Spatial Decay Estimates in Time-Dependent Double-Diffusive Darcy Plane Flow

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This paper considers the time-dependent double-diffusive convective Darcy flow in a semi-infinite channel. An exponential decay estimate in terms of the distance from the finite end of the channel is obtained from a second-order differential inequality for a weighted energy integral. The paper also indicates how to bound the total weighted energy. © 2002 Elsevier Science (USA)

1. INTRODUCTION

In this paper, we investigate the time-dependent double-diffusive convective flow of a fluid through a porous medium in a semi-infinite channel. At the finite end of the channel, time-dependent data are prescribed, and homogeneous data are assumed on the top and bottom of the channel. We derive an explicit inequality which implies exponential decay of a weighted energy integral as a function of the distance from the finite end of the channel for each t. Decay results in steady double-diffusive convective Darcy and Brinkman flows have been studied by Payne and Song [12], who were able to eliminate the pressure term using the method developed by Horgan and Wheeler [7] and Ames et al. [1]. In our time-dependent double-diffusive two-dimensional flow, we introduce a stream function to deal with the pressure term for establishing the decay estimate. Other decay estimates for the plane Navier–Stokes equations and the transient plane Stokes flow equation have been studied by Horgan [3] and Lin [8], respectively.

For a survey of Saint-Venant-type spatial decay results, see Horgan [4, 5] and Horgan and Knowles [6]. More recent work on decay results in porous
medium problems has been carried out by Payne and Song [11, 12], Chadam and Qin [2], and Qin and Kaloni [13].

The outline for this paper is as follows. We formulate the initial–boundary value problem which describes transient double-diffusive convective Darcy flow in the semi-infinite channel in the next section, and define a weighted energy integral to derive a second-order differential inequality in Section 3. In the final section, we establish a bound for the weighted total energy.

2. STATEMENT OF THE PROBLEM

Let $R$ be the interior of the semi-infinite channel where $0 < x_2 < d$, $x_1 > 0$. The subdomain of $R$ for which $x_1 > z$ is denoted by $R_z$, and the line segment containing the points $(z, x_2)$ is denoted by $L_z$ for each fixed $z \geq 0$. Clearly, $R \equiv R_0$ and $L \equiv L_0$.

We consider an initial–boundary value problem for the transient plane double-diffusive Darcy flow in a straight semi-infinite channel. Let $u_\alpha (\alpha = 1, 2)$, $p$, $T$, and $C$ denote velocity, pressure, temperature, and concentration. Then, for the transient double-diffusive Darcy model [14] in the interior of the semi-infinite channel, we have

$$
\begin{align*}
\begin{cases}
  u_\alpha &= -p_\alpha + g_\alpha T + h_\alpha C \\
  \frac{\partial T}{\partial t} + u_\alpha T_{,\alpha} &= \Delta T \\
  \frac{\partial C}{\partial t} + u_\alpha C_{,\alpha} &= \Delta C + \sigma \Delta T \\
  u_{\alpha,\alpha} &= 0
\end{cases}
\end{align*}
$$

in $R \times \{ t > 0 \}, \quad (2.1)$

where $\sigma$ is a material positive constant, $g_\alpha$ and $h_\alpha$ are given bounded functions, and the symbol $\Delta$ is the two-dimensional Laplacian operator. We have used the summation convention summed from 1 to 2, and a comma is used to indicate differentiation. The boundary conditions and initial conditions are

$$
\begin{align*}
  u_2(x_1, 0, t) &= u_2(x_1, d, t) = 0, \\
  T(x_1, 0, t) &= T(x_1, d, t) = 0, \\
  C(x_1, 0, t) &= C(x_1, d, t) = 0, \\
  u_1(0, x_2, t) &= f_1(x_2, t), \\
  T(0, x_2, t) &= q(x_2, t), \\
  C(0, x_2, t) &= r(x_2, t), \\
  T(x_1, x_2, 0) &= C(x_1, x_2, 0) = 0,
\end{align*}
$$

(2.2)
where $f_1$, $q$, and $r$ are prescribed functions. We also assume that

$u_0, p, T, C, \forall T, \forall C \rightarrow 0$ uniformly in $x_2$ and $t$ as $x_1 \rightarrow \infty$. (2.5)

We point out that $u_2$ is not prescribed at the end $x_1 = 0$ [11, p. 177]. A necessary condition for these to be satisfied is of course that

$$\int_{L} f_1 \, dx_2 = 0 \quad \text{for all } t \geq 0,$$

which implies that, for all $z \geq 0$,

$$\int_{L} u_1 \, dx_2 = 0 \quad \text{for all } t \geq 0.$$ 

We define a weighted energy integral for solutions $u_0, T,$ and $C$ of (2.1)–(2.4) by

$$E(z, t) = \frac{\sigma^2}{2} \int_{0}^{t} \int_{R_z} \langle \xi - z \rangle T, T_0 dA d\eta + \frac{1}{2} \int_{0}^{t} \int_{R_z} \langle \xi - z \rangle C, C_0 dA d\eta$$

$$+ \frac{1}{2} \int_{R_z} \langle \xi - z \rangle (\sigma^2 T + C^2) dA |_{\eta = t} + \frac{1}{2} \int_{0}^{t} \int_{R_z} u_1 u_0 dA d\eta.$$ (2.8)

Our aim is to obtain a second-order differential inequality for $E(z, t)$ from which we can deduce an exponential decay estimate of the form

$$E(z, t) \leq E(0, t)e^{-\gamma z}, \quad \gamma > 0,$$ (2.9)

where $0 \leq z < \infty$.

3. ENERGY ESTIMATION

To derive the desired second-order differential inequality for $E(z, t)$, we consider a weighted energy integral

$$F(z, t) = k \int_{0}^{t} \int_{R_z} \langle \xi - z \rangle T, T_0 dA d\eta + \int_{0}^{t} \int_{R_z} \langle \xi - z \rangle C, C_0 dA d\eta$$

$$+ \frac{1}{2} \int_{R_z} \langle \xi - z \rangle (kT^2 + C^2) dA |_{\eta = t} + \int_{0}^{t} \int_{R_z} u_0 u_1 dA d\eta.$$ (3.1)

where $k$ is a positive constant to be chosen later and $dA (= d\xi dx_2)$ is the element of area in $R_z$. On integrating by parts and using the second formula in (2.1), we have

$$\int_{0}^{t} \int_{R_z} \langle \xi - z \rangle T, T_0 dA d\eta = -\int_{0}^{t} \int_{R_z} TT_1 dA d\eta$$

$$- \frac{1}{2} \int_{R_z} \langle \xi - z \rangle T^2 dA |_{\eta = t}$$

$$- \frac{1}{2} \int_{0}^{t} \int_{R_z} u_1 T^2 dA d\eta.$$ (3.2)
Use of the Schwarz inequality, the arithmetic–geometric mean inequality, and the Poincaré inequality leads for arbitrary positive constant $\epsilon_1$ to

\[
\int_0^t \int_{R_z} ((\xi - z)T_\alpha \alpha T_\alpha dA d\eta + \frac{1}{2} \int_{R_z} ((\xi - z)T^2 dA|_{\eta=t}
\leq \frac{1}{2\sqrt{\lambda}} \int_0^t \int_{R_z} T_\alpha T_\alpha dA d\eta + \frac{1}{2} \left( \int_0^t \int_{R_z} u_1^2 dA d\eta \int_0^t \int_{R_z} T^4 dA d\eta \right)^{1/2}
\leq \frac{1}{2\sqrt{\lambda}} \int_0^t \int_{R_z} T_\alpha T_\alpha dA d\eta
\]

\[
+ \frac{q_M}{2\sqrt{\lambda}} \left( \int_0^t \int_{R_z} u_1^2 dA d\eta \right)^{1/2} \left( \int_0^t \int_{R_z} T^2 dA d\eta \right)^{1/2}
\leq \frac{1}{2\sqrt{\lambda}} \int_0^t \int_{R_z} T_\alpha T_\alpha dA d\eta
\]

\[
+ \frac{\epsilon_1 q_M}{4\sqrt{\lambda}} \int_0^t \int_{R_z} u_1^2 dA d\eta + \frac{\epsilon_1^{-1} q_M}{4\sqrt{\lambda}} \int_0^t \int_{R_z} T^2 dA d\eta,
\]

(3.3)

where $\lambda = \pi^2/d^2$, $q_M = \max_{0<\eta<\eta_0} \max_{0<\eta<\eta_0} q(\eta_2, \eta)$. We have used a maximum principle for $T$ in $R$, provided the velocity is uniformly bounded in $R$. It should be pointed out, however, as we shall see later, that for sufficiently small data this assumption would not be necessary if we bound this term just as in deriving (3.5). In a similar manner, it follows that

\[
\int_0^t \int_{R_z} ((\xi - z)C_\alpha C_\alpha dA d\eta + \frac{1}{2} \int_{R_z} ((\xi - z)C^2 dA|_{\eta=t}
\leq -\int_0^t \int_{R_z} C_{,1} dA d\eta + \frac{1}{2} \int_{R_z} u_1 C^2 dA d\eta
\]

\[
- \sigma \int_0^t \int_{R_z} ((\xi - z)C_\alpha T_\alpha dA d\eta - \sigma \int_0^t \int_{R_z} CT_1 dA d\eta.
\]

(3.4)

For the second term on the right-hand side above, we have

\[
\frac{1}{2} \int_0^t \int_{R_z} u_1 C^2 dA d\eta
\]

\[
\leq \frac{1}{2} \int_0^t \int_{R_z} u_1^2 dA d\eta \right)^{1/2} \left( \int_0^t \int_{R_z} C^4 dA d\eta \right)^{1/2}
\leq \frac{1}{2} \int_0^t \int_{R_z} u_1^2 dA d\eta \right)^{1/2} \left[ \int_0^t \int_{R_z} C^2 dA \int_{R_z} C_\alpha C_\alpha dA d\eta \right]^{1/2}
\leq \frac{M_C}{2} \left( \int_0^t \int_{R_z} u_1^2 dA d\eta \right)^{1/2} \left( \int_0^t \int_{R_z} C_\alpha C_\alpha dA d\eta \right)^{1/2}
\leq \frac{M_C}{4} \int_0^t \int_{R_z} C_\alpha C_\alpha dA d\eta,
\]

(3.5)
where we have used the arithmetic–geometric mean inequality for arbitrary positive constant \( \epsilon_2 \) and the following two-dimensional Sobolev inequality for functions \( C \) vanishing on \( \partial R_z \setminus \mathbb{L}_z \):

\[
\int_{R_z} C^4 \, dA \leq \int_{R_z} C^2 \, dA \int_{R_z} C_{,\alpha} C_{,\alpha} \, dA,
\]

(3.6)

with an upper bound for \( M_C \) given later. We establish the Sobolev inequality (3.6) as follows. The divergence theorem and the regularity conditions (2.5) yield, for fixed \( y \) and \( t \),

\[
C^2(z, y, t) = -2 \int_{z}^{\infty} CC_{,\xi} \, d\xi
\]

(3.8)

Since \( C(z, 0, t) = C(z, d, t) = 0 \) for fixed \( z \) and \( t \), we have

\[
C^2(z, x_2, t) = 2 \int_{0}^{x_2} CC_{,\eta} \, d\eta = -2 \int_{x_2}^{d} CC_{,\eta} \, d\eta,
\]

(3.9)

from which it follows that

\[
C^2(z, x_2, t) = \int_{0}^{x_2} CC_{,\eta} \, d\eta - \int_{x_2}^{d} CC_{,\eta} \, d\eta
\]

\[
\leq \left| CC_{,\eta} \right| \, d\eta.
\]

(3.10)

Combining (3.8) and (3.10) and integrating over \( R_z \), we obtain

\[
\int_{R_z} C^4 \, dA \leq 2 \int_{R_z} \left| CC_{,\xi} \right| \, dA \int_{R_z} \left| CC_{,\eta} \right| \, dA
\]

\[
\leq 2 \int_{R_z} C^2 \, dA \left( \int_{R_z} C_{,1}^2 \, dA \right)^{1/2} \left( \int_{R_z} C_{,2}^2 \, dA \right)^{1/2}.
\]

(3.11)

On using the arithmetic–geometric mean inequality, we obtain the inequality (3.6). Employing the inequality (3.5), the Schwarz inequality, the arithmetic–geometric mean inequality, and the Poincaré inequality, we find from (3.4) that

\[
\frac{1}{2} \int_{0}^{t} \int_{R_z} \left( \vec{\xi} - z \right) C_{,\alpha} C_{,\alpha} \, dA \, d\eta + \frac{1}{2} \int_{0}^{t} \left( \vec{\xi} - z \right) C^2 \, dA \bigg|_{\vec{\eta}=t}
\]

\[
\leq \frac{\sigma^2}{2} \int_{0}^{t} \left( \vec{\xi} - z \right) T_{,\alpha} T_{,\alpha} \, dA \, d\eta + \frac{\sigma}{2\sqrt{\lambda}} \int_{0}^{t} T_{,\alpha}^2 \, dA \, d\eta
\]

\[
+ \left( \frac{1}{2\sqrt{\lambda}} + \frac{M_C}{4\epsilon_2} \right) \int_{0}^{t} \int_{R_z} C_{,\alpha} C_{,\alpha} \, dA \, d\eta
\]

\[
+ \frac{\sigma}{2\sqrt{\lambda}} \int_{0}^{t} \int_{R_z} C_{,\beta}^2 \, dA \, d\eta + \frac{\epsilon_2 M_C}{4} \int_{0}^{t} \int_{R_z} u_{,\gamma}^2 \, dA \, d\eta.
\]

(3.12)
Turning to the final term in (3.1), we have

\[
\int_0^t \int_{R_z} u_a u_a \, dA \, d\eta = \int_0^t \int_{R_z} u_a (-p_{,a} + g_a T + h_a C) \, dA \, d\eta \\
= \int_0^t \int_{L_z} u_1 p \, dx_2 \, d\eta + \int_0^t \int_{R_z} u_a g_a T \, dA \, d\eta \\
+ \int_0^t \int_{R_z} u_a h_a C \, dA \, d\eta \\
:= J_1 + J_2 + J_3. \quad (3.13)
\]

To eliminate the pressure term \(p\) in (3.13), we introduce the stream function \(\psi\) that satisfies

\[
u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1} \quad \text{in } R_z, \\
\psi = 0 \quad \text{on } x_2 = 0, d. \quad (3.14)
\]

Using the stream function \(\psi\) in \(J_1\), it follows that

\[
J_1 = \int_0^t \int_{L_z} \psi \psi_{,2} p \, dx_2 \, d\eta \\
= -\int_0^t \int_{L_z} \psi \psi_{,2} \, dx_2 \, d\eta \\
= \int_0^t \int_{L_z} \psi (u_2 - g_2 T - h_2 C) \, dx_2 \, d\eta. \quad (3.15)
\]

Now by means of the Schwarz inequality, the Poincaré inequality, and the arithmetic–geometric mean inequality, we have

\[
J_1 \leq \frac{1}{\sqrt{\lambda}} \left( \int_0^t \int_{L_z} \psi^2 dx_2 \, d\eta \right)^{1/2} \left( \int_0^t \int_{L_z} u_2^2 dx_2 \, d\eta \right)^{1/2} \\
+ \frac{g}{\lambda} \left( \int_0^t \int_{L_z} \psi^2 dx_2 \, d\eta \right)^{1/2} \left( \int_0^t \int_{L_z} T_2^2 dx_2 \, d\eta \right)^{1/2} \\
+ \frac{h}{\lambda} \left( \int_0^t \int_{L_z} \psi^2 dx_2 \, d\eta \right)^{1/2} \left( \int_0^t \int_{L_z} C_2^2 dx_2 \, d\eta \right)^{1/2} \\
\leq \frac{1}{2\sqrt{\lambda}} \int_0^t \int_{L_z} u_a u_a \, dx_2 \, d\eta + \frac{g + h}{2\lambda} \int_0^t \int_{L_z} u_1^2 \, dx_2 \, d\eta \\
+ \frac{g}{2\lambda} \int_0^t \int_{L_z} T_2^2 dx_2 \, d\eta + \frac{h}{2\lambda} \int_0^t \int_{L_z} C_2^2 dx_2 \, d\eta, \quad (3.16)
\]
where $g = \max \sqrt{g_\alpha g_\alpha}$ and $h = \max \sqrt{h_\alpha h_\alpha}$. In the same manner, it follows that, for arbitrary positive constants $\epsilon_3$ and $\epsilon_4$,

$$J_2 + J_3 \leq \frac{\epsilon_3 g + \epsilon_4 h}{2\sqrt{\lambda}} \int_{R_z}^\prime u_a u_a dA d\eta$$

$$+ \frac{\epsilon_3^1 g}{2\sqrt{\lambda}} \int_{R_z}^\prime T_{\alpha z}^2 dA d\eta + \frac{\epsilon_4^1 h}{2\sqrt{\lambda}} \int_{R_z}^\prime C_{\alpha z}^2 dA d\eta. \quad (3.17)$$

Substituting bounds for $\int_{R_z}^\prime (\xi - z)T_{\alpha z} T_{\alpha z} dA d\eta$, $\int_{R_z}^\prime C_{\alpha z} C_{\alpha z} dA d\eta$, and $\int_{R_z}^\prime u_\alpha u_\alpha dA d\eta$ into (3.1), we obtain

$$E(z, t) := \left( k - \frac{\sigma^2}{2} \right) \int_{R_z}^\prime (\xi - z)T_{\alpha z} T_{\alpha z} dA d\eta$$

$$+ \frac{1}{2} \int_{R_z}^\prime (\xi - z)C_{\alpha z} C_{\alpha z} dA d\eta$$

$$+ \left( \frac{k \epsilon_1 q_M}{4\sqrt{\lambda}} - \frac{\epsilon_2 M C}{4} - \frac{\epsilon_3 g + \epsilon_4 h}{2\sqrt{\lambda}} \right) \int_{R_z}^\prime u_a u_a dA d\eta$$

$$+ \frac{1}{2} \int_{R_z}^\prime (\xi - z)(kT^2 + C^2) dA|_{\eta=t}$$

$$\leq \frac{k}{2\sqrt{\lambda}} \int_{R_z}^\prime T_{\alpha z} T_{\alpha z} dA d\eta + \frac{k \epsilon_1 q_M}{4\sqrt{\lambda}} \int_{R_z}^\prime T_{\alpha z}^2 dA d\eta$$

$$+ \frac{\sigma}{2\sqrt{\lambda}} \int_{R_z}^\prime T_{\alpha z}^2 dA d\eta + \frac{\epsilon_3 g}{2\sqrt{\lambda}} \int_{R_z}^\prime T_{\alpha z}^2 dA d\eta$$

$$+ \frac{M C}{4\epsilon_2} \int_{R_z}^\prime C_{\alpha z} C_{\alpha z} dA d\eta + \frac{1}{2\sqrt{\lambda}} \int_{R_z}^\prime C_{\alpha z} C_{\alpha z} dA d\eta$$

$$+ \frac{1}{2\sqrt{\lambda}} (\sigma + \epsilon_4^1 h) \int_{R_z}^\prime C_{\alpha z}^2 dA d\eta$$

$$+ \frac{1}{2\sqrt{\lambda}} \int_{R_z}^\prime u_a u_a dA d\eta + \frac{g + h}{2\lambda} \int_{L_{\alpha}}^\prime u_{\alpha z}^2 dx d\eta$$

$$+ \frac{g}{2\lambda} \int_{L_{\alpha}}^\prime T_{\alpha z}^2 dx d\eta + \frac{h}{2\lambda} \int_{L_{\alpha}}^\prime C_{\alpha z}^2 dx d\eta$$

$$\leq A \int_{R_z}^\prime T_{\alpha z} T_{\alpha z} dA d\eta + B \int_{R_z}^\prime C_{\alpha z} C_{\alpha z} dA d\eta$$

$$+ \frac{1}{2} \left( \frac{1}{\sqrt{\lambda}} + \frac{g + h}{\lambda} \right) \int_{L_{\alpha}}^\prime u_a u_a dA d\eta$$

$$+ \frac{g}{2\lambda} \int_{L_{\alpha}}^\prime T_{\alpha z}^2 dx d\eta + \frac{h}{2\lambda} \int_{L_{\alpha}}^\prime C_{\alpha z}^2 dx d\eta, \quad (3.18)$$
By choosing the arbitrary positive constants $k$ and $\epsilon_i$'s appropriately, we can make the coefficients of \( \int_0^t \int_R (\xi - z) T_{\alpha} T_{\alpha} dA d\eta \) and \( \int_0^t \int_R u_{\alpha} u_{\alpha} dA d\eta \) in (3.18) nonnegative. For instance, taking
\[
k = \sigma^2, \quad \epsilon_1 = \frac{\sqrt{\lambda}}{2\sigma^2 q_M}, \quad \epsilon_2 = \frac{1}{2M_C}, \quad \epsilon_3 = \frac{\sqrt{\lambda}}{4g}, \quad \epsilon_4 = \frac{\sqrt{\lambda}}{4h}, \tag{3.21}
\]
we find
\[
E(z, t) = \frac{\sigma^2}{2} \int_0^t \int_{R_t} (\xi - z) T_{\alpha} T_{\alpha} dA d\eta + \frac{1}{2} \int_0^t \int_{R_t} (\xi - z) C_{\alpha} C_{\alpha} dA d\eta + \frac{1}{2} \int_0^t \int_{R_t} u_{\alpha} u_{\alpha} dA d\eta. \tag{3.22}
\]
From (3.18) we then have the second-order differential inequality
\[
E(z, t) \leq K_1 \frac{\partial^2 E}{\partial z^2} - K_2 \frac{\partial E}{\partial z}, \tag{3.23}
\]
where
\[
K_1 = \max \left( \frac{\frac{g}{\lambda} \frac{1}{\sigma^2} \frac{h}{\lambda}}{\frac{1}{\sigma^2} + \frac{\frac{g}{\lambda}}{\lambda}} \right), \tag{3.24}
\]
\[
K_2 = \max \left[ \frac{2A}{\sigma^2}, 2B, \left( \frac{1}{\sqrt{\lambda}} + \frac{\frac{g}{\lambda}}{\lambda} \right) \right]. \tag{3.25}
\]
We can rewrite (3.23) as
\[
\frac{\partial^2 E}{\partial z^2} - a \frac{\partial E}{\partial z} - bE \geq 0, \tag{3.26}
\]
where $a = K_2/K_1$ and $b = 1/K_1$. Furthermore, we can rewrite (3.23) in the form
\[
\left( \frac{\partial}{\partial z} - \gamma_1 \right) \left( \frac{\partial E}{\partial z} + \gamma_2 E \right) \geq 0, \tag{3.27}
\]
where
\[
\gamma_1 = \frac{a}{2} + \frac{1}{2} \sqrt{a^2 + 4b}, \quad \gamma_2 = -\frac{a}{2} + \frac{1}{2} \sqrt{a^2 + 4b}. \tag{3.28}
\]
Now since
\[ \frac{\partial}{\partial z} \left\{ e^{-\gamma z} \left( \frac{\partial E}{\partial z} + \gamma_2 E \right) \right\} \geq 0, \]
we conclude upon integration from \( z \) to \( \infty \) that
\[ \frac{\partial E}{\partial z} + \gamma_2 E \leq 0, \tag{3.30} \]
and hence that
\[ E(z, t) \leq E(0, t) e^{-\gamma z}. \tag{3.31} \]
This is the exponential decay estimate we seek.

We note that the decay rate \( \gamma_2 \) depends on parameters of the problem \((\sigma, g, h, q_M, \lambda, M_C)\) defined in (3.7). \( M_C \) is treated in Section 4. We note further that the decay rate \( \gamma_2 \) might be improved by a more judicious choice of computable constants in our derivation above. To make the energy estimation explicit, we will indicate a procedure for obtaining bounds for the total weighted energy \( E(0, t) \) in the next section.

4. A BOUND FOR \( E(0, T) \)

We now indicate how we can bound the total weighted energy in terms of data. We shall not determine explicit constants in this bound as we did in the previous sections. First, recalling from (2.8) at \( z = 0 \) and combining (3.3) and (3.12) at \( z = 0 \), we have, for computable positive constants \( \tilde{A}, \tilde{B}, \) and \( \tilde{C} \),
\[ E(0, t) \leq \tilde{A} \int_0^t \int_R T_{\alpha} T_{\alpha} dA d\eta + \tilde{B} \int_0^t \int_R C_{\alpha} C_{\alpha} dA d\eta \]
\[ + \tilde{C} \int_0^t \int_R u_{\alpha} u_{\alpha} dA d\eta, \tag{4.1} \]
Thus we must bound \( \int_0^t \int_R T_{\alpha} T_{\alpha} dA d\eta \), \( M_C = \max(\int_R C^2 d\eta)^{1/2} \) which appears in \( \tilde{B} \), \( \int_0^t \int_R C_{\alpha} C_{\alpha} dA d\eta \), and \( \int_0^t \int_R u_{\alpha} u_{\alpha} dA d\eta \). We first derive a bound for \( \int_0^t \int_R T_{\alpha} T_{\alpha} dA d\eta \). To this end, we note that
\[ \left( \int_0^t \int_R T_{\alpha} T_{\alpha} dA d\eta \right)^{1/2} \leq \left[ \int_0^t \int_R (T - S)_{\alpha} (T - S)_{\alpha} dA d\eta \right]^{1/2} \]
\[ + \left( \int_0^t \int_R S_{\alpha} S_{\alpha} dA d\eta \right)^{1/2}, \tag{4.2} \]
where $S$ is a solution of
\[
\frac{\partial S}{\partial t} = \Delta S \quad \text{in } R,
\] (4.3)
with the same initial–boundary conditions as $T$. First, we see that
\[
\int_{t_0}^t \int_R (T - S).a(T - S).a \, dA \, d\eta = - \frac{1}{2} \int_R (T - S)^2 \, dA \bigg|_{\eta=t} + \int_{t_0}^t \int_R (T - S).a u_a S \, dA \, d\eta. \tag{4.4}
\]
On using the Schwarz inequality and the maximum principle for $S$, we obtain
\[
\int_{t_0}^t \int_R (T - S).a(T - S).a \, dA \, d\eta \leq q^2_M \int_{t_0}^t \int_R u_a u_a \, dA \, d\eta. \tag{4.5}
\]
The bound for $\int_{t_0}^t \int_R u_a u_a \, dA \, d\eta$ is derived following the arguments of Payne and Song [12, Eq. (3.28)] and leads to
\[
\int_{t_0}^t \int_R u_a u_a \, dA \, d\eta \leq \text{data}. \tag{4.6}
\]
Using the work of Lin and Payne [9, Eq. (4.80)], we bound the last term in (4.2)
\[
\int_{t_0}^t \int_R S.a S.a \, dA \, d\eta \leq \int_{t_0}^t \int_L \left[ 4\sqrt{\lambda} r^2 + \frac{\lambda^{-1/2}}{2} r_3^2 + \frac{\lambda^{-3/2}}{16} r_4^2 \right] \, dx_2 \, d\eta. \tag{4.7}
\]
Combining (4.5) and (4.6) and inserting this together with (4.7) into (4.2) lead to a bound for $\int_{t_0}^t \int_R T.a T.a \, dA \, d\eta$ in terms of data.

We next derive a bound for $M_C$. We begin by noting that
\[
\max_i \left( \int_R C^2 \, dA \right)^{1/2} \leq \max_i \left[ \int_R (C - H)^2 \, dA \right]^{1/2} + \max_i \left( \int_R H^2 \, dA \right)^{1/2}, \tag{4.8}
\]
where $H$ is a solution of
\[
\frac{\partial H}{\partial t} + u_a H.a = \Delta H \quad \text{in } R, \tag{4.9}
\]
with the same initial–boundary conditions as $C$. To bound the first term on the right-hand side of (4.8), we use the arguments of Lin and Payne
Then if the maximum of \( \frac{1}{2} \int_R (C - H)^2 \, dA \) occurs at \( t = t^* \) where \( 0 < t^* \leq t \), we write
\[
\frac{1}{2} \max_t \left( \int_R (C - H)^2 \, dA \right) = \frac{1}{2} \left[ \int_R (C - H)^2 \, dA \right]_{t=t^*} \leq -\int_0^{t^*} \int_R (C - H)_{,\alpha} (C - H)_{,\alpha} \, dA \, d\eta \\
+ \sigma \int_0^{t^*} \int_R (C - H)_{,\alpha} T_{,\alpha} \, dA \, d\eta \\
\leq -\frac{1}{2} \int_0^{t^*} \int_R (C - H)_{,\alpha} (C - H)_{,\alpha} \, dA \, d\eta \\
+ \frac{\sigma}{2} \int_0^{t^*} \int_R T_{,\alpha} T_{,\alpha} \, dA \, d\eta \\
\leq \frac{\sigma}{2} \int_0^{t^*} \int_R T_{,\alpha} T_{,\alpha} \, dA \, d\eta. \quad (4.10)
\]

We turn now to the derivation of the bound for the last term on the right-hand side of (4.8). We may use the maximum principle for \( H \), provided the velocity remains bounded. However, we point out that this assumption would not be necessary if we bound this term as in deriving (4.8). Then we write
\[
\max_t \left( \int_R H^2 \, dA \right)^{1/2} \leq \max_t \left( \int_R (H - \tilde{S})^2 \, dA \right)^{1/2} + \max_t \left( \int_R \tilde{S}^2 \, dA \right)^{1/2}, \quad (4.11)
\]
where \( \tilde{S} \) is a solution of
\[
\frac{\partial \tilde{S}}{\partial t} = \Delta \tilde{S} \quad \text{in } R, \quad (4.12)
\]
with the same initial–boundary conditions as \( C \). Similarly as in (4.10), using (4.4) with \( T \) and \( S \) replaced by \( H \) and \( \tilde{S} \), respectively, we find
\[
\max_t \left( \int_R (H - \tilde{S})^2 \, dA \right) = \int_R (H - \tilde{S})^2 \, dA \bigg|_{t=t^*} \leq r_M^2 \int_0^{t^*} \int_R u_{,\alpha} u_{,\alpha} \, dA \, d\eta, \quad (4.13)
\]
where \( r_M = \max_{0 \leq x_2 \leq L, 0 \leq \eta \leq \tau} (x_2, \eta) \). Using the result of Lin and Payne [10 Eq. (4.9)], we have
\[
\max_t \int_R \tilde{S}^2 \, dA \leq 2 \left( \int_0^{t^*} \int_L r^2 \, dx_2 \, d\eta \right)^{1/2} \\
\times \left( \int_0^{t^*} \int_L \left( r_{,x_2}^2 + 8|rr_{,\eta} + 4t r_{,\eta}^2 \right) \, dx_2 \, d\eta \right)^{1/2}. \quad (4.14)
\]
Using the derived bound for \( \int_0^t \int_R T \cdot T \, dA \, d\eta \), we combine inequalities (4.10) and (4.11) to obtain a bound for \( M_C \).

Finally, we derive a bound for \( \int_0^t \int_R C \cdot C \, dA \, d\eta \). Again note that

\[
\left( \int_0^t \int_R C \cdot C \, dA \, d\eta \right)^{1/2} \leq \left[ \int_0^t \int_R (C - H) \cdot (C - H) \, dA \, d\eta \right]^{1/2} + \left( \int_0^t \int_R H \cdot H \, dA \, d\eta \right)^{1/2}.
\]  

(4.15)

On using integration by parts and the differential equation and applying the Schwarz inequality to the first term on the right-hand side of (4.15), we have

\[
\int_0^t \int_R (C - H) \cdot (C - H) \, dA \, d\eta = -\int_0^t \int_R (C - H) \Delta (C - H) \, dA \, d\eta \\
= -\int_0^t \int_R (C - H) \{-\sigma \Delta T + (C - H) \cdot \eta + u_a (C - H) \cdot \eta \} \, dA \, d\eta \\
\leq \sigma^2 \int_0^t \int_R T \cdot T \, dA \, d\eta. \tag{4.16}
\]

To find a bound for \( \int_0^t \int_R H \cdot H \, dA \, d\eta \), we observe that \( H \) satisfies the same type of initial–boundary value problem as \( T \), and thus we can immediately find a bound for the Dirichlet integral in terms of data. Combining these results and using the derived bound for \( \int_0^t \int_R T \cdot T \, dA \, d\eta \), we therefore have a bound for \( \int_0^t \int_R C \cdot C \, dA \, d\eta \). When these results are inserted into (4.1), the bound for \( E(0, t) \) in terms of data is obtained.

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