Approximation on \([0, \infty)\) by Reciprocals of Polynomials with Nonnegative Coefficients*

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A complete theory of best uniform approximation to positive functions decaying to zero on \([0, \infty)\) by reciprocals of polynomials with nonnegative coefficients is presented.

1. INTRODUCTION

Let \(C^+_0(X)\) denote the class of all real-valued continuous functions defined on \(X \subseteq [0, \infty)\), where \(X\) is closed, \(f(x) > 0\) on \(X\) and \(f(x) \to 0\) as \(x \to \infty\) (in \(X\)) if \(X\) is unbounded. Let \(K(X) = \{ p \in \Pi_n : p(x) > 0 \ \forall x \in X \text{ and } p^{(j)}(0) \geq 0, j = 0, 1, \ldots, n \}\), where \(\Pi_n\) denotes the class of all real algebraic polynomials of degree \(\leq n\). Thus, \(K\) consists of positive polynomials with nonnegative coefficients (we suppress the \(X\) whenever possible). We give existence, characterization and (strong) uniqueness results for the problem of best approximating functions \(f \in C^+_0 [0, \infty)\) by reciprocals of elements of \(K\).

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In an earlier paper, Reddy and Shisha [8] showed that the closure of the reciprocals of all polynomials having nonnegative coefficients on \([0, \infty)\) is the set of all reciprocals of entire functions with nonnegative Taylor coefficients.

Although our primary interest is \([0, \infty)\), the theory is developed for \(X\) a closed subset of \([0, \infty)\). The assumption that \(X\) is closed guarantees that 
\[
\|f\|_X = \max\{|f(x)|: x \in X\} < \infty \text{ for each } f \in C_0^+(X).
\]

In Section 2, we begin by establishing an existence theorem. In Section 3, two characterization results are given assuming \(X\) is compact. These characterizations are based upon certain linear functionals in \(\Pi_n^+\), the dual of \(\Pi_n\).

In Section 4 strong uniqueness is shown to hold when \(X\) is compact. In Section 5 it is shown that obtaining the best approximation to \(f \in C_0^+[0, \infty)\) from \(K[0, b]\) is equivalent to finding the best approximation on \([0, b]\) from \(K[0, b]\), where \(b\) may be determined constructively. Combining these results with the results of the previous two sections establishes characterization and uniqueness for the \([0, \infty)\) problem. In Section 6 this theory is then extended to \(X\), a closed subset of \([0, \infty)\), and a discretization result is established. Finally, in Section 7 some numerical examples are given.

2. Existence

We begin by developing an existence theory for this problem. Note that this requires \(\|f\|_x < \infty\) and also requires a little care as it might be possible for \(p\) to become unbounded near where \(f(x)\) is "small."

**Theorem 1 (Existence).** Let \(f \in C_0^+(X)\), where \(X\) is a closed subset of \([0, \infty)\). Then there a \(p^* \in K\) such that

\[
\|f - \frac{1}{p^*}\|_x = \inf \left\{ \|f - \frac{1}{p}\|_x : p \in K \right\}.
\]

**Proof.** If \(n = 0\), then \(1/p^*\) is best with \(1/p^* = \frac{1}{2}(\|f\|_x + \inf_x |f(x)|)\), where we have used the fact that \(\|f\|_x < \infty\). Therefore, assume \(n \geq 1\). Without loss of generality we may assume \(\text{card}(X) \geq n + 2\). Let \(p = \inf_{a \in K} \|f_1 - 1/p\|_x\) and let \(\{p_l\}_{l=1}^{\infty} \subseteq K\) be such that \(\|f - 1/p\|_x \leq p\). Setting \(p_l(x) = \sum_{i=0}^{n} a_{li} x^i\), if we can show that \(\{a_{li}\}\) is bounded, then by using subsequences (relabelled) we can find \(p^*(x) = \sum_{i=0}^{n} a^*_i x^i\) with \(a_{li} \to a^*_i\), so \(a^*_i \geq 0\), \(0 \leq i \leq n\). Furthermore, we must have \(p^*(x) \geq 1/(f(x) + p + 1)\), \(\forall x \in X\) and \(\|f - 1/p^*\|_x \leq p\), so \(1/p^*\) is best.

Therefore, let us assume that \(\{a_{li}\}\) is unbounded so (taking a subsequence of \(\{p_l\}\), if necessary) \(\max_l a_{li} \to \infty\) as \(l \to \infty\). Define \(q_l(x) = (\max_l a_{li})^{-1} p_l(x) = \sum_{i=0}^{n} b_{li} x^i\). Again, using subsequences if necessary, we
can find \( q(x) = \sum_{i=0}^{n} b_i x^i \) with \( b_{ii} \rightarrow b_i, 0 \leq i \leq n \), and \( \max_i b_i = 1, b_i \geq 0, 0 \leq i \leq n \). Hence \( q(x) > 0 \) for \( x > 0 \). For \( x \in X \setminus \{0\} \) we have \( p_l(x) = (\max_i a_{ii}) q_l(x) \rightarrow \infty \) as \( l \rightarrow \infty \). Therefore, since \( 1/p_l(x) \rightarrow 0 \) as \( l \rightarrow \infty \) and

\[
\left| f(x) - \frac{1}{p_l(x)} \right| < \left\| f - \frac{1}{p_l} \right\|, \\
\]

taking the limit as \( l \rightarrow \infty \) yields \( 0 < f(x) < \rho \) (thus \( \rho > 0 \)), \( x \in X \setminus \{0\} \). But this leads to a contradiction since \( p(x) = 2/\rho \) satisfies \( \|f - 1/p\| \leq \rho/2 \) if \( 0 \) is not an isolated point of \( X \), whereas \( p(x) = Mx + (f(0))^{-1} \) satisfies \( \|f - 1/p\| < \rho \) for \( M \) sufficiently large if \( 0 \) is an isolated point of \( X \).  

In closing this section we observe that if \( X \) is unbounded and \( n \geq 1 \) then the best reciprocal approximation to \( f \in C_+^1(X) \) from \( K(X) \) is not a constant. This is easily seen by observing that the best reciprocal constant approximation is \( c^* = 2/\|f\|_X \) and that for a proper choice of \( \varepsilon_1, \varepsilon_2 > 0 \), \( p^*(x) = \varepsilon_2 x + (c^* - \varepsilon_1) \) will belong to \( K \) and satisfy \( \|f - 1/c^*\|_X > \|f - 1/p^*\|_X \).

3. Characterization

In this section we shall assume that \( X \) is compact and establish both a "zero in the convex hull" type of characterization and a generalized alternation characterization. In both cases, these results are analogous to the characterization for approximation as developed in [2]. In order to obtain these results, we use specific linear functionals in \( \Pi_n^* \), the dual space of \( \Pi_n \) with the uniform topology. Basically, two types of linear functionals play a crucial role. They are point evaluations \( e_x \in \Pi_n^* \), where \( e_x(g) = g(x) \), \( \forall g \in C(X), x \in X \), and derivative evaluations at zero \( e^d_0 \in \Pi_n^* \), where \( e^d_0(p) = p^{(j)}(0) \), \( \forall p \in \Pi_n \), \( 0 \leq j \leq n \).

Fix \( f \in C_+^1(X) \) and \( p \in K \). Then we say that \( e \in \Pi_n^* \) is an extreme point for \( f \) and \( p \) if either

(i) \( e = e_x \) for some \( x \in X \) and \( \|e_x(f - 1/p)\| = \|f - 1/p\|_X \), or

(ii) \( e = e^d_j \) for some \( j \), \( 0 \leq j \leq n \) and \( e^d_j(p) = 0 \).

We denote the complete set of all extreme points for \( f \) and \( p \) by \( X_f \), as usual. In addition, we define the sign of an extreme point \( \sigma(e) \) by

(1) \( \sigma(e) = \text{sgn}(f(x) - 1/p(x)) \) if \( e \equiv e_x \), or
(2) \( \sigma(e^d_j) = (-1)^{j+1} \).

We observe that it is not possible for both \( e_0 \) and \( e^d_0 \) to belong to the extreme set of \( f \) and \( p \). In fact, \( e_0 \in X_p \) can occur only if \( 0 \in X \) and \( e^d_0 \in X_p \) can occur only if \( 0 \in X \) (since \( 0 \in X \) implies that \( p(0) > 0 \), as \( p \in K \)).
We note that any \( k \) distinct extreme points for \( f \) and \( p \) with \( k \leq n + 1 \) are linearly independent. Also, any set of \( n + 2 \) extreme points for \( f \) and \( p \) will be linearly dependent as \( \Pi^*_n \) has dimension \( n + 1 \). Finally, we observe that due to the continuity of \( f \) and \( p \) on \( X \), it follows that \( X_p \) is a compact subset of \( \Pi^*_n \). Let

\[
U = \{-e^*_0 : e^*_0 \in X_p \} \cup \{\sigma(e_x) : e_x \in X_p \}.
\]

Then we have the following "zero in the convex hull" characterization theorem.

**Theorem 2.** Let \( f \in C_0^+(X) \) be such that \( 1/f \notin K \). Then \( p^* \in K \) gives a best reciprocal approximation to \( f \) from \( K \) on \( X \) (compact) iff the zero of \( \Pi^*_n \) belongs to the convex hull, \( H(U) \), of \( U \) corresponding to \( X_p \). Furthermore, the convex combination will always consist of precisely \( n + 2 \) nonzero terms.

**Proof.** (\( \Rightarrow \)) By contradiction. Therefore, we assume that \( p^* \in K \) does not give a best approximation to \( f \). Then, \( \exists p \in K \ni \|f - 1/p\| < \|f - 1/p^*\| \). Let \( p(x) = \sum_{i=0}^n a_i x^i \) and set \( p_j(x) = \sum_{i=0}^n (a_i + \varepsilon) x^i \). Since \( X \) is compact, we select \( \varepsilon > 0 \) sufficiently small so that \( \|f - 1/p_\varepsilon\| < \|f - 1/p^*\| \). Then, for \( e^*_0 \in X_{p^*} \), we have that \( -e^*_0(p_\varepsilon - p^*) < 0 \). Also, for \( e_x \in X_{p^*} \), we have from the inequality

\[
\sigma(e_x) \left( \frac{1}{p^*(x)} - \frac{1}{p_j(x)} \right) < 0,
\]

that \( \sigma(e_x) e_x(p_\varepsilon - p^*) < 0 \). Thus, the system of linear inequalities \( e(p) < 0 \), \( e \in U \), is consistent. Since \( U \) is compact (as is \( X_{p^*} \)) we have, by the Theorem on Linear Inequalities (see, e.g., [3, p. 191]) (identifying \( \Pi^*_n \) and \( \Pi_n \) with \( R^n \)), that zero does not belong to the convex hull of \( U \). This is a contradiction establishing the desired result.

(\( \Rightarrow \)) By contradiction. Therefore, we assume \( 0 \notin H(U) \). Again, by the Theorem on Linear Inequalities, we have that \( \exists q \in \Pi_n \) such that \( -e^*_0(q) < 0 \) for all \( e^*_0 \in X_{p^*} \) and \( \sigma(e_x) e_x(q) < 0 \) for all \( e_x \in X_{p^*} \). Set \( p_\varepsilon = p^* + \varepsilon q \), where \( \varepsilon > 0 \) is chosen sufficiently small so that \( p_j(x) > 0 \) for all \( x \in X \). Now, for \( e^*_0 \in X_{p^*} \), we have that \( q^{(j)}(0) > 0 \) so that \( p_\varepsilon^{(j)}(0) > 0 \). By taking \( \varepsilon > 0 \) smaller, if necessary, we can also guarantee that \( p_\varepsilon^{(j)}(0) > 0 \) for all \( j, 0 \leq j \leq n \), such that \( e^*_0 \in X_{p^*} \), since \( p^{(j)}(0) > 0 \) for these indices. Hence \( p_\varepsilon \in K \).

We now claim that for \( \varepsilon > 0 \) (chosen smaller yet, if necessary), we must have that \( \|f - 1/p_\varepsilon\| < \|f - 1/p^*\| \) giving the desired contradiction. A standard compactness argument gives this result since at the positive extremals \( e_x \) (i.e., \( \sigma(e_x) = 1 \)) we have that \( q(x) < 0 \) so that \( 1/p^*(x) < 1/p_\varepsilon(x) \) and at the negative extremals \( 1/p^*(x) > 1/p_\varepsilon(x) \).

Finally, since \( \Pi^*_n \) is \( n + 1 \) dimensional, we have that the zero in the convex hull result will hold with \( s \leq n + 2 \) terms. In order to see that it is not
possible for this to hold with less than \( n + 2 \) terms, we simply note that for a set \( S \) of \( s < n + 2 \) distinct elements of \( X_P \), we can always find \( p \in \Pi_{s-1} \) for which \( e_0(p) = -1 \) if \( e_0 \in S \) and \( e_x(p) = \sigma(e_x) \) if \( e_x \in S \). This follows from the fact that the Hermite–Birkhoff problem associated with these equations is poised (i.e., all supported blocks are even, see [1]).

We now turn to developing our generalized alternation theorem. To this end, fix \( f \) and let \( p \in K \). We say that \( \{e^p_i\}_{i=1}^{s} \cup \{e_{x_{\mu}}\}_{\mu=s+1}^{k} \subset X_p \) is an alternant of length \( k \) for \( f - 1/p \) provided \( n \geq j_1 > j_2 > \cdots > j_s > 0; \ x_{s+1} < x_{s+2} < \cdots < x_k \) with

1. \( j_v - j_{v+1} \) an odd integer for \( v = 1, 2, \ldots, s - 1 \) (if \( s \leq 1 \), then this requirement is vacuous),
2. \( \sigma(e_{x_{s+1}}) = (-1)^{j_s} \) (if \( s = 0 \), or \( s = k \), then this requirement is vacuous), and
3. \( \sigma(e_{x_{\mu}}) = -\sigma(e_x) \) for \( \mu = s + 1, \ldots, k - 1 \) (vacuous if \( k < s + 1 \)).

Thus, (1)–(3) imply that if \( \{e_i\}_{i=1}^{s} \cup \{e_{x_{\mu}}\}_{\mu=s+1}^{k} \), listed in this order, then \( \sigma(e_{x_{i+1}}) = -\sigma(e_{x_i}) \) for \( i = 1, \ldots, k - 1 \).

With this definition, we have

**Theorem 3.** Suppose \( f \in C^+_c(X) \) and \( 1/f \notin K \). Then \( p^* \in K \) gives a best reciprocal approximation to \( f \) from \( K \) on \( X \) (compact) iff \( f - 1/p^* \) has an alternant of length \( n + 2 \).

**Proof.** The method of proof is to show that this alternant is precisely a basis for the “zero in the convex hull” result of Theorem 2. The specific proof given here is patterned after one given by B. Chalmers [2, Theorem 2, Section 4].

(\( \Leftarrow \)) Suppose that \( p^* \) gives a best reciprocal approximation to \( f \) from \( K \) on \( X \). Then, there exist positive constant \( \lambda_1, \ldots, \lambda_{n+2} \) with \( \sum_{i=1}^{n+2} \lambda_i = 1 \), and a set of \( n + 2 \) distinct extremals in \( X_{p^*} \), \( \{e_0^p\}_{i=1}^{s} \cup \{e_{x_{\mu}}\}_{\mu=s+1}^{n+2} \), ordered as above (i.e., \( n \geq j_1 > j_2 > \cdots > j_s > 0; \ x_{s+1} < x_{s+2} < \cdots < x_{n+1} \)) such that

\[
\sum_{v=1}^{s} \lambda_v (-e_0^p) + \sum_{\mu=s+1}^{n+2} \lambda_{\mu} \sigma(e_{x_{\mu}}) e_{x_{\mu}} = 0
\]

in \( \Pi_n^F \). Now set \( J = \{j_s, j_{s-1}, \ldots, j_1\} \) and \( I = \{0, 1, \ldots, n\} \setminus J \). Now apply the linear combination (1) to the functions \( x^{j_k}, k = s, s - 1, \ldots, 1 \), which yields

\[
\sum_{\mu=s+1}^{n+2} \lambda_{\mu} \sigma(e_{x_{\mu}}) x^{j_k} = (j_k! \lambda_j), \quad k = s, s - 1, \ldots, 1.
\]

Applying (1) to the function \( x^m, m \in I \), gives

\[
\sum_{\mu=s+1}^{n+2} \lambda_{\mu} \sigma(e_{x_{\mu}}) x^m = 0, \quad m \in I.
\]
Note that (3) consists of precisely \( n + 1 - s \) equations and \( n + 2 - s \) coefficients. Now, using the fact that \( \det [(x^m_{ij})_{i,j=1}] > 0 \) for \( 0 < x_1 < \cdots < x_i < \infty \) and \( -\infty < \rho_1 < \cdots < \rho_i < \infty \) (see, e.g., [4, p. 9]) and Cramer's rule we have by standard techniques (see, e.g., [3, p. 74])

\[
\text{sgn} \lambda_\mu \sigma(e_{x_\mu}) = -\text{sgn} \lambda_{\mu+1} \sigma(e_{x_{\mu+1}}), \quad \mu = s + 1, \ldots, n + 1,
\]

or

\[
\sigma(e_{x_\mu}) = -\sigma(e_{x_{\mu+1}}), \quad \mu = s + 1, \ldots, n + 1, \text{ as } \lambda_j > 0, \forall i.
\]

Next, in system (2), observe that the functions \( \phi_1(t) = x_{s+1}, \phi_2(t) = x_{s+2}, \ldots, \phi_{n+2-s}(t) = x_{n+2} \) (use \( \phi_2, \phi_3, \ldots, \phi_{n+2-s} \) if \( x_{s+1} = 0 \)) form a Chebyshev system for \( t \in [0, \infty) \). Thus,

\[
F(t) = \sum_{\mu=s+1}^{n+2} [\lambda_\mu \sigma(e_{x_\mu})] x'_\mu
\]

can have at most \( n + 1 - s \) zeros in \([0, \infty)\) counting a zero at which \( F(t) \) does not change sign as two zeros (for \( x_{s+1} = 0 \), use \( F(t) = \sum_{\mu=s+2}^{n+2} [\lambda_\mu \sigma(e_{x_\mu})] x'_\mu \) which can have only \( n - s \) zeros in \([0, \infty)\). Note that \( F(0) = -\lambda_{s+1} \sigma(e_{x_{s+1}}) \neq 0 \). This is the equation of (3) corresponds to \( m = 0 \). Recall that \( 0 \in X \) implies that \( j_s > 0 \).

Now \( F(t) \) vanishes at \( t = m, m \in I \), for a total of \( n + 1 - s \) points. (For the case \( x_{s+1} = 0, F(t) \) vanishes at \( t = m, m \in I, m \neq 0 \), for \( n - s \) points.) Thus, each point of \( I \setminus \{0\} \) must be a point where \( F(t) \) changes sign and \( F(t) \) can have no additional positive zeros. Now, since \( (j_s) \lambda_{j_s} > 0 \) for \( k = s, s - 1, \ldots, 1 \) we see that for \( j_k \in J, j_{k+1} \) and \( j_k \) must have an even number of elements of \( I \) between them (0 is allowed). That is, \( j_k - j_{k+1} \) must be an odd integer for \( k = 1, \ldots, s - 1 \).

Finally, define \( p \in \Pi_n \) by \( p^{(j)}(0) = 0, j \in I \setminus \{j_s\}, p(x_\mu) = 0 \), \( \mu = s + 2, \ldots, n + 2 \) and \( p(x_{s+1}) = 1 \), where \( p(x) = \sum_{l=0}^{n} a_l x^l \). Observing that \( \{0, 1, \ldots, j_s - 1\} \subset I \), we shall enumerate \( I \setminus \{j_s\} \) by \( I \setminus \{j_s\} = \{0, 1, \ldots, j_s, l_s+1, \ldots, l_{n+1-s}\} \), where \( j_s < l_{s+1} < \cdots < l_{n+1-s} \leq n \). Then \( p \) satisfies the system

\[
\sum_{m=0}^{l_{n+1-s}} a_m x_{\mu}^m = \delta_{s+1,\mu}, \quad \mu = s + 1, \ldots, n + 2.
\]

Solving for \( a_{j_s} \) by Cramer's rule and using the fact that \( \det [(x^m_{ij})_{i,j=1}] > 0 \) for \( 0 < x_1 < \cdots < x_i < \infty, -\infty < \rho_1 < \cdots < \rho_i < \infty \) again, we see, after \( j_s \) column interchanges in the numerator determinant, that \( \text{sgn}(a_{j_s}) = (-1)^{j_s} \).

Now, applying (1) to \( p \) we find that

\[
-\lambda_{j_s} (j_s!) a_{j_s} + \lambda_{s+1} \sigma(e_{x_{s+1}}) = 0,
\]
or that $\sigma(e_{x_{s+1}}) = (-1)^i$. This shows that the extreme points of the "zero in the convex hull" characterization form an alternant of length $n + 2$ for $f - 1/p^*$.

$(\Rightarrow)$ Conversely, let $\{e^{j_v}\}_{v=1}^s \cup \{e_{x_{\mu}}\}_{\mu=s+1}^{n+2}$ be an alternant of length $n + 2$ for $f - 1/p^*$. Then, since $\Pi^*_n$ is $n + 1$ dimensional and any $n + 1$ of the above extremals form a basis for $\Pi^*_n$ we have that $\exists$ constants $\theta_1, \ldots, \theta_{n+2}$, all not zero, such that

$$
\sum_{v=1}^s \theta_v (e^{j_v}) + \sum_{\mu=s+1}^{n+2} \theta_\mu \sigma(e_{x_\mu}) e_{x_\mu} = 0
$$

(5) in $\Pi^*_n$. Define $J$ and $I$ as above and apply (5) to $x^m, m = 0, 1, \ldots, n$ to obtain

$$
\sum_{\mu=s+1}^{n+2} \theta_\mu \sigma(e_{x_\mu}) x^{j_k}_\mu = \theta_k (j_k!), \quad k = s, s - 1, \ldots, 1
$$

(6) and

$$
\sum_{\mu=s+1}^{n+2} \theta_\mu \sigma(e_{x_\mu}) x^m_\mu = 0, \quad m \in I.
$$

(7)

Now, as above, (7) implies that $\text{sgn}(\theta_\mu \sigma(e_{x_\mu})) = -\text{sgn}(\theta_{\mu+1} \sigma(e_{x_{\mu+1}}))$, $\mu = s + 1, \ldots, n + 1$. Since $\sigma(e_{x_\mu}) = -\sigma(e_{x_{\mu+1}})$ for $\mu = s + 1, \ldots, n + 1$ we have that $\text{sgn} \theta_\mu = \text{sgn} \theta_{\mu+1}, \mu = s + 1, \ldots, n + 1$. Next, for the special function $p$ defined by $p^{(j)}(0) = 0, k = s - 1, \ldots, 1, p(x_\mu) = 0, \mu = s + 2, \ldots, n + 2$ and $p(x_{s+1}) = 1$, we get, after applying (5) to this $p$, that $\theta_{s+1} \sigma(e_{x_{s+1}}) = \theta_s p^{(j)}(0)$. Since $\sigma(e_{x_{s+1}}) = (-1)^i$ and $\text{sgn} p^{(j)}(0) = (-1)^i$ from above, we have that $\text{sgn} \theta_{s+1} = \text{sgn} \theta_s$. Finally, by repeating the $F(t)$ argument appearing in the first half of this proof we have that $\text{sgn} \theta_v = \text{sgn} \theta_{v-1}$ for $v = s, s - 1, \ldots, 2$ as desired. Thus, $\theta_i \neq 0, i = 1, 2, \ldots, n + 2$, and all are of the same sign. Hence (using a suitable normalization), we have that the zero of $\Pi^*_n$ belongs to the convex hull of $U, U$ corresponding to $X_p^*$, as above (in fact, we know a specific convex combination from $U$ for $0$). Thus, $p^* \in K$ gives a best reciprocal approximation to $f$ from $K$ on $X$ as desired. 

We observe that in an alternant of length $n + 2$ for $f - 1/p^*$, we must have $s \leq n$, so that there will always exist at least two standard extremals and normal alternation between them; if $p^*$ is not a constant and $0 \in X$ then $s \leq n - 1$, so there will be at least three standard extremals and normal alternation between them.
4. Uniqueness

Best approximations in our setting are unique; in fact, the zero in the convex hull theorem enables us to prove strong uniqueness. Lipschitz continuity of the best approximation operator then follows as in [3, p. 82]. In this section we shall write $\| \cdot \|$ for $\| \cdot \|_X$.

**Theorem 4.** Let $f \in C^*_c(X)$, where $X$ is compact, and let $p^* \in K$ satisfy $\| f - 1/p^* \| = \inf_{p \in K} \| f - 1/p \|$. Then there exists a positive constant $\gamma = \gamma(f)$ such that

$$ \| f - \frac{1}{p} \| \geq \| f - \frac{1}{p^*} \| + \gamma \frac{1}{p - \frac{1}{p^*}} $$

for all $p \in K$.

**Proof.** Without loss of generality we may assume $\| f - 1/p^* \| > 0$, since otherwise the theorem holds with $\gamma = 1$. For $p \in K$, $p \neq p^*$, define

$$ \gamma(p) = \frac{\| f - \frac{1}{p} \| - \| f - \frac{1}{p^*} \|}{\| p - \frac{1}{p^*} \|} . $$

Assume (by way of contradiction) that there exist a sequence $\{ p_k \} \subseteq K$, $p_k \neq p^*$, with $\gamma(p_k) \to 0$. Then $\| 1/p_k \|$ is bounded (otherwise $\gamma(p_k) \to 0$), and thus $\| f - 1/p_k \| - \| f - 1/p^* \| \to 0$ (otherwise $\gamma(p_k) \to 0$), so from the proof of Theorem 1 we have that $\| p_k \|$ must be bounded. By Theorem 2 there is a set of $n + 2$ distinct extremals

$$ U = \{ e_{i0} \}_{i=1}^s \cup \{ e_{x^i} \}_{i=s+1}^{n+2} \subseteq X_p , $$

and a set $\{ \lambda_i \}_{i=1}^{n+2}$ of positive constants such that

$$ \sum_{i=1}^s \lambda_i e_{i0}^j + \sum_{u=s+1}^{n+2} \lambda_u \sigma(e_{x_u}) e_{x_u} = 0 \in \Pi_n^* . $$

Now let $p \in K$ satisfy

$$ -e_{i0}(p) \leq 0, \quad v = 1, \ldots, s , $$

and

$$ \sigma(e_{x_u}) e_{x_u}(p) \leq 0, \quad \mu = s + 1, \ldots, n + 2 . $$
Then from
\[ \sum_{\nu=1}^{s} \lambda_{\nu}(-e_{0}^{\nu}(p)) + \sum_{\mu=s+1}^{n+2} \lambda_{\mu} \sigma(e_{\mu}) e_{\mu}(p) = 0 \]
and the fact that \( \lambda_{i} > 0 \) for \( i = 1, \ldots, n+2 \), we get
\[ e_{0}^{\nu}(p) = 0, \quad \nu = 1, \ldots, s, \]
and
\[ e_{\mu}(p) = 0, \quad \mu = s+1, \ldots, n+2. \]

But any \( n+1 \) of these conditions imply that \( p = 0 \), since the associated Hermite–Birkhoff interpolation problem is poised. Thus, if \( p \in K \) satisfies \( p \neq 0 \) and \(-e_{0}^{\nu}(p) \leq 0, \nu = 1, \ldots, s\), then for some \( \omega \) with \( s+1 \leq \omega \leq n+2 \) we must have \( \sigma(e_{\omega}) e_{\omega}(p) > 0 \). Let
\[ c = \inf \{ \max_{s+1 \leq \mu \leq n+2} \sigma(e_{\mu}) p(x_{\mu}) : p \in K, \| p \| = 1 \text{ and } -e_{0}^{\nu}(p) \leq 0, \nu = 1, \ldots, s \} > 0. \]

Then for all \( \mu = s+1, \ldots, n+2 \), we have
\[
\gamma(p_{k}) \left\| \frac{1}{p_{k}} - \frac{1}{p^{*}} \right\| = \left\| f - \frac{1}{p_{k}} \right\| - \left\| f - \frac{1}{p^{*}} \right\|
\geq \sigma(e_{\mu}) \left( f(x_{\mu}) - \frac{1}{p_{k}(x_{\mu})} \right) - \sigma(e_{\mu}) \left( f(x_{\mu}) - \frac{1}{p^{*}(x_{\mu})} \right)
= \sigma(e_{\mu}) \left( \frac{1}{p_{k}(x_{\mu})} - \frac{1}{p^{*}(x_{\mu})} \right)
= \sigma(e_{\mu}) \frac{p_{k}(x_{\mu}) - p^{*}(x_{\mu})}{p^{*}(x_{\mu}) \ p_{k}(x_{\mu})}
= \left\| p_{k} - p^{*} \right\| \ p_{k}(x_{\mu}) \ p^{*}(x_{\mu}) \left( \sigma(e_{\mu}) \cdot \frac{p_{k}(x_{\mu}) - p^{*}(x_{\mu})}{\| p_{k} - p^{*} \|} \right). \]

So for some \( \omega = s+1, \ldots, n+2 \), we have
\[ \gamma(p_{k}) \left\| \frac{1}{p_{k}} - \frac{1}{p^{*}} \right\| \geq \frac{\left\| p_{k} - p^{*} \right\| \sigma(e_{\omega})}{p^{*}(x_{\omega}) \ p_{k}(x_{\omega})} \cdot c. \]

Now for each \( k \) select \( y_{k} \in X \) such that
\[
\left\| \frac{1}{p_{k}(y_{k})} - \frac{1}{p^{*}(y_{k})} \right\| = \left\| \frac{1}{p_{k}} - \frac{1}{p^{*}} \right\|. \]
Then,
\[ \frac{1}{p_k} - \frac{1}{p^*} \leq \frac{\|p_k - p^*\|}{p_k(y_k) p^*(y_k)} \]
so that
\[ \gamma(p_k) \frac{\|p_k - p^*\|}{p^*(y_k) p_k(y_k)} \geq \frac{\|p_k - p^*\|}{p^*(x_\infty) p_k(x_\infty)} \cdot c. \]

Hence,
\[ \gamma(p_k) \geq \frac{p^*(y_k) p_k(y_k)}{p^*(x_\infty) p_k(x_\infty)} \cdot c \to 0, \]
as \( \|p_k\| \) and \( 1/p_k \) are bounded independent of \( k \) and \( X \) is compact. This gives us our desired contradiction, completing the proof.

5. APPROXIMATION ON \([0, \infty)\)

We now state and prove a central result which shows that for \( n \geq 1 \), approximation on \([0, \infty)\) with reciprocals of elements of \( K \) is completely equivalent to approximation on \([0, b]\) for some \( b > 0 \). This result allows us to apply the theory of the previous sections to this problem. Also, this proof can be made constructive, giving a procedure for calculating \( b \).

**Theorem 5.** Let \( f \in C^+_0[0, \infty) \) and assume \( n \geq 1 \). Then there exists \( b > 0, p^* \in K[0, \infty) = K \) such that
\[
\left\| f - \frac{1}{p^*} \right\|_{[0, b]} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_{[0, b]} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_{[0, \infty)} = \frac{1}{p^*} \right\|_{[0, \infty)} = \lambda_{\infty}.
\]

**Proof.** We can assume \( 1/f \notin K \). For each \( 0 < b \leq \infty \), choose \( p_b \in K \) satisfying
\[
\left\| f - \frac{1}{p_b} \right\|_{[0, b]} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_{[0, b]} = \lambda_b.
\]
Assume \( p_b \) cannot serve as \( p_m \) for all finite positive \( b \). Then by uniqueness of such \( p_b, p_\infty \) cannot serve as \( p_b \) for any finite positive \( b \). Hence for all \( 0 < b < \infty \),
\[
\left\| f - \frac{1}{p_b} \right\|_{[0, \infty)} > \left\| f - \frac{1}{p_\infty} \right\|_{[0, \infty)} \geq \left\| f - \frac{1}{p_\infty} \right\|_{[0, b]} \geq \left\| f - \frac{1}{p_b} \right\|_{[0, b]}.
\]

(8)
Then for some \( y_b > b \),

\[
\left| f(y_b) - \frac{1}{p_b(y_b)} \right| > \left\| f - \frac{1}{p_\infty} \right\|_{[0, \infty)} > \left| f(b) - \frac{1}{p_b(b)} \right|.
\]

But \( p_b(y_b) \geq P_b(b) \) and \( \max\{f(x) : x \geq b\} \to 0 \) as \( b \to \infty \). We deduce

\[
\lim_{b \to \infty} \frac{1}{p_b(b)} = \left\| f - \frac{1}{p_\infty} \right\|_{[0, \infty)}.
\]

Write \( p_b(x) = \sum_{j=0}^{n} a_{jb} x^j \). Then if \( y > 0 \) is given, and \( b > y \),

\[
\left\| \sum_{j=1}^{n} a_{jb} x^j \right\|_{[0, y]} = \sum_{j=1}^{n} a_{jb} \left( \frac{y}{b} \right)^j b^j < \frac{y}{b} p_b(b) \to 0 \quad \text{as} \quad b \to \infty.
\]

Further \( a_{0b} \leq M \), some \( M < \infty \), for all \( b > 0 \) as \( \| f - 1/p_\infty \|_{[0, \infty)} < \frac{1}{2} \| f \|_{[0, \infty)} \).

Choose a sequence \( B \) of values for \( b \) such that as \( b \to \infty \) through \( B \), \( a_{0b} \to c \).

Then we see, as \( c \) is independent of \( y \), that

\[
\lim_{b \to \infty} \left\| f - \frac{1}{p_b} \right\|_{[0, y]} = \left\| f - \frac{1}{c} \right\|_{[0, y]} \quad \text{for each} \quad y > 0.
\]

Then using the last inequality in (8), we see

\[
\left\| f - \frac{1}{c} \right\|_{[0, y]} \leq \limsup_{b \to \infty} \left\| f - \frac{1}{p_b} \right\|_{[0, b]} \leq \left\| f - \frac{1}{p_\infty} \right\|_{[0, \infty)}
\]

for each \( y > 0 \). We deduce that

\[
\left\| f - \frac{1}{c} \right\|_{[0, \infty)} = \left\| f - \frac{1}{p_\infty} \right\|_{[0, \infty)}
\]

so that a constant \( c \) is a best approximation; as after Theorem 1, this is impossible.

**Remark.** A constructive proof can be given for calculating \( b \) in which at most four best reciprocal approximations need be calculated. A copy of this is available upon request.

**Corollary.** The best approximation to \( f \in C^1_0[0, \infty) \), for \( n \geq 1 \), exists, is unique, and is characterized by the alternation of Theorem 3.

Note that strong uniqueness need not hold in the \([0, \infty)\) setting. For
example, if $n = 3$, $p^*(x) = x + 1$ is readily seen to be the unique best reciprocal approximation to $f(x)$ by the standard alternating theorem where $f(x)$ is defined to be piecewise linear on $[0, \frac{3}{4}]$ with vertices $(v/4, (1 + v/4)^{-1} - \frac{1}{4}(-1)^v)$, $v = 0, \ldots, 4$ and $(\frac{3}{4}, (1 + \frac{3}{4})^{-1})$. For $x \geq \frac{3}{4}$, $f(x)$ is defined to be $(x + 1)^{-1}$. Setting $p_k(x) = 1 + x + x^k/k$, one can show that strong uniqueness fails to hold in this case.

6. DISCRETIZATION RESULTS

Suppose $X$ is a nonvoid closed subset of $[0, \infty)$. Define $|X| = \sup_{x \in [0, \infty)} \inf_{y \in X} |x - y|$ = density of $X$ in $[0, \infty)$. Then we have

**Theorem 6.** If $f \in C_0^+(X)$, $n \geq 1$, then there exists a $b > 0$ and a $p^* \in K = K(X)$ such that

$$
\left\| f - \frac{1}{p^*} \right\|_{[0,b] \cap X} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_{[0,b] \cap X} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_X = \left\| f - \frac{1}{p^*} \right\|_X.
$$

**Proof.** The proof follows the proof of Theorem 5 where each interval is replaced by its intersection with $X$, and where each point mentioned is in $X$.

**Corollary.** The best reciprocal approximation to $f \in C_0^+(X)$ on $X$, for $n \geq 1$, exists, is unique, and is characterized by the alternation of Theorem 3.

Now, let $n \geq 1$, $f \in C_0^+[0, \infty)$ and $1/f \notin K[0, \infty)$. Define $\lambda_b$, $\lambda_\infty$ as in Theorem 5 (note $\lambda_\infty > 0$) and define

$$
\lambda_b^X = \inf_{p \in K(X)} \left\| f - \frac{1}{p} \right\|_{[0,b] \cap X},
$$

$$
\lambda_\infty^X = \inf_{p \in K(X)} \left\| f - \frac{1}{p} \right\|_X,
$$

$1/p_b^X$ = best approximation to $f$ on $[0,b] \cap X$ where $p_b^X \in K(X)$,

$1/p_\infty^X$ = best approximation to $f$ on $X$ where $p_\infty^X \in K(X)$,

$1/p_\infty$ = best approximation to $f$ on $[0, \infty)$ where $p_\infty \in K[0, \infty)$.

$b^* = \inf \{ b \in \mathbb{R} : \lambda_b = \lambda_\infty \},$

$$
b^{**} = \sup \left\{ b \in \mathbb{R} : \left| f(b) - \frac{1}{p_\infty(b)} \right| = \lambda_\infty \right\},
$$

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and

\[ b^*_X = \inf \{ b \in \mathbb{R} : \lambda^X_b = \lambda^\infty_b \}. \]

Observe that \( 0 < b^* \leq b^{**} < \infty \), \( \lambda^*_b = \lambda^\infty_b = \lambda^{**}_b \), \( \lambda^X_b = \lambda^\infty_b \).

**Theorem 7.** Let \( f \in C^+_0 [0, \infty) \), \( 1/f \in K[0, \infty) \), \( n \geq 1 \). Suppose \( X \subseteq [0, \infty) \) with \( |X| < \delta \) for some \( \delta > 0 \). Then

(i) For any \( \varepsilon > 0 \), \( b^*_X \in (b^* - \varepsilon, b^{**} + \varepsilon) \), for all \( \delta > 0 \) sufficiently small. (Thus, if \( b^* = b^{**} \), then \( b^*_X \rightarrow b^* \) as \( \delta \rightarrow 0 \).)

(ii) For every \( \delta > 0 \), sufficiently small, there exists a constant \( \gamma \) independent of \( X \) such that

\[ \left\{ f - \frac{1}{p^X_{\infty}} \right\}_{(0, \infty)} - \left\{ f - \frac{1}{p_{\infty}} \right\}_{(0, \infty)} \leq \omega(\delta) + \gamma \delta, \]

where \( \omega(\delta) = \max_{x, y \in [0, \infty), |x - y| \leq \delta} |f(x) - f(y)| \).

(iii) \( 1/p^X_{\infty} \) converges uniformly to \( 1/p_{\infty} \) on \( [0, \infty) \) as \( \delta \rightarrow 0 \).

**Proof.** (i) (by contradiction) Suppose there exist sets \( \{X_i\}_{i=1}^\infty \) with \( |X_i| < \delta_i \), \( \delta_i \rightarrow 0 \) and \( b^*_X \in (b^* - \varepsilon, b^{**} + \varepsilon) \) for some \( \varepsilon > 0 \) fixed. For notational convenience, let \( p_i = p^X_{X_i}, b^*_i = b^*_X \) and \( \lambda^i_{\infty} = \lambda^X_{\infty} \) so that \( p_i \) gives the best reciprocal approximation to \( f \) on \( [0, b^*_X] \cap X_i \) and \( X_i \) from \( K(X_i) \). If \( p_i = \sum_{i=0}^n a_{it} x^t \) then by arguments similar to those of Theorem 1 we have that \( \{a_{it}\} \) is bounded, so going to further subsequences, if necessary, we have that \( a_{it} \rightarrow a_t \) as \( i \rightarrow \infty \) for \( 0 \leq t \leq n \). Set \( p(x) = \sum_{i=0}^n a_t x^t \). Again, using arguments as in Theorem 1, it can be show that \( p \equiv p_{\infty} \). Thus, \( p \) is not a constant so choosing a nonzero coefficient \( a_k \) with \( k \geq 1 \) we must have \( a_{ki} \geq a_k/2 \) for \( i \geq i_1 \) (say) implying there exists \( c \geq b^* \) such that \( 1/p_i(x) \leq \lambda^i_{\infty}/2 \) and \( f(x) \leq \lambda^i_{\infty}/2 \) for all \( x \geq c \). By the uniform convergence of \( \{p_i\} \) to \( p_{\infty} \) on \( [0, b^*_X] \) and the assumption that \( |X_i| \rightarrow 0 \) it follows that for \( i \) sufficiently large \( \lambda^i_{\infty} \geq \frac{1}{2} \lambda_{\infty} \). Thus, for \( i \) sufficiently large we have that \( b^*_i \leq c \). Therefore, \( \{b^*_i\} \) is bounded.

Choose a subsequence (note relabelled) so that \( b^*_i \rightarrow b \) (say), and choose \( i_2 \) so large that \( b^*_i \in [0, L] \) for all \( i \geq i_2 \), where \( L = \max(b^{**} + \varepsilon, b) + 1 \). Then

\[ \inf_{p \in K[0, \infty)} \left\{ \left\| f - \frac{1}{p} \right\|_{(0, L)} \right\} = \lambda_{\infty} \quad \text{and} \quad \inf_{p \in K(X_i)} \left\{ \left\| f - \frac{1}{p} \right\|_{(0, L) \cap X_i} \right\} = \lambda^i_{\infty}, \quad i \geq i_2. \]

Now, by the uniform convergence of \( \{p_i\} \) to \( p_{\infty} \) on \( [0, L] \) we have that \( \lambda^i_{\infty} \rightarrow \lambda_{\infty} \) as \( i \rightarrow \infty \).

Now suppose \( b \geq b^{**} + \varepsilon \). Then, we must have that \( |f(b) - 1/p_{\infty}(b)| < \lambda_{\infty} \).
by the definition of \( b^{**} \). Thus, there exists \( \eta > 0 \) such that for all \( i \) sufficiently large, \( |f(y) - 1/p_i(y)| < \lambda_i^\eta \), \( \forall y \in (b - \eta, b + \eta) \cap X_i \), contradicting the fact that \( b_i^* \in (b - \eta, b + \eta) \cap X_i \) for all \( i \) sufficiently large.

On the other hand, suppose \( b \leq b^* - \varepsilon \). Then, by the definition of \( b^* \) we have that

\[
\alpha = \inf \left\{ \left\| \frac{f - \frac{1}{p}}{p} \right\|_{[0,b+\varepsilon/2]} : p \in K[0, \infty) \right\} < \lambda_\infty.
\]

However, this implies \( \lambda_i \leq \alpha \) for all \( i \) sufficiently large (so that \( b^*_i \leq b + \varepsilon/2 \)) which contradicts the fact that \( \lambda_i \to \lambda_\infty \). This contradiction then proves part (i) of the theorem.

For parts (ii) and (iii), since \( b^*_i \in [0, b^{**} + 1] \) for all \( \delta > 0 \) sufficiently small, we have

\[
\lambda_\infty = \left\| \frac{f - \frac{1}{p}}{p} \right\|_{[0,b^{**}+1]}, \quad \lambda_\infty^X = \left\| \frac{f - \frac{1}{p}}{p} \right\|_{[0,b^{**}+1] \cap X}.
\]

Parts (ii) and (iii) then follow since the coefficients of \( p^X_\infty \) are bounded and \( p^X_\infty \) is bounded away from zero on \([0, b^{**} + 1]\) so that arguments similar to those in [3, pp. 84–88] can be applied.

We give the following example.

**Example.** Define \( f(x) = 1/(x + 1) + g(x) \), where

\[
\begin{align*}
g(x) &= \begin{cases}
3/16, & x = 0, \\
-3/16, & x = 1, \\
3/16, & x = 2, \\
-3/16, & x = 3, \\
0, & x \geq 4,
\end{cases}
\end{align*}
\]

and \( g(x) \) is linear in \([0, 1], [1, 2], [2, 3] \) and \([3, 4] \), so \( f(x) \in C_0^+ [0, \infty) \). Let \( n = 1 \). Then \( 1/p_\infty = 1/(x + 1) \), \( \lambda_\infty = 3/16, b^* = 2, b^{**} = 3 \).

(a) If \( X_i = [0, \infty) \setminus (3 - 1/2i, 3 + 1/2i) \), \( i \geq 1 \), we have \( 1/p^X_\infty = 1/(x + 1) \), \( b^*_i = 2 \), for all \( i \).

(b) If \( X_i = [0, \infty) \setminus [0, 1/2i) \), \( i \geq 1 \), we have \( 1/p^X_\infty = 1/(x + 1) \); \( b^*_i = 3 \), for all \( i \).

Using other choices of \( X_i \), we can make \( b^*_i < 2 \) or \( b^*_i > 3 \).

**7. Numerical Examples**

We show here some examples which were run on a CDC Cyber 172 in single precision (approximately 15 digits of accuracy). The program used
was a combined First Remes-differential correction algorithm program (see [5-7]) with minor changes in two subroutines to force $0 \leq q_j \leq 1$ instead of $-1 \leq q_j \leq 1$. The computed approximations of the form $p_1/(q_0 + q_1 x + \cdots + q_n x^n)$ were then normalized by dividing all coefficients by $p_1$.

**Example 1.** Let $f(x) = ((x + 1)/2) e^{(1-x)/2}$ and $n = 2$. This function has a maximum at $x = 1$, with $f(1) = 1$ (not the type of function that should be approximated by this sort of theory, in general). Let $X = \{0.01l: 0 \leq l \leq \infty \}$. Taking $b1 = 1$, the computed approximation on $[0, 1] \cap X$ (101 points) is

$$
\frac{1}{p_1(x)} = \frac{1}{1.09627448},
$$

with error norm $\lambda = 0.08781968$ and alternant $\{e_0, e_1\} \cup \{e_0, e_1\}$ (in particular, $p''_1(0) = 0$, $p'_1(0) = 0$, $f(0) = 1/p_1(0) = 0.08781968$, $f(1) = 1/p_1(1) = 0.08781968$). Now using Newton’s method to approximate a solution of $f(x) - 0.08781968 = 0$, we get $x = 9.105$ and $f(9.11) = 0.08763095$. Since $f$ is decreasing for $x \geq 1$, we take $b2 = 9.11$.

The computed approximation on $[0, 9.11] \cap X$ is

$$
\frac{1}{p_{9.11}(x)} = \frac{1}{1.06281016 + 0.04620946x^2},
$$

with error norm $\lambda_{9.11} = 0.11654117$ and alternant $\{e_0, e_1\} \cup \{e_0, e_1, e_{9.11}\}$. This is not best on $[0, \infty)$, since $f(12.75) = 0.11654154$. We observe that $p_{9.11}$ is not a constant, so searching for $b3$ (which will be the required $b$ here) with $1/p_{9.11}(b3) \leq \lambda_{9.11}$ (which can be done by solving $1/p_{9.11}(b3) = \lambda_{9.11}$, the solution is 12.755), we take $b3 = 12.76$. The computed approximation on $[0, 12.76] \cap X$ is

$$
\frac{1}{p_{12.76}(x)} = \frac{1}{1.06281009 + 0.04620952x^2},
$$

with error norm $\lambda_{12.76} = 0.11654123$ and alternant $\{e_0, e_1\} \cup \{e_0, e_{12}, e_{9.12}\}$; this is best on $X$. By comparison, if we remove the nonnegativity restriction on the denominator coefficients, the best computed approximation on $X$ is

$$(1.21587901 - 0.33317116x + 0.12629914x^2)^{-1}$$

with error norm 0.03835538, achieved at the extreme points 0.44+, 1.97-, 4.62+, 11.92-, where the sign indicates the sign of $1/p$.

**Example 2.** Let $f(x) = (\ln(x + 2))^{-1}$, $n = 2$. We first tried $X = \{0.01l: l \in \text{integer, } 0 \leq l < \infty \}$ as above; the computed approximation on $[0, 1] \cap X$ was $(0.69955039 + 0.41523483x)^{-1}$ with error norm 0.01320544 and alternant $\{e_0, e_{3.5}, e_1\}$. Solving $f(x) = 0.01320544$ we got
\[ x \approx e^{75.7} - 2 \approx 7.52 \times 10^{32} \] which is too large for practical computation. Replacing \( b1 \) by 100, and replacing \( X \) by \( X' = \{ l : l \text{ integer, } 0 \leq l < \infty \} \) to save computer time, our computed approximation on \([0, 100] \cap X'\) was \((6.78068253 + 0.17583824x)^{-1}\) with error norm 0.16176462 and alternant \( \{ e_0^2 \} \cup \{ e_0, e_2, e_{100} \} \). Solving \( f(x) = 0.16176462 \) yielded \( x \approx 481.9 \); the computed approximation on \([0, 482] \cap X'\) was \((0.78105035 + 0.17526786x)^{-1}\) with error norm 0.16236785 and alternant \( \{ e_0^3 \} \cup \{ e_0, e_2, e_{128} \} \). This is the best approximation on \( X' \). Having found an approximate location for \( b^* \), we refined the approximation using \([0, 130] \cap X (13,001 \text{ points}) \); the computed approximation after 22.4 second execution time was \((0.78109464 + 0.17557370x)^{-1}\) with error norm 0.16244044 and alternant \( \{ e_0^3 \} \cup \{ e_0, e_{1.84}, e_{128} \} \). This we verified to be best on \( X \) by directly checking the error on \([0, 469.59] \) and noting that \( f(x), 1/p(x) < 0.16244044 \) for \( x > 469.59 \). By comparison, removing the nonnegativity restriction on the denominator coefficients yielded \((0.75913982 + 0.21799463x - 0.00154261x^2)^{-1}\) as the best approximation on \([0, 130] \cap X \), with error norm 0.1254 1465 achieved at the extreme points \(0^+, 1.50^-, 44.82^+, 130^-\). This is not best on \( X^* \) due to pole near 144.72.

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**REFERENCES**

8. A. R. REDDY AND O. SHISHA, A characterization of entire functions \( \sum_{k=0}^{\infty} a_k x^k \) with all \( a_k \geq 0 \), *J. Approx. Theory* 15 (1975), 83–84.