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The endomorphism monoid of $\overline{P_n}$ ☆

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Abstract

In this paper, the endomorphism monoid of $\overline{P_n}$, the complement of a path P_n with n vertices, is explored explicitly. It is shown that $\mathrm{End}(\overline{P_n})$ is orthodox. Some enumerative problems concerning $\mathrm{End}(\overline{P_n})$ are solved. In particular, the endomorphism spectrum and the endomorphism type of $\overline{P_n}$ are given.

1. Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is try to establish the relationship between graph theory and algebra theory of semigroups and to apply the theory of semigroups to graph theory. Just as Petrich and Reilly pointed out in [9], in the great range of special classes of semigroups, regular semigroups take a central position from the point of view of the richness of their structural "regularity". So it is natural to ask for which graph G the endomorphism monoid of G is regular (such an open question was raised by Marki in [8]). However, it seems difficult to obtain a general answer to this question. So the strategy for solving this question is finding various kinds of regularity conditions for various kinds of graphs. In [10], the connected bipartite graphs whose endomorphism monoids are regular were explicitly found. The split graphs with regular endomorphism monoids were studied in [6]. Graphs whose endomorphism monoids are regular (orthodox) were obtained in [4] by means of lexicographic product of graphs with

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regular (orthodox) endomorphism monoids. In this paper, we present an infinite family of End-orthodox (and of course End-regular) graphs, known as $\overline{P_n}$, the complements of the paths P_n with n vertices. We explore $\mathrm{End}(\overline{P_n})$ explicitly and give some enumerative problems concerning $\mathrm{End}(\overline{P_n})$. In particular, we obtain the endomorphism spectra and the endomorphism types of $\overline{P_n}$.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. The cardinality of the set $V(X)$ is called the *order* of X. If two vertices x_1 and x_2 are adjacent in graph X, the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. A subgraph H is called an *induced subgraph* of X if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph is called *complete* if for any $a, b \in V(X)$, $\{a, b\} \in E(X)$. We denote by K_n a complete graph with n vertices. A *clique* of a graph X is the maximal complete subgraph of X. The *clique number* of X, denoted by $\omega(X)$, is the maximal order among the cliques of X. A graph is called a *path graph* with n vertices if its edges form a path of length $n-1$, denoted by P_n. The *complement graph* $\overline{X}$ of X is a graph such that $V(\overline{X}) = V(X)$ and $E(\overline{X}) = (V \times V) \setminus E(X)$.

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* (from X to Y) if $\{x_1, x_2\} \in E(X)$ implies $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is called a *half-strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. A homomorphism f is called a *locally strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that for every preimage $x_1 \in V(X)$ of $f(a)$ there exists a preimage $x_2 \in V(X)$ of $f(b)$ such that $\{x_1, x_2\} \in E(X)$ and analogously for every preimage of $f(b)$. A homomorphism f is called a *quasi-strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that there exists a preimage $x_1 \in V(X)$ of $f(a)$ which is adjacent to every preimage of $f(b)$ and analogously for a preimage of $f(b)$. A homomorphism f is called a *strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that any preimage of $f(a)$ is adjacent to any preimage of $f(b)$. A homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is a homomorphism. A homomorphism (resp. isomorphism) f from X to itself is called an *endomorphism* (resp. *automorphism*) of X (see [3]). The sets of all endomorphisms, half-strong endomorphisms, locally strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms and automorphisms of X are denoted by $\mathrm{End}(X)$, $h\mathrm{End}(X)$, $l\mathrm{End}(X)$, $q\mathrm{End}(X)$, $s\mathrm{End}(X)$ and $\mathrm{Aut}(X)$, respectively. Clearly, for any graph X, we always have

$$\mathrm{Aut}(X) \subseteq s\mathrm{End}(X) \subseteq q\mathrm{End}(X) \subseteq l\mathrm{End}(X) \subseteq h\mathrm{End}(X) \subseteq \mathrm{End}(X).$$

It is well known that $\mathrm{End}(X)$ and $s\mathrm{End}\,X$ form monoids with respect to composition of mappings and $\mathrm{Aut}\,X$ is a group. Recall from Proposition 2.1 [3] that $h\mathrm{End}\,X$, $l\mathrm{End}\,X$ and $q\mathrm{End}\,X$ do not form monoids in general. Various endomorphisms were investigated by many authors (see [3] and its references). To pursue a more systematic treatment of different endomorphisms, Böttcher and Knauer in [3] introduce the concepts of the endomorphism spectrum and the endomorphism type of a graph. For a graph X, the 6-tuple

$$(|\mathrm{End}\,X|, |h\mathrm{End}\,X|, |l\mathrm{End}\,X|, |q\mathrm{End}\,X|, |s\mathrm{End}\,X|, |\mathrm{Aut}\,X|)$$

is called the endomorphism spectrum of X and is denoted by $\mathrm{Endospec}\,X$, that is,

$$\mathrm{Endospec}\,X = (|\mathrm{End}\,X|, |h\mathrm{End}\,X|, |l\mathrm{End}\,X|, |q\mathrm{End}\,X|, |s\mathrm{End}\,X|, |\mathrm{Aut}\,X|).$$

Associate with $\mathrm{Endospec}\,X$ a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$ with $s_i \in \{0, 1\}$, $i = 1, 2, 3, 4, 5$, where $s_i = 0$ indicates that the ith coordinate is equal to the $(i+1)$th coordinate in $\mathrm{Endospec}\,X$; $s_i = 1$

otherwise. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the *endomorphism type* of X and is denoted by Endotype$X$.

There are 32 possibilities, that is, endotype 0 up to endotype 31. It is known that Endotype 0 describes unretractive graphs, endotype 0 up to 15 describe S-unretractive graphs, endotype 16 describes $E-S$-unretractive graphs which are not unretractive, endotype 31 describes graphs for which all six sets are different (see [3] and its references).

Let f be an endomorphism of a graph X. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e., for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f. The partition π of $V(X)$ corresponding to ρ_f is called the *kernel* of f. By X/ρ_f we denote the *factor graph* of X under ρ_f, that is a graph with $V(X/\rho_f) = V(X)/\rho_f$ and $\{[a]_{\rho_f}, [b]_{\rho_f}\} \in E(X/\rho_f)$ if and only if there exist $c \in [a]_{\rho_f}$ and $d \in [b]_{\rho_f}$ such that $\{c, d\} \in E(X)$ (see [7]).

Let S be a semigroup. An element a of S is called *regular* if there exists $x \in S$ such that $axa = a$. Moreover, if $xax = x$, then x is called a *inverse* of a in S. Denote by $V(f)$ the set of all inverses of $f \in S$. An element e of S is called *idempotent* if $e^2 = e$. Denote by Idpt(S) the set of all idempotent elements of S. A semigroup S is called regular if all its elements are regular. A regular semigroup S is called *orthodox* if Idpt(S) forms a semigroup under the operation of S. A graph X is said to be End-regular (End-orthodox) if its endomorphism monoid End(X) is regular (orthodox). Clearly, orthodox semigroups are regular; hence End-orthodox graphs are End-regular.

We shall use the standard terminology and notation of semigroup theory as in [2] and of graph theory as in [1,3]. The following results will be used in this paper.

Lemma 1.1 ([6]). *Let X be a graph and $f \in \mathrm{End}(X)$. Then:*

(1) *$f \in h\mathrm{End}(X)$ if and only if I_f is an induced subgraph of X.*
(2) *If f is regular, then $f \in h\mathrm{End}(X)$.*

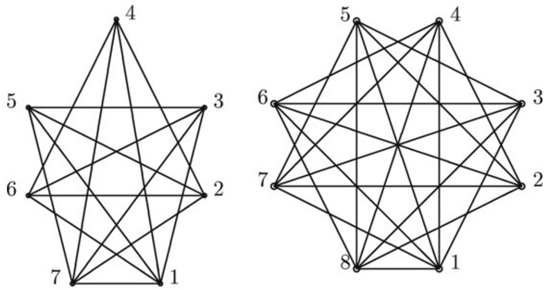
Lemma 1.2 ([7]). *Let X be a graph and $f \in \mathrm{End}(X)$. Then f is regular if and only if there exist $g, h \in \mathrm{Idpt}(\mathrm{End}(X))$ such that $\rho_g = \rho_f$ and $I_h = I_f$.*

2. End($\overline{P_n}$) is orthodox

In this section, we will investigate the endomorphisms of $\overline{P_n}$ (the complement of a path P_n) and prove that End($\overline{P_n}$) is orthodox. We label the graph $\overline{P_n}$ by the numbers $1, 2, 3, \ldots, n$ in a counterclockwise manner; for example, see Fig. 1.

It is trivial that $\{i, j\} \in E(\overline{P_n})$ if and only if $2 \le |i - j| \le n - 1$ for any $i, j \in \{1, 2, \ldots, n\}$. Note that $\overline{P_n}$ can be viewed as a graph obtained by adding an edge $\{1, n\}$ to $\overline{C_n}$, where C_n is a cycle with n vertices. Thus $\overline{P_n}$ looks like $\overline{C_n}$, but the endomorphism monoids of $\overline{P_n}$ and $\overline{C_n}$ are quite different. By K_n^* we denote a graph obtained by deleting an edge from K_n, that is $K_n^* = K_n - e$, where e is any edge of K_n.

To prove that End($\overline{P_n}$) is orthodox, there are two cases to be considered, namely where $\overline{P_n}$ has odd number of vertices or even number of vertices. We first consider the case for $\overline{P_n}$ having odd number of vertices and suppose that $n = 2m + 1$ for some positive integer m. Let



$\overline{P_7}$ $\overline{P_8}$

Fig. 1. Graphs $\overline{P_7}$ and $\overline{P_8}$.

$S_1 = \{1, 3, 5, \dots, 2m+1\}$. Then S_1 is the only independent set with $m+1$ vertices in P_{2m+1}. Clearly End($\overline{P_3}$) is orthodox, so we may assume that $m \geq 2$.

Lemma 2.1. (1) *The induced subgraph of $\overline{P_{2m+1}}$ determined by S_1, namely $\langle S_1 \rangle$, is the only clique of order $m+1$ in $\overline{P_{2m+1}}$.*
(2) *$\overline{P_{2m+1}}$ does not contain a subgraph isomorphic to K^*_{m+2}.*

Proof. (1) Since S_1 is the only independent set with $m+1$ vertices in P_{2m+1}, the induced subgraph $\langle S_1 \rangle$ of $\overline{P_{2m+1}}$ is the only clique of order $m+1$ in $\overline{P_{2m+1}}$.

(2) Assume $\overline{P_{2m+1}}$ contains a subgraph isomorphic to K^*_{m+2}. Then it contains more than one clique of order $m+1$, which contradicts (1).

Note that $\langle S_1 \rangle$ is isomorphic to K_{m+1}, the complete graph of $m+1$ vertices. We may identify $\langle S_1 \rangle$ with K_{m+1}.

Lemma 2.2. *Let f be an endomorphism of $\overline{P_{2m+1}}$. Then*

(1) *$f(K_{m+1}) = K_{m+1}$,*
(2) *if $x_1, x_2 \in V(\overline{P_{2m+1}})$ such that $f(x_1) = f(x_2)$, then $|x_1 - x_2| = 1$,*
(3) *there are no three vertices $x_1, x_2, x_3 \in V(\overline{P_{2m+1}})$ such that $f(x_1) = f(x_2) = f(x_3)$.*

Proof. (1) Since any endomorphism f maps a clique to a clique of the same size and K_{m+1} is the only clique of size $m+1$ in $\overline{P_{2m+1}}$, we have $f(K_{m+1}) = K_{m+1}$.

(2) Note that $\{x_1, x_2\} \in E(\overline{P_{2m+1}})$ if and only if $|x_1 - x_2| \neq 1$. If $f(x_1) = f(x_2)$, then $\{x_1, x_2\} \notin E(\overline{P_{m+1}})$ and so $|x_1 - x_2| = 1$.

(3) It follows immediately from (2).

Lemma 2.3. *Let $f \in$ End($\overline{P_{2m+1}}$) and $f(i) = f(i+1)$ for some $i \in \{1, 2, \dots, 2m\}$.*

(1) *If i is even, then $f(i) = f(i+1), f(i+2) = f(i+3), \dots, f(2m) = f(2m+1)$.*
(2) *If i is odd, then $f(i+1) = f(i), f(i-1) = f(i-2), \dots, f(2) = f(1)$.*

Proof. (1) If $f(i) = f(i+1)$ and i is even, then by Lemma 2.2 either $f(i+2) = f(i+3)$ or $[i+2]_{\rho_f} = \{i+2\}$. If $[i+2]_{\rho_f} = \{i+2\}$, since $i+2$ is adjacent to all vertices in $S_1 \setminus \{i+1, i+3\}$, we thus have that $f(i+2)$ is adjacent to all vertices in $f(S_1) \setminus \{f(i+1), f(i+3)\}$. On the other hand, $\{i, i+2\} \in E(\overline{P_{2m+1}})$ implies that $\{f(i+1), f(i+2)\} = \{f(i), f(i+2)\} \in E(\overline{P_{2m+1}})$. Hence the induced subgraph of $\overline{P_{2m+1}}$ determined by $f(S_1) \cup \{f(i+2)\}$ is isomorphic to either

K_{m+2} or K^*_{m+2}. Both cases will yield a contradiction. Thus we must have $f(i+2) = f(i+3)$. With a similar argument, we may show that $f(i+4) = f(i+5), \cdots, f(2m) = f(2m+1)$.

(2) It is the dual of (1).

Lemma 2.4. *Let $f \in \mathrm{End}(\overline{P_{2m+1}})$. Then I_f is an induced subgraph of $\overline{P_{2m+1}}$.*

Proof. Let $f \in \mathrm{End}(\overline{P_{2m+1}})$. To prove that I_f is an induced subgraph of $\overline{P_{2m+1}}$, let $a, b \in V(I_f)$ with $\{a, b\} \in E(\overline{P_{2m+1}})$. We need to show that there exist $c \in f^{-1}(a)$ and $d \in f^{-1}(b)$ such that $\{c, d\} \in E(\overline{P_{2m+1}})$.

If both of a and b are odd, since $f(S_1) = S_1$, there exist two odd numbers c and d such that $c \in f^{-1}(a)$, $d \in f^{-1}(b)$ and $\{c, d\} \in E(\overline{P_{2m+1}})$.

If both of a and b are even, then both of $f^{-1}(a)$ and $f^{-1}(b)$ contain only one even number. Let $c = f^{-1}(a)$ and $d = f^{-1}(b)$. Then $\{c, d\} \in E(\overline{P_{2m+1}})$.

If exactly one of a and b is even, without loss of generality, suppose a is even and b is odd. Then $|a - b| \geq 3$ and $|f^{-1}(a)| = 1$ and so $c = f^{-1}(a)$ is even. If $|f^{-1}(b)| = 1$ and $|c - f^{-1}(b)| = 1$, then $a = f(c)$ is adjacent to at least m vertices in $f(S_1)$ and so $f(\overline{P_{2m+1}})$ contains a subgraph which is isomorphic to K^*_{m+2}; this is a contradiction. Hence we must have $|c - f^{-1}(b)| \geq 2$ and so $\{c, f^{-1}(b)\} \in E(\overline{P_{2m+1}})$. If $|f^{-1}(b)| \geq 2$; since $f(S_1) = S_1$, there is an even number $d \in f^{-1}(b)$ and therefore $\{c, d\} \in E(\overline{P_{2m+1}})$.

For any $f \in \mathrm{End}(\overline{P_{2m+1}})$, let $S_f = \{i \in V(\overline{P_{2m+1}}) \mid |[i]_{\rho_f}| = 1\}$. We have

Lemma 2.5. *Let $f \in \mathrm{End}(\overline{P_{2m+1}})$. Then*

(1) *S_f $(f(S_f))$ is not empty and forms an interval of $V(\overline{P_{2m+1}})$. In particular, the even numbers that appeared in $f(\overline{P_{2m+1}})$ are consecutive even numbers in $V(\overline{P_{2m+1}})$.*

(2) *Let i (j) be the least (greatest) number in S_f. Then i and j are odd. If this is the case, the partition on $V(\overline{P_{2m+1}})$ induced by ρ_f is*

$$\{1, 2\}, \ldots, \{i-2, i-1\}, \{i\}, \{i+1\}, \ldots, \{j\}, \{j+1, j+2\}, \ldots, \{2m, 2m+1\}.$$

(3) *Let i_0 (j_0) be the least (greatest) number in $f(S_f)$. Then i_0 and j_0 are odd. If this is the case, $V(I_f) = \{1, 3, \ldots, i_0 - 2, i_0, i_0 + 1, \ldots, j_0, j_0 + 1, \ldots, 2m + 1\}$.*

Proof. (1) and (2) follow from Lemma 2.3 immediately. Note that $f(S_1) = S_1$. For any $a \in V(I_f)$ with a being even, $|f^{-1}(a)| = 1$ and so $f^{-1}(a)$ contains only an even number. Now (3) follows from (1), (2) and Lemma 2.4 directly.

Proposition 2.6. $\mathrm{End}(\overline{P_{2m+1}})$ *$(m \geq 2)$ is regular.*

Proof. To prove that $\mathrm{End}(\overline{P_{2m+1}})$ is regular, let $f \in \mathrm{End}(\overline{P_{2m+1}})$. We only need to show that there exist two idempotent endomorphisms g and h such that $\rho_g = \rho_f$ and $I_h = I_f$.

Without loss of generality, suppose that $V(\overline{P_{2m+1}})/\rho_f = \{[1]_{\rho_f}, [3]_{\rho_f}, \ldots, [2m+1]_{\rho_f}, [2i_1]_{\rho_f}, [2i_2]_{\rho_f}, \ldots, [2i_n]_{\rho_f}\}$ for some n and $i_1, i_2, \ldots, i_n$. Then by Lemmas 2.3 and 2.5 $[2i_1]_{\rho_f}, [2i_2]_{\rho_f}, \ldots, [2i_n]_{\rho_f}$ are singleton and $i_1, i_2, \ldots, i_n$ are consecutive numbers. Without loss of generality, we may suppose that $i_1 < i_2 < \cdots < i_n$. Also by Lemmas 2.3 and 2.5 we have

$$[k]_{\rho_f} = \begin{cases} \{k, k+1\} & \text{if } k \in \{1, 3, \ldots, 2i_1 - 3\}, \\ \{k\} & \text{if } k \in \{2i_1 - 1, 2i_1, \ldots, 2i_n + 1\}, \\ \{k-1, k\} & \text{if } k \in \{2i_n + 3, 2i_n + 5, \ldots, 2m + 1\}. \end{cases}$$

Define a mapping g from $V(\overline{P_{2m+1}})$ to itself by

$$g(x) = \begin{cases} 2k+1 & \text{if } x \in [2k+1]_{\rho_f}, k \in \{0, 1, \ldots, m\}, \\ x & \text{if } x \in [2j]_{\rho_f}, j \in \{i_1, i_2, \ldots, i_n\}. \end{cases}$$

Then $g \in \mathrm{End}(\overline{P_{2m+1}})$ and $\rho_f = \rho_g$. If $x \in [2k+1]_{\rho_f}$ for $k \in \{0, 1, \ldots, m\}$, then $g^2(x) = g(2k+1) = 2k+1 = g(x)$. If $x \in [2j]_{\rho_f}$ for $j \in \{i_1, i_2, \ldots, i_n\}$, then $g^2(x) = g(x) = x$. Hence $g^2 = g$ and g is an idempotent endomorphism.

By Lemma 2.5 we know the even vertices in $V(I_f)$ must be consecutive even numbers; without loss of generality, suppose that they are $2j_1, 2j_2, \ldots, 2j_n$ with $j_1 < j_2 < \cdots < j_n$. Define a mapping h from $V(\overline{P_{2m+1}})$ to itself by

$$h(x) = \begin{cases} x & \text{if } x \in V(I_f), \\ x-1 & \text{if } x \in V(\overline{P_{2m+1}}) \setminus V(I_f) \text{ and } x < 2j_1, \\ x+1 & \text{if } x \in V(\overline{P_{2m+1}}) \setminus V(I_f) \text{ and } x > 2j_n. \end{cases}$$

Then $h \in \mathrm{End}(\overline{P_{2m+1}})$ and $I_h = I_f$. If $x \in V(I_f)$, then $h^2(x) = h(x) = x$. If $x \in V(\overline{P_{2m+1}}) \setminus V(I_f)$ and $x < 2j_1$, then $h^2(x) = h(x-1) = x-1 = h(x)$. If $x \in V(\overline{P_{2m+1}}) \setminus V(I_f)$ and $x > 2j_n$, then $h^2(x) = h(x+1) = x+1 = h(x)$. Hence $h^2 = h$ and h is an idempotent endomorphism.

The next lemma describes the idempotent endomorphisms of $\overline{P_{2m+1}}$.

Lemma 2.7. *Element $f \in \mathrm{End}(\overline{P_{2m+1}})$ is idempotent if and only if*

$$f = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i \\ 1 & 1 & \cdots & i-1 & i-1 \end{pmatrix} \begin{pmatrix} j & j+1 & \cdots & 2m & 2m+1 \\ j+1 & j+1 & \cdots & 2m+1 & 2m+1 \end{pmatrix}$$

for some i, j with $1 \le i < j \le 2m+1$, where i (j) must be even whenever $i > 1$ ($j < 2m+1$), $i + 2 \le j$ whenever $i > 1$ and $j < 2m+1$.

Proof. Note that $f(S_1) = S_1$ and f is idempotent if and only if $f(x) = x$ for all $x \in V(I_f)$. The assertion follows immediately from Lemma 2.3 and the proof of Proposition 2.6.

Theorem 2.8. $\mathrm{End}(\overline{P_{2m+1}})$ *is orthodox.*

Proof. By Proposition 2.6, we only need to show that the composition of any two idempotent endomorphisms is also an idempotent endomorphism.

Let

$$f = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i \\ 1 & 1 & \cdots & i-1 & i-1 \end{pmatrix} \begin{pmatrix} j & j+1 & \cdots & 2m & 2m+1 \\ j+1 & j+1 & \cdots & 2m+1 & 2m+1 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 1 & 2 & \cdots & s-1 & s \\ 1 & 1 & \cdots & s-1 & s-1 \end{pmatrix} \begin{pmatrix} t & t+1 & \cdots & 2m & 2m+1 \\ t+1 & t+1 & \cdots & 2m+1 & 2m+1 \end{pmatrix}$$

be two idempotent endomorphisms of $\overline{P_{2m+1}}$. Without loss of generality, we may assume that $i, s > 1$ and $j, t < 2m+1$. Then i, j, s and t are even numbers. If $i < s < j < t$, then

$$fg = \begin{pmatrix} 1 & 2 & \cdots & s-1 & s \\ 1 & 1 & \cdots & s-1 & s-1 \end{pmatrix} \begin{pmatrix} j & j+1 & \cdots & 2m & 2m+1 \\ j+1 & j+1 & \cdots & 2m+1 & 2m+1 \end{pmatrix}$$

is also an idempotent endomorphism. If $i < j < s < t$, then

$$fg = \begin{pmatrix} 1 & 2 & \cdots & s-3 & s-2 \\ 1 & 1 & \cdots & s-3 & s-3 \end{pmatrix} \begin{pmatrix} s & s+1 & \cdots & 2m & 2m+1 \\ s+1 & s+1 & \cdots & 2m+1 & 2m+1 \end{pmatrix}$$

is also an idempotent endomorphism. Similarly we can show that fg is an idempotent endomorphism for the other cases.

Next we consider the case for $\overline{P_n}$ having even number of vertices and suppose that $n = 2m$ for some positive integer m. It is clear that $\mathrm{End}(K_2)$ is orthodox, so we may assume that $m \geq 2$.

Lemma 2.9. (1) *$\omega(\overline{P_{2m}}) = m$. In particular, the cliques of $\overline{P_{2m}}$ of order m are of the form $\langle 1, 3, \ldots, 2k-1, 2(k+1), 2(k+2), \ldots, 2m \rangle$ where $0 \leq k \leq m$ ($k = 0$ means that the clique is $\langle 2, 4, \ldots, 2m \rangle$, $k = m$ means that the clique is $\langle 1, 3, \ldots, 2m-1 \rangle$).*

(2) *Any two adjacent vertices in $V(\overline{P_{2m}}) \setminus V(K)$ are not both adjacent to $m-1$ vertices in K for any clique K of order m.*

Proof. (1) Note that P_{2m} has no independent set which contains more than m vertices and the sets $\{1, 3, \ldots, 2k-1, 2(k+1), 2(k+2), \ldots, 2m\}$, $k = 0, 1, \ldots, m$, are the only independent sets which contains m vertices. The assertion follows immediately.

(2) Let K be a clique of order m in $\overline{P_{2m}}$. If K contains both of 1 and $2m$, then there are only two vertices in $V(\overline{P_{2m}}) \setminus V(K)$ which are adjacent to $m-1$ vertices in K; they are two consecutive numbers and they are not adjacent. If K contains exactly one of 1 and $2m$, then there is only one vertex in $V(\overline{P_{2m}}) \setminus V(K)$ which is adjacent to $m-1$ vertices of K.

Lemma 2.10. *Let $f \in \mathrm{End}(\overline{P_{2m}})$ and $x_1, x_2 \in V(\overline{P_{2m}})$ be such that $f(x_1) = f(x_2)$. Then $|x_1 - x_2| = 1$. Moreover, if $x_1 < x_2$, then x_1 is odd.*

Proof. The first assertion is obvious.

Suppose that there exists a vertex $2k$ in $V(\overline{P_{2m}})$ such that $f(2k) = f(2k+1)$. Assume without loss of generality that k is minimal among the numbers with such a property. Consider the following cases:

(1) If $k = 1$, then since the subgraph of $\overline{P_{2m}}$ induced by $A = \{2, 4, \ldots, 2m\}$ is a clique isomorphic to K_m and $\{1, 3\} \in E(\overline{P_{2m}})$, we have that $f(1)$ is adjacent to every vertex in $f(A)$. Hence the subgraph of $\overline{P_{2m}}$ induced by $\{f(1), f(2), f(4), \ldots, f(2m)\}$ is isomorphic to K_{m+1}, which is a contradiction.

(2) If $k > 1$, then $f(2k-1) \neq f(2k)$ and $[2k-1]_{\rho_f} = \{2k-1\}$ by the minimality.

(2a) Assume $f(2k-2) = f(2k-3)$. Then $f(\{2k-1\} \cup \{2, 4, \ldots, 2m\}) \cong K_{m+1}$, which is impossible.

(2b) Assume $f(2k-2) \neq f(2k-3)$.

(i) If $k = 2$, then $f(1)$ and $f(3)$ are not in $f(A)$ and each is adjacent to at least $m-1$ vertices in $f(A)$, which contradicts Lemma 2.9.

(ii) If $k \geq 3$, then $f(2k-3) \neq f(2k-4)$ by the minimality. Take the clique B of order m in $\overline{P_{2m}}$ not containing $2k-3$ and $2k-4$. Then $f(2k-1)$ and $f(2k-3)$ are not in $f(B)$ and both adjacent to at least $m-1$ vertices of $f(B)$. This contradiction completes the proof.

Lemma 2.11. *Let $f \in \mathrm{End}(\overline{P_{2m}})$.*

(1) *If $[2i]_{\rho_f} = \{2i\}$ and $f(2i+1) = f(2i+2)$, then $f(2i+3) = f(2i+4)$, $f(2i+5) = f(2i+6), \ldots, f(2m-1) = f(2m)$.*

(2) *If* $f(2i+1) = f(2i+2)$ *and* $[2i+3]_{\rho_f} = \{2i+3\}$*, then* $f(2i-1) = f(2i)$, $f(2i-3) = f(2i-2), \ldots, f(1) = f(2)$.

Proof. We only need to prove (1). Let $A = \{2, 4, \ldots, 2m\}$. Suppose that there exists a vertex $2k+1$ with $2i+3 \leq 2k+1 < 2m$ such that $[2k+1]_{\rho_f} = \{2k+1\}$ and $f(2j+1) = f(2j+2)$ for $j = i, i+1, \ldots, k-1$. In this case, $f(2k+1)$ is adjacent to every vertex in $f(A) \setminus \{f(2k+2)\}$. Now $\omega(\overline{P_{2m}}) = m$ implies that the subgraph of $\overline{P_{2m}}$ induced by $f(A) \cup \{f(2k+1)\}$ is isomorphic to K^*_{m+1}.

By Lemma 2.10, if $[2s]_{\rho_f} = \{2s\}$ for some s, then $[2s-1]_{\rho_f} = \{2s-1\}$. We claim that $[t]_{\rho_f} = \{t\}$ for $t = 1, 2, \ldots, 2i-1$. Otherwise, there exists $j < i$ such that $[2j-1]_{\rho_f} = \{2j-1\}$ and $f(2j-3) = f(2j-2)$. Hence the subgraph of $\overline{P_{2m}}$ induced by $f(A) \cup \{f(2j-1)\}$ is isomorphic to K^*_{m+1}. Thus $f(2j-1)$ and $f(2k+1)$ are adjacent to $m-1$ vertices of $f(A)$. Note that $\{2k+1, 2j-1\} \in E(\overline{P_{2m}})$. This is a contradiction to Lemma 2.9(2). Now $[1]_{\rho_f} = \{1\}$ implies that the subgraph of $\overline{P_{2m}}$ induced by $f(A) \cup \{f(1)\}$ is isomorphic to K^*_{m+1}. This yields a contradiction to Lemma 2.9(2). Therefore (1) holds.

Lemma 2.12. *Let* $f \in \mathrm{End}(\overline{P_{2m}})$*. Then* I_f *is an induced subgraph of* $\overline{P_{2m}}$.

Proof. Let $f \in \mathrm{End}(\overline{P_{2m}})$ and suppose that I_f is not an induced subgraph of $\overline{P_{2m}}$. Then there exist two vertices $a, b \in I_f$ with $\{a, b\} \in E(\overline{P_{2m}})$ such that $\{x, y\} \notin E(\overline{P_{2m}})$ for any $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$. In this case, it is easy to see that $|f^{-1}(a)| = 1$ and $|f^{-1}(b)| = 1$. Let $f^{-1}(a) = \{c\}$ and $f^{-1}(b) = \{d\}$. Then $|c-d| = 1$. Without loss of generality, we may suppose that $c < d$.

If c is odd, let $A = \{1, 3, 5, \ldots, c, d+2, d+4, \ldots, 2m\}$. Then the subgraph of $\overline{P_{2m}}$ induced by A is a clique of order m which is isomorphic to K_m and d is adjacent to every vertex of $A \setminus \{c\}$. Thus $f(d)$ is adjacent to every vertex of $f(A) \setminus \{f(c)\}$ and so the subgraph of $\overline{P_{2m}}$ induced by $f(A) \cup \{b\}$ is isomorphic to K_{m+1} since $\{f(c), f(d)\} = \{a, b\} \in E(\overline{P_{2m}})$. This is a contradiction to $\omega(\overline{P_{2m}}) = m$.

If c is even, then by Lemma 2.10, $[c-1]_{\rho_f}$, $[c]_{\rho_f}$, $[d]_{\rho_f}$ and $[d+1]_{\rho_f}$ are singleton. Let $B = \{1, 3, \ldots, c-3, c, d+1, d+3, \ldots, 2m\}$. Then the subgraph of $\overline{P_{2m}}$ induced by B is a clique of order m which is isomorphic to K_m and d is adjacent to every vertex of $B \setminus \{c, d+1\}$. Thus $f(d)$ is adjacent to every vertex of $f(B) \setminus \{f(c), f(d+1)\}$. Note that $\{f(c), f(d)\} = \{a, b\} \in E(\overline{P_{2m}})$ and $\omega(\overline{P_{2m}}) = m$. We obtain that the subgraph of $\overline{P_{2m}}$ induced by $f(B) \cup \{f(d)\}$ is isomorphic to K^*_{m+1}. Since $c-1$ is adjacent to every vertex of $B \setminus \{c\}$, $f(c-1)$ is adjacent to every vertex of $f(B) \setminus \{f(c)\}$. Therefore we conclude that $f(c-1)$ and $f(d)$ are adjacent to exactly $m-1$ vertices of the clique $f(B)$. This is a contradiction to Lemma 2.9(2) since $\{c-1, d\} \in E(\overline{P_{2m}})$. Consequently, I_f is an induced subgraph of $\overline{P_{2m}}$.

Lemma 2.13. *Let* $f \in \mathrm{End}(\overline{P_{2m}})$ *and* $S_f = \{i \in \overline{P_{2m}} \,||[i]_{\rho_f}| = 1\}$*. If* S_f *is not empty, then both of* S_f *and* $f(S_f)$ *consist of consecutive numbers of* $V(\overline{P_{2m}})$.

Proof. Let $f \in \mathrm{End}(\overline{P_{2m}})$. Then by Lemmas 1.1 and 2.12 f is a half-strong endomorphism. That S_f consists of consecutive numbers of $V(\overline{P_{2m}})$ follows directly from Lemma 2.11. Let $i, i+1 \in S_f$. Since $\{i, i+1\} \notin E(\overline{P_{2m}})$, we have $\{f(i), f(i+1)\} \notin E(\overline{P_{2m}})$. So $|f(i) - f(i+1)| = 1$, that is, $f(i)$ and $f(i+1)$ are two consecutive numbers. Observe that f is an isomorphism from the subgraph $\langle S_f \rangle$ to the subgraph $\langle f(S_f) \rangle$. Therefore $f(S_f)$ consists of consecutive numbers.

Lemma 2.14. *Let* $f \in \mathrm{End}(\overline{P_{2m}})$.

(1) *If S_f is not empty, let i (j) be the least (greatest) number in S_f. Then i is odd and j is even. If this is the case, the partition on $V(\overline{P_{2m}})$ induced by ρ_f is*

$$\{1,2\},\ldots,\{i-2,i-1\},\{i\},\{i+1\},\ldots,\{j\},\{j+1,j+2\},\ldots,\{2m-1,2m\}.$$

(2) *Let i_0 (j_0) be the least (greatest) number in $f(S_f)$. Then i_0 is odd and j_0 is even. Moreover, $V(I_f)=\{1,3,\ldots,i_0-2,i_0,i_0+1,\ldots,j_0,j_0+2,j_0+4,\ldots,2m\}$.*

(3) *If S_f is empty, then the partition on $V(\overline{P_{2m}})$ induced by ρ_f is*

$$\{1,2\},\{3,4\},\ldots,\{2m-1,2m\}$$

and I_f is a clique of order m.

Proof. Observe that $|V(\overline{P_{2m}})\setminus S_f|$ is even and the cliques of $f(\overline{P_{2m}})$ are of order m.

(1) It follows from Lemma 2.11 immediately.

(2) If S_f is not empty, then by Lemmas 2.11 and 2.13, the subgraph of $\overline{P_{2m}}$ induced by $f(V(\overline{P_{2m}})\setminus S_f)$ is a clique of order $|V(\overline{P_{2m}})\setminus S_f|/2$ and $\{i_1,i_2\}\in E(\overline{P_{2m}})$ for any vertex $i_1\in f(S_f)$ and any vertex $i_2\in f(V(\overline{P_{2m}})\setminus S_f)$. Hence $i_0-1,\ j_0+1\notin f(V(\overline{P_{2m}})\setminus S_f)$ whenever $i_0-1\geq 1$ and $j_0+1\leq 2m$.

Suppose that i_0 is even. Then j_0 is odd and $|f(V(\overline{P_{2m}})\setminus S_f)|=\frac{i_0-2}{2}+\frac{2m-j_0-1}{2}<\frac{2m+i_0-j_0-1}{2}$. It follows that $\omega(f(\overline{P_{2m}}))<\frac{2m+i_0-j_0-1}{2}+\frac{j_0-i_0+1}{2}=m$. This is a contradiction. Therefore i_0 is odd and j_0 is even. In this case, the clique $f(V(\overline{P_{2m}})\setminus S_f)=\{1,3,\ldots,i_0-2\}\cup\{j_0+2,j_0+4,\ldots,2m\}$, where $f(V(\overline{P_{2m}})\setminus S_f)=\{j_0+2,j_0+4,\ldots,2m\}$ if $i_0=1$ and $f(V(\overline{P_{2m}})\setminus S_f)=\{1,3,\ldots,i_0-2\}$ if $j_0=2m$. Otherwise, without loss of generality, suppose there exists $2k\in\{1,2,\ldots,i_0-2\}$ such that $2k\in V(I_f)$ or $2r+1\in\{j_0+3,j_0+4,\ldots,2m\}$ such that $2r+1\in V(I_f)$, then $|f(V(\overline{P_{2m}})\setminus S_f)|<\frac{2m+i_0-j_0-1}{2}=|V(\overline{P_{2m}})\setminus S_f|/2$. This is also a contradiction.

(3) If S_f is empty, then every ρ_f-class contains two consecutive numbers and $|I_f|=m$; thus I_f is a clique of order m since $\omega(I_f)=\omega(\overline{P_{2m}})=m$.

Proposition 2.15. End$(\overline{P_{2m}})$ *is regular.*

Proof. Let $f\in \mathrm{End}(\overline{P_{2m}})$. We only need to show that there exist idempotent endomorphisms g and h such that $\rho_g=\rho_f$ and $I_h=I_f$. To this end, without loss of generality, suppose $V(\overline{P_{2m}})/\rho_f=\{[k_1]_{\rho_f},\ldots,[k_r]_{\rho_f},[j_1]_{\rho_f},\ldots,[j_s]_{\rho_f},[k_{r+1}]_{\rho_f},\ldots,[k_t]_{\rho_f}\}$, where k_u is odd, $k_1<\cdots<k_r<j_1<\cdots<j_s<k_{r+1}<\cdots<k_t$ and $[k_u]_{\rho_f}=\{k_u,k_u+1\}$, $[j_v]_{\rho_f}=\{j_v\}$ for $u\in\{1,2,\ldots,t\}$, $v\in\{1,2,\ldots,s\}$. Define a mapping g from $V(\overline{P_{2m}})$ to itself by

$$g(x)=\begin{cases}k_u & \text{if } x\in[k_u]_{\rho_f},\ u\in\{1,\ldots,r\},\\ j_v & \text{if } x\in[j_v]_{\rho_f},\ v\in\{1,\ldots,s\},\\ k_u+1 & \text{if } x\in[k_u]_{\rho_f},\ u\in\{r+1,\ldots,t\}.\end{cases}$$

Then it is easy to see that $g\in \mathrm{End}(\overline{P_{2m}})$, $g^2=g$ and $\rho_g=\rho_f$. With the same notation as in Lemma 2.14, let $V(I_f)=f(S_f)\cup\{1,3,\ldots,i_0-2\}\cup\{j_0+2,j_0+4,\ldots,2m\}$. Define a mapping h from $V(\overline{P_{2m}})$ to itself by

$$h(x)=\begin{cases}x & \text{if } x\in V(I_f),\\ x-1 & \text{if } x=2k\in\{2,\ldots,i_0-1\},\\ x+1 & \text{if } x=2k+1\in\{j_0+1,\ldots,2m-1\}.\end{cases}$$

Then it is easy to see that $h\in \mathrm{End}(\overline{P_{2m}})$, $h^2=h$ and $I_h=I_f$.

The following lemma describes the idempotents in $\text{End}(\overline{P_{2m}})$.

Lemma 2.16. *Element $f \in \text{End}(\overline{P_{2m}})$ is idempotent if and only if*

$$f = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i \\ 1 & 1 & \cdots & i-1 & i-1 \end{pmatrix} \begin{pmatrix} j & j+1 & \cdots & 2m-1 & 2m \\ j+1 & j+1 & \cdots & 2m & 2m \end{pmatrix}$$

for some i, j with $1 \le i < j \le 2m$. If $i \neq 1$ and $j \neq 2m$, then i is even, j is odd.

Proof. Note that an endomorphism f of $\overline{P_{2m}}$ is an idempotent if and only if $f(x) = x$ for all $x \in V(I_f)$. This follows from Lemmas 2.11 and 2.14.

Theorem 2.17. $\text{End}(\overline{P_{2m}})$ *is orthodox.*

Proof. The proof follows the same lines as the proof for Theorem 2.8.

Now combining Theorems 2.8 and 2.17, we have

Theorem 2.18. $\text{End}(\overline{P_n})$ *is orthodox.*

3. Endospec$\overline{P_n}$ and Endotype$\overline{P_n}$

In this section, we will give some enumerative results on $\text{End}(\overline{P_n})$ and determine Endospec$\overline{P_n}$ and Endotype$\overline{P_n}$ for $n \ge 4$.

Let S be a semigroup. Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ on S are defined by

$$a \,\mathcal{L}\, b \Leftrightarrow S^1 a = S^1 b,$$
$$a \,\mathcal{R}\, b \Leftrightarrow a S^1 = b S^1,$$
$$\mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

The next lemma characterizes Green's relations on the endomorphism monoid of a graph which is due to Fan [5].

Lemma 3.1 (*[5]*). *Let X be a graph. If f and g are regular endomorphisms of X, then*

(1) *$f \,\mathcal{L}\, g$ if and only if $\rho_f = \rho_g$;*
(2) *$f \,\mathcal{R}\, g$ if and only if $I_f = I_g$;*
(3) *$f \,\mathcal{D}\, g$ if and only if I_f is isomorphic to I_g.*

Applying Lemma 3.1 to $\overline{P_n}$, we have the following more explicit characterization of Green's relations on $\text{End}(\overline{P_n})$.

Lemma 3.2. *Let $f, g \in \text{End}(\overline{P_n})$. Then*

(1) *$f \,\mathcal{L}\, g$ if and only if $S_f = S_g$;*
(2) *$f \,\mathcal{R}\, g$ if and only if $f(S_f) = g(S_g)$;*
(3) *$f \,\mathcal{D}\, g$ if and only if $|V(I_f)| = |V(I_g)|$.*

Proof. (1) Let $f, g \in \text{End}(\overline{P_n})$. Then by Lemma 2.5(2) and Lemma 2.14(1), (3) $\rho_f = \rho_g$ if and only if $S_f = S_g$. Now it follows from Lemma 3.1(1) immediately.

(2) Let $f, g \in \text{End}(\overline{P_n})$. Then by Lemma 2.5(3) and Lemma 2.14(2) $I_f = I_g$ if and only if $f(S_f) = g(S_g)$. Now it follows from Lemma 3.1(2) immediately.

(3) By Lemma 3.1, we only need to prove I_f is isomorphic to I_g if and only if $|V(I_f)| = |V(I_g)|$. The direct part is obvious.

Conversely, if $|V(I_f)| = |V(I_g)|$, then by Lemmas 2.5 and 2.14 $|f(S_f)| = |g(S_g)|$. By Lemma 2.5(1) and Lemma 2.13 suppose that $f(S_f) = \{i, i+1, \ldots, i+t\}$ and $g(S_g) = \{j, j+1, \ldots, j+t\}$ for some nonnegative integers i, j and t. Define a mapping h from $V(I_f)$ to $V(I_g)$ such that $h(i+k) = j+k$ for any $0 \le k \le t$; for any $x \in V(I_f) \setminus f(S_f)$, $h(x) = p(x)$, where p is any bijective from $V(I_f) \setminus f(S_f)$ to $V(I_g) \setminus g(S_g)$). It is a routine matter to verify that h is an isomorphism from I_f to I_g. Hence $I_f \cong I_g$.

Let $f \in \mathrm{End}(\overline{P_n})$ and $t = |V(I_f)|$. Denote by D_t the $\mathcal{D}$-class containing f. In the following, we first consider the case of $n = 2m+1$ with $m \ge 2$ and give the order of $\mathrm{End}(\overline{P_n})$, the endomorphism spectrum and endomorphism type of $\overline{P_{2m+1}}$.

Lemma 3.3. $|\mathrm{End}(\overline{P_{2m+1}})| = 2\sum_{i=1}^{m} i\,i! + (m+1)(m+1)!$

Proof. It is clear that $\mathrm{End}(\overline{P_{2m+1}})$ has $m+1$ $\mathcal{D}$-classes: $D_{m+1}, D_{m+2}, \ldots, D_{2m+1}$. First let $f \in D_{m+1}$. Then $|S_f| = 1$ and S_f contains only an odd number. Thus by Lemma 3.2 there are $m+1$ $\mathcal{L}$-classes in D_{m+1}. Note that $f(S_1) = S_1$. $f(S_f)$ contains only an odd number. On the other hand, $f|_{S_1 \setminus S_f}$, the restriction of f to $S_1 \setminus S_f$, is a bijection from $S_1 \setminus S_f$ to $S_1 \setminus f(S_f)$. Hence there are $(m+1)!$ elements in every $\mathcal{L}$-class and therefore there are $(m+1)(m+1)!$ elements in D_{m+1}. Secondly, for $i = 2, \ldots, m$, let $f \in D_{m+i}$. Then $|S_f| = 2i-1$ and S_f consists of $2i-1$ consecutive numbers which both start and end with an odd number. Since there are $m-i+2$ such subsets of $V(\overline{P_{2m+1}})$, by Lemma 3.2 there are $m-i+2$ $\mathcal{L}$-classes in D_{m+i}. As in the case for D_{m+1}, $f|_{S_1 \setminus S_f}$ is a bijection from $S_1 \setminus S_f$ to $S_1 \setminus f(S_f)$. Observe that f can map S_f to $f(S_f)$ in two ways and there are $m-i+2$ possibilities for $f(S_f)$ in $V(I_f)$. There are $2(m-i+2)(m-i+1)!$ elements in every $\mathcal{L}$-class and therefore there are $2(m-i+2)(m-i+2)!$ elements in D_{m+i}. Finally, let $f \in D_{2m+1}$. Then $|S_f| = 2m+1$ and so f is an automorphism of $\overline{P_{2m+1}}$. Clearly, $D_{2m+1} = \mathrm{Aut}(\overline{P_{2m+1}})$ and D_{2m+1} has two elements. Summing up, we form the following table:

D	$\lvert D/\mathcal{L}\rvert$	$\lvert L\rvert$	$\lvert D\rvert$
D_{m+1}	$m+1$	$(m+1)!$	$(m+1)(m+1)!$
D_{m+2}	m	$2m!$	$2mm!$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{m+i}	$m-i+2$	$2(m-i+2)!$	$2(m-i+2)(m-i+2)!$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{2m}	2	$2 \cdot 2!$	$2 \cdot 2 \cdot 2!$
D_{2m+1}	1	2	2

Consequently $|\mathrm{End}(\overline{P_{2m+1}})| = 2\sum_{i=1}^{m} i\,i! + (m+1)(m+1)!$.

Lemma 3.4. $\mathrm{End}(\overline{P_{2m+1}}) = h\mathrm{End}(\overline{P_{2m+1}})$.

Proof. It follows from Lemma 1.1 and Proposition 2.6.

Lemma 3.5. $\mathrm{Aut}(\overline{P_{2m+1}}) = l\mathrm{End}(\overline{P_{2m+1}})$.

Proof. Let $f \in l\mathrm{End}(\overline{P_{2m+1}})$ and suppose that $f \notin \mathrm{Aut}(\overline{P_{2m+1}})$. Then $S_f \neq V(\overline{P_{2m+1}})$ and $f(i) = f(i+1)$ for some $1 \le i < 2m+1$. Since $|\overline{P_{2m+1}}| = 2m+1$ is odd, without loss of generality, there exists $t \geq 1$ such that $[t]_{\rho_f} = \{t\}$ and $[t+1]_{\rho_f} = \{t+1, t+2\}$. It is clear that $\{f(t), f(t+1)\} \in I_f$, $f^{-1}(f(t)) = \{t\}$, $f^{-1}(f(t+1)) = \{t+1, t+2\}$ and $\{t+1, t\} \notin E(\overline{P_{2m+1}})$. Hence $f \notin l\mathrm{End}(\overline{P_{2m+1}})$. Thus we have $\mathrm{Aut}(\overline{P_{2m+1}}) = l\mathrm{End}(\overline{P_{2m+1}})$.

Theorem 3.6. $\mathrm{Endospec}(\overline{P_{2m+1}}) = (2\sum_{i=1}^{m} ii! + (m+1)(m+1)!, 2\sum_{i=1}^{m} ii! + (m+1)(m+1)!, 2, 2, 2, 2)$.

Proof. It follows immediately from Lemmas 3.3–3.5.

Corollary 3.7. $\mathrm{Endotype}(\overline{P_{2m+1}}) = 2$.

Proof. By Theorem 3.6, $|h\mathrm{End}(\overline{P_{2m+1}})| \neq 2$ when $m \geq 2$ and so $h\mathrm{End}(\overline{P_{2m+1}}) \neq l\mathrm{End}(\overline{P_{2m+1}})$. Now the assertion follows from the definition of endomorphism type and Lemmas 3.4 and 3.5.

Next we consider the case of $n = 2m$ with $m \geq 2$.

Lemma 3.8. $|\mathrm{End}(\overline{P_{2m}})| = 2\sum_{i=1}^{m} ii! + (m+1)!$

Proof. It is clear that $\mathrm{End}(\overline{P_{2m}})$ has $m+1$ $\mathcal{D}$-classes: $D_m, D_{m+1}, \ldots, D_{2m}$. First let $f \in D_m$. Then S_f is empty and by Lemma 2.14(3) and 3.2 there is only one $\mathcal{L}$-class in D_m. In this case, I_f is a clique of order m and so there are $m!$ elements in L_f such that their images are I_f. By Lemma 2.9(1) there are $m+1$ cliques in $\overline{P_{2m}}$ and thus there are $m+1$ possibilities of I_f in $\overline{P_{2m}}$. Hence there are $(m+1)!$ elements in L_f and therefore D_m has $(m+1)!$ elements. Secondly, for $i = 1, \ldots, m-1$, let $f \in D_{m+i}$. Then $|S_f| = 2i$ and S_f consists of $2i$ consecutive numbers which start with an odd number and end with an even number. Since there are $m-i+1$ such subsets of $V(\overline{P_{2m}})$, by Lemma 3.2 there are $m-i+1$ $\mathcal{L}$-classes in D_{m+i}. By Lemma 2.14(1) the restriction of f to $V(\overline{P_{2m}}/\rho_f) \setminus \{[s]_{\rho_f} | s \in S_f\}$ is a bijection from $V(\overline{P_{2m}}/\rho_f) \setminus \{[s]_{\rho_f} | s \in S_f\}$ to $V(I_f) \setminus f(S_f)$. Observe that there are two ways to map S_f to $f(S_f)$ and there are $m-i+1$ possibilities for $f(S_f)$ in $V(I_f)$. Hence there are $2(m-i+1)(m-i)!$ elements in every $\mathcal{L}$-class and therefore D_{m+i} has $2(m-i+1)(m-i+1)!$ elements. Finally, let $f \in D_{2m}$. Then $|S_f| = 2m$ and so f is an automorphism of $\overline{P_{2m}}$. Clearly, $D_{2m} = \mathrm{Aut}(\overline{P_{2m}})$ and D_{2m} has two elements. Summing up, we form the following table:

D	$\|D/\mathcal{L}\|$	$\|L\|$	$\|D\|$
D_m	1	$(m+1)!$	$(m+1)!$
D_{m+1}	m	$2m(m-1)!$	$2mm!$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{m+i}	$m-i+1$	$2(m-i+1)(m-i)!$	$2(m-i+1)(m-i+1)!$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{2m-1}	2	$2 \cdot 2!$	$2 \cdot 2 \cdot 2!$
D_{2m}	1	2	2

Consequently, $|\mathrm{End}(\overline{P_{2m}})| = 2\sum_{i=1}^{m} ii! + (m+1)!$.

Lemma 3.9. $\mathrm{End}(\overline{P_{2m}}) = h\mathrm{End}(\overline{P_{2m}})$.

Proof. It follows from Lemma 1.1 and Proposition 2.15.

Lemma 3.10. $l\mathrm{End}(\overline{P_{2m}}) = q\mathrm{End}(\overline{P_{2m}}) = D_m \cup D_{2m}$.

Proof. Clearly $q\text{End}(\overline{P_{2m}}) \subseteq l\text{End}(\overline{P_{2m}})$. Let $f \in \text{End}(\overline{P_{2m}})$ be such that $f \notin D_{2m}$ and $f \notin D_m$. Then S_f is not empty and $S_f \neq V(\overline{P_{2m}})$. As in the proof for Lemma 3.5, we may show that $f \notin l\text{End}(\overline{P_{2m}})$. Hence $l\text{End}(\overline{P_{2m}}) \subseteq D_m \cup D_{2m}$.

Let $f \in D_m$. Then for each $i \in V(\overline{P_{2m}})$, $|[i]_{\rho_f}| = 2$. Now for $a, b \in I_f$ with $a \neq b$ (note that I_f is a clique), there are two numbers $i, j \in V(\overline{P_{2m}})$ such that $f^{-1}(a) = \{i-1, i\}$ and $f^{-1}(b) = \{j, j+1\}$. It is clear that $i-1$ is adjacent to both j and $j+1$ and $j+1$ is adjacent to both $i-1$ and i. This implies that $f \in q\text{End}(\overline{P_{2m}})$. If $f \in D_{2m}$, then f is an automorphism of $\overline{P_{2m}}$ and so $f \in q\text{End}(\overline{P_{2m}})$. We have proved that $D_m \cup D_{2m} \subseteq q\text{End}(\overline{P_{2m}})$. Consequently $l\text{End}(\overline{P_{2m}}) = q\text{End}(\overline{P_{2m}}) = D_m \cup D_{2m}$.

Lemma 3.11. $s\text{End}(\overline{P_{2m}}) = \text{Aut}(\overline{P_{2m}})$.

Proof. It is clear that $\text{Aut}(\overline{P_{2m}}) \subseteq s\text{End}(\overline{P_{2m}})$. Let f be a strong endomorphism but not an automorphism. Then there are $i, j \in V(\overline{P_{2m}})$ such that $f(i) = f(j)$ and so $|i-j| = 1$ by Lemma 2.10. Without loss of generality, suppose that $i < j < 2m$ (note that $m \geq 2$). Clearly, $\{f(i), f(j+1)\} \in E(I_f)$ and $\{j, j+1\} \notin E(\overline{P_{2m}})$. This implies that $f \notin s\text{End}(\overline{P_{2m}})$, a contradiction. Therefore $s\text{End}(\overline{P_{2m}}) \subseteq \text{Aut}(\overline{P_{2m}})$ and $s\text{End}(\overline{P_{2m}}) = \text{Aut}(\overline{P_{2m}})$.

Theorem 3.12. $\text{Endospec}(\overline{P_{2m}}) = (2\sum_{i=1}^{m} ii! + (m+1)!, 2\sum_{i=1}^{m} ii! + (m+1)!, (m+1)! + 2, (m+1)! + 2, 2, 2)$.

Proof. From the proof for Lemma 3.8, we have $|D_m \cup D_{2m}| = (m+1)! + 2$ and $|\text{Aut}(\overline{P_{2m}})| = |D_{2m}| = 2$. Now the assertion follows from Lemmas 3.8–3.11.

Corollary 3.13. $\text{Endotype}(\overline{P_{2m}}) = 10$.

Proof. By Theorem 3.12, it is easy to see that $\text{End}(\overline{P_{2m}}) = h\text{End}(\overline{P_{2m}}) \neq l\text{End}(\overline{P_{2m}}) = q\text{End}(\overline{P_{2m}}) \neq s\text{End}(\overline{P_{2m}}) = \text{Aut}(\overline{P_{2m}})$ whenever $m \geq 2$. Thus $\text{Endotype}(\overline{P_{2m}}) = 10$.

Recall that a graph X is S-unretractive if $s\text{End}(X) = \text{Aut}(X)$ (see [3]).

Corollary 3.14. *For any positive integer n, $\overline{P_n}$ is S-unretractive.*

Proof. It follows from Lemmas 3.5 and 3.11.

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