. If not, stop!

Choose n linearly independent rows $T_{i_1}, \ldots, T_{i_n}$ of T, and denote the other $m - n$ rows of T as $T_{j_1}, \ldots, T_{j_{m-n}}$.

Define

$$A^1 = \begin{bmatrix} T_{j_1} \\ \vdots \\ T_{j_{m-n}} \end{bmatrix} \begin{bmatrix} T_{i_1} \\ \vdots \\ T_{i_n} \end{bmatrix}^{-1} \in \mathcal{R}^{(m-n) \times n}.$$

Check: each column of A^1 has at least one negative element. If not, stop!

Step k ($2 \leq k \leq n$). Denote the Fourier-Motzkin elimination of A^{k-1} as A^k.

The second algorithm is based on (2) of Theorem 1.

ALGORITHM 2.

Step 1. Same as Step 1 of Algorithm 1. Denote $\mathcal{M}_1 = \{A^1\}$.

Step k ($2 \leq k \leq n$). Denote $\mathcal{M}_k$ as the collection of all the Fourier-Motzkin reductions of all matrices in $\mathcal{M}_{k-1}$.

Check: each column of each of the matrix in $\mathcal{M}_k$ has at least one negative element. If not, stop!

REMARK 1. Algorithm 1 is based on the original Fourier-Motzkin eliminations, the row size of the Fourier-Motzkin transformation is generally less than the sum of the row size of all Fourier-Motzkin descendents. But because of the block structure of Algorithm 1, it is better for parallel computing. The second algorithm is sometimes more effective when verifying a polyhedral cone is not zero, as demonstrated in [2] for the well-posedness of piecewise linear systems.

REMARK 2. Both algorithms can be improved by arguments of extreme rays, as done in the dual algorithms of double description [5].

We look at some special cases.

When $m = n+1$, $P(T) = \{0\}$ is very easy to characterize.

LEMMA 4. *When $A = (a_1, \dots, a_n)$ is a row vector, then $P^+(A) = \{0\}$ if and only if $a_i < 0$ for $i = 1, \dots, n$.*

PROOF. Sufficiency is clear. Necessity is implied by Lemma 3.

When $m = n+2$, a more dedicated result can be obtained.

In this case,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \end{bmatrix}. \tag{8}$$

From Lemma 3, for each $i = 1, \dots, n$, a_{1i} or a_{2i} are negative. Without loss of generality, assume that in the first row: $a_{11}, \dots, a_{1r}$ are negative and $a_{1r+1}, \dots, a_{1n}$ are nonnegative. Then necessarily, $a_{2r+1}, \dots, a_{2n}$ are negative.

LEMMA 5. *When A is given by (8), and $a_{11}, \dots, a_{1r}$ are negative and $a_{1r+1}, \dots, a_{1n}$ are nonnegative, then $P^+(A) = \{0\}$ if and only if*

$$0 < \max_{i=r+1,\dots,n} \frac{a_{1i}}{a_{2i}} \max_{i=1,\dots,r} \frac{a_{2i}}{a_{1i}} < 1. \tag{9}$$

PROOF.

NECESSITY. First of all,

$$\max_{i=r+1,\dots,n} \frac{a_{1i}}{a_{2i}} \max_{i=1,\dots,r} \frac{a_{2i}}{a_{1i}} > 0,$$

since otherwise there is at least one nonnegative element from $a_{2r+1}, \dots, a_{2n}$, contradicting Lemma 3.

Since $a_{11} < 0$, we use Fourier-Motzkin transformation to eliminate x_1, and we get a matrix

$$B = \begin{bmatrix} a_{12} & \cdots & a_{1r} & a_{1r+1} & \cdots & a_{1n} \\ a_{22} - a_{12}\frac{a_{21}}{a_{11}} & \cdots & a_{2r} - a_{1r}\frac{a_{21}}{a_{11}} & a_{2r+1} - a_{1r+1}\frac{a_{21}}{a_{11}} & \cdots & a_{2n} - a_{1n}\frac{a_{21}}{a_{11}} \end{bmatrix}.$$

By (2) of Theorem 1, for $i = r+1, \dots, n$,

$$a_{2i} - a_{1i}\frac{a_{21}}{a_{11}} < 0,$$

or

$$\frac{a_{1i}}{a_{2i}}\frac{a_{21}}{a_{11}} < 1.$$

Therefore,

$$\max_{i=r+1,\dots,n} \frac{a_{1i}}{a_{2i}}\frac{a_{21}}{a_{11}} < 1.$$

By symmetricy, for $i = 1, \dots, r$,

$$\max_{i=r+1,\dots,n} \frac{a_{1i}}{a_{2i}}\frac{a_{2i}}{a_{1i}} < 1.$$

So

$$\max_{i=r+1,\ldots,n} \frac{a_{1i}}{a_{2i}} \max_{i=1,\ldots,r} \frac{a_{2i}}{a_{1i}} < 1.$$

SUFFICIENCY. When (9) holds, it is possible to find an $\alpha > 0$ such that

$$\alpha > \max_{i=r+2,\ldots,n} \frac{a_{1i}}{a_{2i}}$$
$$\frac{1}{\alpha} > \max_{i=1,\ldots,r} \frac{a_{2i}}{a_{1i}}$$

For this choice of α,

$$A_1 + \alpha A_2 = (a_{11} + \alpha a_{21}, \ldots, a_{1n} + \alpha a_{2n}),$$

in which A_1 and A_2 denote the two rows of matrix A.

For $i = 1, \ldots, r$,

$$\begin{aligned} a_{1i} + \alpha a_{2i} &= -|a_{1i}| + \alpha a_{2i} \\ &= -|a_{1i}|\alpha \left(\frac{1}{\alpha} - \frac{a_{2i}}{|a_{1i}|} \right) \\ &< 0, \end{aligned}$$

and for $i = r + 1, \ldots, n$,

$$\begin{aligned} a_{1i} + \alpha a_{2i} &= a_{1i} - \alpha |a_{2i}| \\ &= -|a_{2i}|\alpha \left(\alpha - \frac{a_{1i}}{|a_{2i}|} \right) \\ &< 0, \end{aligned}$$

For any $x \in P^+(A)$, since $x \geq 0$, $A_1 x \geq 0$ and $A_2 x \geq 0$, we have

$$(A_1 + \alpha A_2)\, x \geq 0.$$

However, since each element of $A_1 + \alpha A_2$ are negative, we have that $x = 0$.

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