JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **198,** 879–892 (1996)
ARTICLE NO. 0119

Nonclassical Symmetry Reduction and Riemann Wave Solutions

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Submitted by Thanasis Fokas

Received February 8, 1994

In this paper we employ the "nonclassical symmetry method" in order to obtain Riemann multiple wave solutions of a system of first-order quasilinear differential equations. We show how to construct a Lie module of vector fields which are symmetries of the system supplemented by certain first-order differential constraints. We demonstrate the usefulness of our approach on several examples.

1. INTRODUCTION

The objective of this paper is an adaptation of the symmetry reduction method for the purpose of constructing Riemann wave solutions and their superpositions (multiple waves) [1–5] for first-order quasilinear hyperbolic systems of partial differential equations (PDE). The traditional symmetry reduction method is not a proper tool for obtaining these types of solutions because in this case the Lie point symmetry groups of the basic system are too small. To overcome this difficulty we make use of the "nonclassical" symmetry reduction, introduced by Bluman and Cole [6]. This technique has been later developed by several authors (i.e., Olver and Rosenau [7, 8], Levi and Winternitz [9], Pucci and Saccomandi [10, 11], Vorobev [12]). The most important element of this approach is the introduction of the so-called nonclassical (or conditional) symmetries. These

0022-247X/96 $18.00

are symmetries of the basic equations supplemented by certain first-order differential constraints. The constraints are chosen in such a way as to weaken the invariance criterion of the basic system and to provide us with the larger Lie-point symmetry groups for the augmented system. This approach is particularly useful in the application to our problem, i.e., constructing the Riemann wave type solutions.

The plan of this paper is as follows. In Section 2 we describe infinitesimal symmetries of Riemann wave solutions. Section 3 contains a detailed account of necessary conditions for the existence of nonclassical symmetries of differential equations. In Section 4 we present a description of how to construct single and double wave solutions. Section 5 deals with examples which illustrate the theoretical considerations.

2. PROPERTIES OF RIEMANN WAVE SOLUTIONS

Let X and U be two differential manifolds of dimension p and q, respectively, equipped with coordinates x^{μ} (on X) and u^j (on U). Let $\Delta_j^{s\mu}$, $s = 1, \ldots, r$ be real valued functions on U (components of tensors $\Delta_j^{s\mu}(u)\partial_{\mu} \otimes du^j$ on $X \times U$). Consider the system of first-order differential equations

$$\Delta_j^{s\mu}(u)u^j_{,\mu} = 0, \tag{2.1}$$

where $u^j_{,\mu} = \partial u^j / \partial x^{\mu}$ and we adopt the convention that repeated indices are summed unless one of them is in a bracket. For simplicity we assume that all considered functions and manifolds are of the class C^{∞}. All our considerations are of a local character. In the language of jet spaces, solving Eqs. (2.1) is equivalent to finding a map f: $X \mapsto J^1$ (where $J^1 = J^1(X \times U)$ denotes the first jet space over X) which annihilates the contact forms, i.e.,

$$f^*\left(du^j - u^j_{,\mu}\, dx^{\mu}\right) = 0 \tag{2.2}$$

and which has an image in a submanifold of J^1 given by (2.1) (now x^j, u^j, $u^j_{,\mu}$ play a role of coordinates of J^1).

Let us suppose that the system (2.1) is hyperbolic (a weaker assumption is the existence of real valued functions λ_{μ}, γ^j: $U \to \mathbb{R}^1$ satisfying (2.4)). Then, as is well known [1–3], this system admits solutions in the form of Riemann waves. The common property of these solutions is that the image of f lies in a submanifold of J^1 given by

$$u^j_{,\mu} = \sum_{A=1}^{k} h_A \gamma^j_A(u)\lambda^A_\mu(u), \tag{2.3}$$

where h_A are functions on J^1 and $\lambda^A = (\lambda^A_1, \dots, \lambda^A_p)$, $\gamma_A = (\gamma^1_A, \dots, \gamma^q_A)$ are fields on U satisfying the so-called wave relation

$$\Delta^{s\mu}_j \lambda^A_\mu \gamma^j_{(A)} = 0 \tag{2.4}$$

(no sum over A). The functions λ^A and γ_A can be constructed as follows. First, we seek functions λ^A_μ, $A = 1, \dots, k$, such that

$$\operatorname{rank}\left(\Delta^{s\mu}_j \lambda^A_\mu\right) < q. \tag{2.5}$$

Then, for each such choice of λ^A_μ, we find corresponding solutions γ^j_A of (2.4).

Let us assume, that at each point of $X \times U$, the 1-forms $\lambda^A = \lambda^A_\mu\, dx^\mu$ are linearly independent and $k < p$. Then it follows from (2.3) that there exist linearly independent vectors

$$\xi_a = \xi^\mu_a(u)\,\partial_\mu, \qquad a = k+1, \dots, p, \tag{2.6}$$

such that

$$\xi^\mu_a(u) u^j_{,\mu} = 0. \tag{2.7}$$

The fields ξ_a are defined by

$$\xi^\mu_a(u)\,\lambda^A_\mu(u) = 0. \tag{2.8}$$

In general, the condition (2.7) is weaker than (2.3), since (2.7) implies merely

$$u^j_{,\mu} = \eta^j_A \lambda^A_\mu, \tag{2.9}$$

where η^j_A are some functions on J^1, not necessarily satisfying the wave relation (2.4).

In this paper we study Eqs. (2.1) augmented by the characteristic equations

$$Q^j_a \equiv -\xi^\mu_a(u) u^j_{,\mu} = 0, \tag{2.10}$$

associated with the vector fields (2.6).

Different approaches to this augmented system can lead to generalizations or modifications of the Riemann invariants method of solving PDEs.

Here we shall investigate Eqs. (2.1) and (2.10) from the point of view of the nonclassical symmetry method [6–9].

3. CONDITIONS FOR THE EXISTENCE OF NONCLASSICAL SYMMETRIES

Let $\mathscr{G}$ be a $(p-k)$-dimensional Lie vector module over $C^\infty(X \times U)$ with generators of the form $\xi = \xi^\mu(u)\partial_\mu$. Let Λ be the k-dimensional module generated by 1-forms $\lambda = \lambda_\mu(u)\,dx^\mu$ annihilated by $\xi \in \mathscr{G}$. We assume, that the fields (ξ_a) and (λ^A) related by (2.8) form a basis of $\mathscr{G}$ and Λ, respectively. However, at the moment, we do not require that Λ admits a basis satisfying the wave relation (2.4) (with some γ's). We shall call ξ_a and $\mathscr{G}$ nonclassical symmetries and the nonclassical symmetry module of (2.1), respectively, iff ξ_a are ordinary (classical) Lie point symmetries [13, 14] of the original system (2.1) augmented by Eqs. (2.10). Note that one can choose ξ_a to form a basis of a finite dimensional subalgebra of $\mathscr{G}$ (one can even assume that ξ_a commute). We will call ξ_a genuine nonclassical symmetries of Eqs. (2.1) if $\mathscr{G}$ is not spanned by classical symmetries of (2.1). Equations (2.1) and (2.10) form an overdetermined system. It was proved [11, 15] that a "large part" (if not all) of the integrability conditions for this system is identically satisfied if ξ_a are nonclassical symmetries of (2.1).

Equations (2.10) say that the vector fields ξ_a are tangent to the graph $\Gamma = \{(x, u(x))\} \subset X \times U$. For ξ_a to be nonclassical symmetries of (2.1) one requires that the first prolongation of ξ_a to J^1 [13, 14] is tangent to $\mathscr{R}_Q = \{(x, u^{(1)})\colon \Delta^s(x, u^{(1)}) = Q^j_a(x, u^{(1)}) = 0\}$. So $\mathscr{G}$ is a nonclassical symmetry module of (2.1) iff equations

$$\mathrm{pr}^{(1)}\xi_a\Big(\Delta^{s\mu}_j u^j_{,\,\mu}\Big) = 0 \tag{3.1}$$

are satisfied on J^1 modulo Eqs. (2.1) and (2.7). It follows from (2.6) that

$$\mathrm{pr}^{(1)}\xi_a = \xi_a - \xi^\mu_{a,\,u^k} u^k_{,\,\nu} u^j_{,\,\mu} \frac{\partial}{\partial u^j_{,\,\nu}}. \tag{3.2}$$

Substituting (3.2) into (3.1) yields

$$\Delta^{s\nu}_j \xi^\mu_{a,\,u^k} u^k_{,\,\nu} u^j_{,\,\mu} = 0. \tag{3.3}$$

Equations (3.3) should be satisfied whenever (2.1) and (2.7) are satisfied. They can be treated as equations for ξ_a (or λ^A since (2.8) holds). These equations always admit solutions $\xi^\mu_a = \text{const}$, which correspond to translational (classical) symmetries of (2.1). By virtue of Eqs. (2.9) Eqs. (3.3) are equivalent to

$$Q_B^{sA}\ \xi_a^\mu\ \eta_A\left(\lambda_\mu^B\right) = 0, \tag{3.4}$$

where

$$Q_B^{sA} = \Delta_j^{s\nu}\eta_B^j\lambda_\nu^A \tag{3.5}$$

and we define the vector fields η_A on U according to

$$\eta_A = \eta_A^j \frac{\partial}{\partial u^j}. \tag{3.6}$$

4. INVARIANT SOLUTIONS

In this section we describe conditions which guarantee the existence of nonclassical symmetries of (2.1) and the corresponding invariant solutions of rank k. For the vector fields ξ_a to be nonclassical symmetries of (2.1) the conditions (3.4) (or (3.3)) have to be satisfied. Consider first the case when $k = 1$. Then $A = B = 1$ and (2.1) and (2.9) yield

$$\Delta_j^{s\nu}\lambda_\nu\eta^j = 0. \tag{4.1}$$

Hence $Q_B^{sA} = 0$ and Eq. (3.4) is satisfied. Thus, in this case, any set of vectors ξ_a generate a nonclassical symmetry module. However, not every such module admits nonconstant solutions of (2.1). If $\eta = 0$ is the only solution of (4.1), then evidently $u^j_{,\mu} = 0$ and this implies $u^j = \text{const}$. To have $\eta \neq 0$ one requires

$$\operatorname{rank}\left(\Delta_j^{s\nu}\lambda_\nu\right) \leq q - 1. \tag{4.2}$$

If λ satisfies condition (4.2), then the space of solutions η of (4.1) is spanned by vectors γ satisfying (2.4). Hence the decomposition (2.3), with possible degeneracy of λ, is valid. We have proved the following.

THEOREM 1. *Any* $(p-1)$*-dimensional module* $\mathcal{G}$ *generated by* $\xi_a^\mu(u)\partial_\mu$ *is a nonclassical symmetry module of* (2.1). *Nonconstant* $\mathcal{G}$*-invariant solutions of* (2.1) *exist iff* λ *satisfies condition* (4.2). *They are single Riemann waves*.

Let us now consider the case $k \geq 2$. Then, in general, the existence of k-waves, satisfying (2.3) and (2.4), does not imply that the corresponding ξ_a are nonclassical symmetries of the system (2.1) (e.g., ξ_a can be symmetries of (2.1) and (2.7) augmented by (3.1) (see [10, 11]). However, this can be true in special cases. For instance, let $q = k = 2$ and let λ^A admit γ_A such that (2.4) is satisfied. If the vector fields $\gamma_A = \gamma_A^i \partial/\partial u^i$ are linearly independent then

$$\eta_A = h_A^B \gamma_B, \tag{4.3}$$

where h_A^B are some coefficients. Equations (2.1) and (2.9) yield

$$\tilde{Q}_B^{sA} h_A^B = 0, \tag{4.4}$$

where

$$\tilde{Q}_B^{sA} = \Delta_j^{s\mu} \lambda_\mu^A \gamma_B^j.$$

Equations (4.4) represent two linear equations for quantities h_1^2 and h_2^1 (note that $\tilde{Q}_1^{s1} = \tilde{Q}_2^{s2} = 0$). Assume that

$$\tilde{Q}_2^{s_1 1} \tilde{Q}_1^{s_2 2} - \tilde{Q}_2^{s_2 1} \tilde{Q}_1^{s_1 2} \neq 0 \tag{4.5}$$

for some indices s_1, s_2. Then it follows from (4.4) that $h_1^2 = h_2^1 = 0$. Hence we get

$$\eta_A = h_{(A)} \gamma_A \tag{4.6}$$

and the decomposition (2.3). The condition (3.4) becomes

$$\tilde{Q}_B^{sA} h_A h_B \xi_a^\mu \gamma_A \left(\lambda_\mu^B \right) = 0;$$

hence (due to (4.5))

$$\xi_a^\mu \gamma_A \left(\lambda_\mu^B \right) = 0 \qquad \text{for } A \neq B \tag{4.7}$$

(we assume that, in general, $h_1 \cdot h_2 \neq 0$, since Eqs. (3.4) have to be satisfied for all solutions of Eqs. (2.1) and (2.7)).

The requirement (4.5) is equivalent to the nondecomposability of $\tilde{Q}_B^{sA}$ into the product $v^s P_B^A$ (with some v^s and P_B^A). By virtue of the linear independence of vector fields γ_A, this condition reads

$$\Delta_j^{s\mu} \lambda_\mu^A \neq v^s Q_j^A. \tag{4.8}$$

Note, that (4.8) does not depend on a choice of the basis λ^A of Λ. It says that Eqs. (2.1) reduce to more than one equation when (2.7) is satisfied.

Condition (4.7) coincides with one of the sufficient conditions (the other one is identically satisfied, since in this case γ_A form a vector basis on X) which guarantees the existence of double Riemann waves [1–3]. Therefore, we have proved the following theorem.

THEOREM 2. *Let a system* (2.1) *be given with* $p \geq 3$ *independent variables and* $q = 2$ *dependent variables. Assume there exist linearly independent vector fields* $\gamma_A = \gamma_A^j \partial/\partial u^j$ *and* 1-*forms* $\lambda^A = \lambda_\mu^A \, dx^\mu$, $A = 1, 2$, *satisfying* (2.4) *and* (4.8). *Then* ξ_a, *given by* (2.8), *span a module* $\mathcal{G}$ *of nonclassical symmetries of* (2.1) *iff condition* (4.7) *is satisfied. The rank*-2 $\mathcal{G}$-*invariant solutions of* (2.1) *are precisely the Riemann double waves.*

Relations (2.4) and (4.7) are rather complicated conditions that have to be satisfied by the basic system of Eqs. (2.1) in order that they admit nonclassical symmetries. In the following we show how these conditions can be simplified (generalizations to $q \geq 2$ are also possible).

Let $q = k = 2$ and ξ_a be related to λ^A via relation (2.8). Now the λ^A are not required to admit γ's such that (2.4) is satisfied. It is convenient to write Eqs. (3.4) in the form

$$\operatorname{Tr}\left(\Delta^s \eta \theta_a \eta \lambda \right) = 0, \tag{4.9}$$

where matrices in the bracket are given by

$$\Delta^s = \left(\Delta_j^{s\mu} \right), \qquad \eta = \left(\eta_A^j \right), \qquad \theta_a = \left(\lambda_{\mu, u^k}^A \xi_a^\mu \right), \qquad \lambda = \left(\lambda_\mu^A \right). \tag{4.10}$$

Since η and θ_a are 2×2 matrices, they satisfy the identities

$$\begin{aligned} (\eta - \operatorname{Tr} \eta) \eta &= -\det \eta, \\ \eta \theta_a - \operatorname{Tr}(\eta \theta_a) &= -(\theta_a - \operatorname{Tr} \theta_a)(\eta - \operatorname{Tr} \eta). \end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.9) and taking into account that $\operatorname{Tr}(\Delta^s \eta \lambda) = 0$ is equivalent to (2.1) yields

$$\operatorname{Tr}\left[\Delta^s (\theta_a - \operatorname{Tr} \theta_a) \lambda \right] = 0. \tag{4.12}$$

When looking for solutions of (4.12) and corresponding invariant solutions of the original system (2.1) it is convenient to split coordinates x^μ into x^A and x^a and to assume that

$$\xi_a = \partial_a + \xi_a^A \partial_A, \tag{4.13}$$

$$\lambda^A = dx^A - \xi_a^A \, dx^a \tag{4.14}$$

(one can obtain these expressions by making u-dependent linear transformations of the fields ξ_a and λ^A). Then

$$\theta_a = -\left(\xi_{a, u^k}^A \right); \tag{4.15}$$

hence (4.12) becomes a system of quasilinear differential equations for ξ_a^A.

Assume that a solution $\{\xi_a^A\}$ (hence ξ_a) of this system is given. In order to find $\mathscr{G}$-invariant solutions of Eqs. (2.1) one can first solve conditions (2.10). It follows from them that

$$x^A - \xi_a^A(u)x^a = r^A(u), \tag{4.16}$$

where r^A are functions of two variables. The implicit relations (4.16) determine the matrix of derivatives of u^i in the form

$$\left(u^i_{,\mu}\right) = \left(R(u) - \theta_a(u)x^a\right)^{-1}\lambda, \tag{4.17}$$

where $R = (r^i_{,u^k})$. By virtue of (4.11) and (4.12) substituting (4.17) into Eqs. (2.1) yields the following linear equations for the functions $r^A(u)$:

$$\mathrm{Tr}\left[\Delta^s(R - \mathrm{Tr}\,R)\lambda\right] = 0. \tag{4.18}$$

Thus we have proved the following.

THEOREM 3. *A system of PDEs* (2.1) *with* $p \geq 3$ *independent variables and* $q = 2$ *dependent variables admits* $(p-2)$*-dimensional Lie module* $\mathscr{G} = \mathrm{Span}\{\xi_a = \xi_a^\mu(u)\partial_\mu\}$ *of nonclassical symmetries iff the fields* ξ_a *satisfy Eqs.* (4.12). *Given* ξ_a *and* λ^A *in the form* (4.13), (4.14) *rank*-2 $\mathscr{G}$*-invariant solutions of* (2.1) *are defined implicitly by* (4.16), *where functions* r^A *are required to satisfy the linear equations* (4.18).

If the system (2.1) is properly determined (i.e., $q = r = 2$), then, generically, given ξ_a one can find λ^A and γ_A (possibly complex) such that γ_A are linearly independent and Eqs. (2.4) and (2.8) are satisfied. To this end, one can choose first any pair of real 1-forms λ'^A satisfying (2.8) and then look for solutions of

$$\det\left(\Delta_j^{s\mu}\lambda_\mu\right) = 0 \tag{4.19}$$

in the form $\lambda = \alpha_A\lambda'^A$, where α_A are unknown coefficients. Then Eq. (4.19) becomes a quadratic equation for the ratio of α_1 to α_2, which, in most cases, possesses two different solutions (real or complex). The eigenvectors γ of the matrix $\Delta_j^{s\mu}\lambda_\mu$ corresponding to the zero eigenvalue will usually be linearly independent (real or complex, respectively). In the case $q = r = 2$, Eqs. (4.12) form a system of $2(p-2)$ equations for $2(p-2)$ functions ξ_a^A (see (4.13)). So, in the generic case, Eqs. (4.12) admit solutions depending on arbitrary functions of one variable. Hence they are not equivalent to translations or dilations, which are, in general, the only classical symmetries of (2.1). Note also that the inequality (4.8) is satisfied for most systems. Combining these facts with Theorems 2 and 3 we obtain Theorem 4.

THEOREM 4. *Generically, the system of two equations* (2.1), *with* $p \geq 3$ *independent variables and* $q = 2$ *dependent variables, admits* $(p-2)$-*dimensional genuine nonclassical symmetry modules* $\mathcal{G}$. $\mathcal{G}$-*invariant solutions of* (2.1) *are* (*generically*) *double Riemann wave solutions* (*possibly with complex single composite waves*).

5. EXAMPLES

In this section we demonstrate the usefulness of Theorem 3 on several examples.

EXAMPLE 1. In this example, we consider a special case of the fluid dynamic equations which describe the potential of solutions of a conservative system in Lagrangian coordinates [5],

$$u_{i,t} + u_j u_{i,j} + A_{ijk}(u) u_{k,j} = 0, \qquad i, j, k = 1, 2. \tag{5.1}$$

Here A_{ijk} are functions of u satisfying

$$A_{ijj} = 0. \tag{5.2}$$

It is easy to check that Eq. (4.12) is satisfied in this case by the vector field

$$\xi = \partial_t + u_j \partial_j, \tag{5.3}$$

which is not proportional to any classical symmetry of (5.1). Theorem 3 says that ξ-invariant solutions of (5.1) are given implicitly by

$$x^i - t u_i = r_i(u), \tag{5.4}$$

where r^i satisfy the linear equations

$$A_{ijk} r_{k,u_j} = 0. \tag{5.5}$$

It is worth noting that one can choose a specific form of coefficients A_{ijk} such that Eq. (4.13) does not admit two linearly independent vectors λ^A (hence, there are no double Riemann waves). For instance, this is the case for the equations

$$\begin{aligned} u_{1,t} + u_j u_{1,j} + a(u)(u_{1,1} - u_{2,2}) + b(u) u_{1,2} &= 0 \\ u_{2,t} + u_j u_{2,j} + c(u) u_{1,2} &= 0, \end{aligned} \tag{5.6}$$

where a, b, c are functions of two variables such that $a \cdot c \neq 0$. In this case, solutions invariant under (5.3) are defined by

$$x^1 - tu_1 + g(u_1) = 0, \tag{5.7}$$

$$u_2 = \frac{x^2 + h(u_1)}{t - g_{,u_1}}, \tag{5.8}$$

where g, h are arbitrary functions of u^1. These solutions have rank 2, but they are not double Riemann waves.

EXAMPLE 2. In order to apply Theorem 3 to a more realistic physical situation let us consider equations describing an isentropic flow of a politropic gas with no external forces [5]. For simplicity we assume that the velocity of the gas along the third axis vanishes. In this case the equations read

$$u_{i,t} + u_j u_{i,j} + \rho^{-1} p_{,i} = 0, \tag{5.9}$$

$$\rho_t + (\rho u_i)_{,i} = 0, \tag{5.10}$$

$$p = \kappa \rho^{\gamma}, \qquad \kappa, \gamma = \text{const}, \gamma > 1, \tag{5.11}$$

where $i, j = 1, 2$. In order to reduce the number of dependent variables to two we assume that $\rho = \rho(u_i)$. Then Eqs. (5.9) and (5.10) yield

$$u_{i,t} + u_j u_{i,j} + f_{,u_j} u_{j,i} = 0, \tag{5.12}$$

$$f_{,u_i} f_{,u_j} u_{i,j} - (\gamma - 1) f u_{i,i} = 0, \tag{5.13}$$

where f is a function of the velocities u_i. The pressure p and the energy density ρ are given in terms of f

$$p = \kappa_1 (1 - \gamma^{-1}) f^{\gamma/(\gamma - 1)}, \tag{5.14}$$

$$\rho = \kappa_1 f^{1/(\gamma - 1)}, \tag{5.15}$$

where

$$\kappa = (1 - \gamma^{-1}) \kappa_1^{1-\gamma}. \tag{5.16}$$

Given f, Eqs. (5.12) and (5.13) form an overdetermined system of equations for u_i with three independent variables. According to Theorem 3 this system admits a nonclassical symmetry of the form $\xi = \partial_t + \xi_i(u) \partial_i$ iff the equations

$$q_{ij}(u_j - \xi_j) + q_{ji} f_{,u_j} = 0, \tag{5.17}$$

$$f_{,u_i} f_{,u_j} q_{ij} - (\gamma - 1) f q_{ii} = 0 \tag{5.18}$$

are satisfied, where

$$q_{ij} = \xi_{i,u_j} - \xi_{k,u_k} \delta_{ij}. \tag{5.19}$$

One can choose the function f in such way that Eqs. (5.17) and (5.18) admit nontrivial solutions ξ_i. In other words, these equations can be considered as a system for the functions ξ_i and f. The following simple solutions of Eqs. (5.17) and (5.18) can be obtained under the assumption that the functions ξ_i and f depend, respectively, linearly and quadratically on the velocities u_j:

(i) (for any γ) $\quad f = \frac{1}{2}(\gamma - 1) u_i u_i, \qquad \xi_i = \gamma u_i,$ (5.20)

(ii) (for $1 < \gamma < 3$)

$$f = \frac{\gamma - 1}{3 - \gamma}(u_1)^2, \qquad \xi_1 = \frac{\gamma + 1}{3 - \gamma} u_1, \qquad \xi_2 = u_2, \tag{5.21}$$

(iii) (for $\gamma = 2$)

$$f = \tfrac{1}{2} u_i u_i, \qquad \xi_1 = u_1 - u_2, \qquad \xi_2 = u_1 + u_2. \tag{5.22}$$

In all these cases the Galilean transformations were used to simplify the expressions. The resulting vectors ξ are not proportional to any of the classical symmetries of Eqs. (5.12) and (5.13) (for this choice of f). Thus ξ represents a genuine nonclassical symmetry of these equations. Rank-2 solutions $u_i(t, x_j)$, which are invariant under ξ, are given implicitly by

$$x_i - \xi_i(u) t = r_i(u), \tag{5.23}$$

where $\det(r_{i,u_j}) \neq 0$ and the functions r_i are required to satisfy, respectively, the linear equations:

(i) $\quad r_i = r_{,u_i}, \qquad u_i u_j r_{,u_i,u_j} - \frac{1}{2} u_i u_i r_{,u_j,u_j} = 0$ (5.24)

(ii) $\quad r_i = r_{,u_i}, \qquad (3 - \gamma) r_{,u_1,u_1} - (\gamma + 1) r_{,u_2,u_2} = 0,$ (5.25)

(iii)
$$\begin{aligned} (u_1 + u_2) r_{2,u_2} + u_1 r_{1,u_2} - u_2 r_{2,u_1} &= 0, \\ (u_1 - u_2) r_{1,u_1} + u_1 r_{1,u_2} + u_2 r_{2,u_1} &= 0. \end{aligned} \tag{5.26}$$

The method of separation of the variables in (5.24) (in polar coordinates) yields solutions of the form

$$r = \left(c_1 u^{2a} + c_2 u^{2(1-a)}\right)\left(c_3 \sin\sqrt{2a(a-1)}\,\phi + c_4 \cos\sqrt{2a(a-1)}\,\phi\right), \tag{5.27}$$

where $u_1 = u\cos\phi$, $u_2 = u\sin\phi$ and a, c_1, c_2, c_3, c_4 are constants, $a(a-1) > 0$. These solutions can be superimposed. Equation (5.25) possesses the general solution

$$r = g(u_1 + cu_2) + h(u_1 - cu_2), \tag{5.28}$$

where g, h are arbitrary functions of one variable and $c^2 = (3-\gamma)/(1+\gamma)$. Equation (5.26) is more difficult to solve. The only nontrivial solution known to the authors is given by

$$r_i = u_i\left(c_1 e^{\operatorname{arctg}(u_1/u_2)} + c_2 e^{\operatorname{arctg}(-u_2/u_1)}\right). \tag{5.29}$$

In all these cases solutions u_i of the original equations are defined implicitly by relations (5.23). They have the form of Riemann double waves. The functions p, ρ are given by (5.14), (5.15) and (5.20)–(5.22).

EXAMPLE 3. In this example we consider the modification of the (2 + 1)-dimensional wave equation,

$$\phi_{,tt} = \phi_{,xx} + \rho(\phi_{,y}, \phi_{,t} - \phi_{,x})\phi_{,yy}, \tag{5.30}$$

where ρ is any function of two variables. Let us define new coordinates x_1, x_2, x_3 and new dependent variables u_1, u_2 by

$$x_1 = \tfrac{1}{2}(t+x), \qquad x_2 = \tfrac{1}{2}(t-x), \qquad x_3 = y, \tag{5.31}$$

$$u_1 = \phi_{,y}, \quad u_2 = \phi_{,t} - \phi_{,x}. \tag{5.32}$$

In these new variables Eq. (5.30) reads

$$\rho(u_1, u_2)u_{1,3} = u_{2,1}. \tag{5.33}$$

Solutions of (5.33) yield solutions of (5.30), provided

$$u_{2,3} = u_{1,2}. \tag{5.34}$$

We look for a nonclassical symmetry of Eqs. (5.33) and (5.34) of the form

$$\xi = \partial_3 + \xi_1(u)\,\partial_1 + \xi_2(u)\,\partial_2. \tag{5.35}$$

Condition (4.12) yields

$$\rho\xi_1\,\xi_{2,u_2} - \rho\xi_2\,\xi_{1,u_2} - \xi_{2,u_1} = 0, \tag{5.36}$$

$$\xi_1\,\xi_{2,u_1} - \xi_2\,\xi_{1,u_1} + \xi_{1,u_2} = 0. \tag{5.37}$$

Simple solutions of Eqs. (5.36), (5.37) are given by

(i) $\xi_1 = 0, \qquad \xi_2 = f_2(u_2),$

(ii) $\xi_1 = f_1(u_1), \qquad \xi_2 = 0.$

In both cases ξ is not proportional to any classical symmetry of Eqs. (5.33), (5.34). In case (i) all ξ-invariant solutions of these equations are given implicitly by

$$x_1 = h_1\big(u_1 + f(u_2)\big), \qquad x_2 - tf_{,u_2} = -\big(f_{,u_2}\big)^2 \int \rho h_{1,u_1}\, du_1 + h_2(u_2), \tag{5.38}$$

where f, h_1, h_2 are arbitrary functions of one variable. In case (ii) the ξ-invariant solutions are given by

$$x_1 - tf_1 = -f_1 n_{,u_1}, \qquad x_2 = n_{,u_2}, \tag{5.39}$$

where $n(u)$ satisfies the linear equation

$$\rho f_1 n_{,u_2u_2} + n_{,u_1u_2} = 0. \tag{5.40}$$

For $\rho = \rho(u_1)$ one can find the general solution of Eq. (5.40). The resulting solutions of (5.30) are given implicitly by

$$\rho x_1 - tf_{,u_1} = -\big(f_{,u_1}\big)^2 x_2 + h_1(u_1), \qquad u_2 = h_2(x_2) - f(u_1), \tag{5.41}$$

where f, h_1, h_2 are arbitrary functions of one variable.

ACKNOWLEDGMENTS

This work has been partially supported by research grants from NSERC of Canada, FCAR du Québec, and the Polish Committee for Scientific Research (Grant 201689101).

REFERENCES

1. M. Burnat, The hyperbolic double wave, *Bull. Acad. Polon. Sci. Sér. Sci. Tech.* **17** (1969), 10–16.
2. Z. Peradzynski, Nonlinear plane *k*-waves and Riemann invariants, *Bull. Acad. Polon. Sci. Sér. Sci. Tech.* **19** (1969), 59–69.
3. A. M. Grundland, Riemann invariants for nonhomogeneous systems of quasilinear partial differential equations. Conditions of involution, *Bull. Acad. Polon. Sci. Sér. Sci. Tech.* **22** (1974), 177–185.
4. A. Jeffrey, ''Quasilinear Hyperbolic Systems and Wave Propagation,'' Longman Scientific & Technical, London, 1976.
5. B. L. Rozdestvenskii and N. N. Janenko, Systems of quasilinear equations and their applications to gas dynamics, *in* ''Translations of Math. Monographs,'' Vol. 55, Amer. Math. Soc., Providence, RI, 1983.
6. G. W. Bluman and J. D. Cole, The general similarity solution of the heat equation, *J. Math. Mech.* **18** (1969), 1025–1042.
7. P. J. Olver and P. Rosenau, The construction of special solutions to partial differential equations, *Phys. Lett. A* **114** (1986), 107–112.
8. P. J. Olver and P. Rosenau, Group-invariant solutions of differential equations, *SIAM J. Appl. Math.* **47** (1987), 263–278.
9. D. Levi and P. Winternitz, Nonclassical symmetry reduction: Example of the Boussinesq equation, *J. Phys. A: Math. Gen.* **22** (1989), 2915–2924.
10. E. Pucci, Similarity reductions of partial differential equations, *J. Phys. A: Math. Gen.* **25** (1992), 2631–2646.
11. E. Pucci and G. Saccomandi, On the weak symmetry groups of partial differential equations, *J. Math. Anal. Appl.* **163** (1992), 588–600.
12. E. Vorobev, Reduction and quotient equations for differential equations with symmetries, *Acta Appl. Math.* **23** (1991), 1–24.
13. P. J. Olver, ''Application of Lie Groups to Differential Equations,'' Springer-Verlag, Berlin, 1986.
14. G. W. Bluman and S. Kumei, ''Symmetries and Differential Equations,'' Springer-Verlag, Berlin, 1989.
15. A. M. Grundland and J. Tafel, On the existence of nonclassical symmetries of partial differential equations, *J. Math. Phys.* **36** (1995), 1426–1434.