Path partition number in tough graphs

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Abstract

In this paper we present some results for path partition number \( \pi(G) \) in graphs \( G \) with
toughness \( t(G) \geq 1 \).

1. Introduction

In this paper we consider only simple graphs \( G \). For undefined terminology and
notation we refer to [3]. If \( S \subseteq V(G) \) then \( \omega(G - S) \) denotes the number of compo-
nents of \( G - S \). Following Chvátal [4] a graph \( G \) is called \( t \)-tough (\( t \in \mathbb{R}, \ t > 0 \)) if
\( |S| \geq t\omega(G - S) \) for every subset \( S \subseteq V(G) \) with \( \omega(G - S) \geq 2 \). The toughness of \( G \),
denoted by \( \tau(G) \), is the maximum value of \( t \) for which \( G \) is \( t \)-tough. For \( n \geq 1 \) we set
\( \tau(K_n) = \infty \). A graph \( G \) is called hamiltonian if it contains a Hamilton cycle (a cycle
containing every vertex of \( G \)); \( G \) is called hamiltonian-connected if for every pair of
distinct vertices \( x \) and \( y \) of \( G \) there is a Hamilton path (a path containing all vertices
of \( G \) with endvertices \( x \) and \( y \)). Results on the relationship between toughness and hamiltonicity have already appeared in Thomassen (see [2]), and Enomoto et al. [5].

As Skupièn defined in [9], \( \pi(G) \) denotes the path partition number of \( G \), i.e.
\( \pi(G) = -1 \) if \( G \) is hamiltonian connected; and \( \pi(G) = 0 \) if \( G \) is hamiltonian but
not hamiltonian-connected; and \( \pi(G) \) is the smallest integer \( t \) such that there exists
a system of pairwise disjoint paths \( P_1, \ldots, P_t \) containing all vertices of \( G \) if \( G \) is not
hamiltonian. In [9] and [10] some relations between path partition number and hamiltonicity of graphs are proved which allow us to get some upper bounds for \( \pi(G) \). Similarly, we can ask for an upper bound of \( \pi(G) \) if \( G \) satisfies toughness conditions in the well-known conjecture of Chvátal on 2-tough graphs or in a conjecture of Stibitz [11] for bipartite graphs. In fact, it is a special case of the question posed by Skupièn
in the problem 1 in [9].

In this paper we also consider a special class of graphs with \( \pi(G) \leq 2 \) as follows. Let
\( G \) be a graph with a Hamilton cycle and let \( S \) be a nonempty subset of \( V(G) \). Then
we have \( \pi(G - S) \leq |S| \) and therefore \( \pi(G - S) \leq |S| \) for all nonempty sets \( S \subseteq V(G) \) is a necessary condition for Hamiltonicity. This condition appears in Schiermeyer’s definition of path-tough graphs [8]. In order to preclude trivialities we put that definition as follows. A graph \( G \) is called path tough if \( G \) has at least three vertices and \( \pi(G - S) \leq |S| \) for all nonempty sets \( S \subseteq V(G) \). Clearly, \( \pi(G) \leq 2 \) if \( G \) is a path-tough graph since for an arbitrary vertex \( v \) of \( G \) the graph \( G - \{v\} \) contains a Hamiltonian path. For integers \( n_1, n_2, \) and \( m \) with \( n_1, n_2 \geq 2m + 1 \) we give an example of a path-tough non-Hamiltonian graph obtained from the complete graphs \( K_{n_1}, K_{n_2} \) by choosing \( 2m + 1 \) vertices \( x_1, x_2, \ldots, x_{2m+1} \) in \( K_{n_1} \) and \( y_1, \ldots, y_{2m+1} \) in \( K_{n_2} \) and by adding \( 2m + 1 \) new vertices \( z_1, \ldots, z_{2m+1} \) and \( 3(2m+1) \) edges \((x_i, y_i), (z_i, x_i), (z_i, y_i) \) \( (i = 1, \ldots, 2m+1) \). More examples of path-tough graphs can be found in [6] and [8].

2. Problems and results

The following conjecture is attributed to Chvátal.

**Conjecture 1.** A 2-tough graph on at least 3 vertices is Hamiltonian.

A similar conjecture for a bipartite graph is stated in Stiebitz [11].

**Conjecture 2.** Let \( G = (A, B; E) \) be a bipartite graph satisfying the following conditions:

1. \(|A| + 1 = |B|\), and
2. \( \omega(G - S) \leq |S| \) for every nonempty proper subset \( S \) of \( A \).

Then \( G \) contains a Hamilton path.

The following question was posed by Jackson [7].

**Problem 3.** Is a graph path-tough if it is 2-tough?

Conjecture 2, however, is false and, in fact, \( \pi(G) \) may be made arbitrarily large as indicated by the following class of graphs. For \( m \geq 5 \) construct the graph \( G_m \) from \( m \) disjoint edges \((a_i, b_i) \) \( (1 \leq i \leq m) \) by adding three vertices \( \{a_{m+1}, b_{m+1}, b_{m+2}\} \) and the edges \((a_{m+1}, b_i), (b_{m+1}, a_i) \) and \((b_{m+2}, a_i) \) \( (1 \leq i \leq m) \). It is easily seen that \( G_m \) satisfies conditions (1) and (2) in Conjecture 2 and \( \pi(G_m) = m - 3 \).

Recently Bauer et al. [1] have shown that Conjecture 1 is equivalent with other statements about 2-tough graphs.

**Theorem 4** (Theorem 6 in [1]). The following statements are equivalent:

a. Every graph \( G \) with \( \tau(G) \geq 2 \) is Hamiltonian-connected.

b. Every graph \( G \) on at least 3 vertices with \( \tau(G) \geq 2 \) is Hamiltonian.

c. Every graph \( G \) with \( \tau(G) > 2 \) contains a Hamilton path.

The proof of Theorem 4 relies on the following lemma. Let \( H \) be a graph with \( \tau(H) \geq 2 \). Choose \( x, y \in V(H) \). Now a graph \( G_{l,m}(H) \) \( (l, m \in \mathbb{N}) \) is constructed as
follows. Take \( m \) disjoint copies \( H_1, \ldots, H_m \) of \( H \), with vertices \( x_i, y_i \) in \( H_i \) corresponding to the vertices \( x \) and \( y \) in \( H \) (\( i = 1, 2, \ldots, m \)). Let \( F_m \) be the graph obtained from \( H_1 \cup \cdots \cup H_m \) by adding all possible edges between pairs of vertices in \( \{x_1, \ldots, x_m, y_1, \ldots, y_m\} \). Let \( T = K_l \) and let \( G_{l,m}(H) \) be the joint \( T + F_m \) of \( T \) and \( F_m \).

**Lemma 5** (Claim 1 and Claim 2 in the proof of Theorem 6 in [1]).

1. If \( l \geq 3 \), then \( \tau(G_{l,m}(H)) > 2 \).
2. If \( m \geq 2l + 3 \) and if there exists no Hamilton path in \( H \) between \( x \) and \( y \), then \( G_{l,m}(H) \) contains no Hamilton path.

We shall now establish a relation between the path partition number and hamiltonicity in 2-tough graphs to show that the affirmative answer to Problem 3 confirms Conjecture 1. The proof of our result is partly based on the following lemma (observation (1) in [9]).

**Lemma 6.** If \( p \geq 0 \) is an integer such that \( \pi(G + K_p) \geq 0 \) then \( p + \pi(G + K_p) = \pi(G) \).

**Theorem 7.** The following statements are equivalent:

(a) Every 2-tough graph on at least 3 vertices is hamiltonian.
(b) There exists an integer \( c \) such that \( c \geq \pi(G) \) if \( \tau(G) > 2 \).

**Proof.** By Theorem 4, (a) is equivalent to the following statement:

\( (**) \) Every graph \( G \) with \( \tau(G) \geq 2 \) is hamiltonian-connected.

It is obvious that \( (**) \) implies (b), so we only have to show that (b) implies \( (**) \).

To prove that (b) implies \( (**) \) we assume that there exists a graph \( H \) with \( \tau(H) \geq 2 \) that is non-hamiltonian-connected. We consider the graph \( G_{l,m}(H) \) with \( l \geq 3 \) and odd \( m \geq 2l + 3 \). By (1) of Lemma 5, \( \tau(G_{l,m}(H)) > 2 \). Since clearly \( G_{l,m}(H) + K_{((m-3)/2)-1} = G_{(m-3)/2,m}(H) \) and \( \pi(G_{(m-3)/2,m}(H)) \geq 2 \) by (2) of Lemma 5, we have, by Lemma 6, \( \pi(G_{l,m}(H)) = ((m-3)/2) - l + \pi(G_{l,m}(H) + K_{((m-3)/2)-1}) = ((m-3)/2) - l + \pi(G_{(m-3)/2,m}) \geq (m-3)/2 - l + 2 = (m+1)/2 - l \) which contradicts (b). \( \square \)

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**References**
