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Tame Algebras with Sincere Directing Modules

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Let k be an algebraically closed field. Let Λ be a finite dimensional k-algebra.

Following [12], an indecomposable Λ-module X is said to be *directing* if it does not belong to a cycle $X \to X_1 \to \cdots \to X_{n-1} \to X$ of nonzero nonisomorphisms between indecomposable Λ-modules. Preprojective components as well as connecting components of tilted algebras are formed by directing modules. An indecomposable Λ-module X is *sincere* if $\operatorname{Hom}_\Lambda(P, X) \neq 0$ for every projective Λ-module $P \neq 0$. Representation-finite algebras with sincere directing modules have been extensively studied. They are tilted of tree type [1, 2, 12]; those algebras with more than 13 vertices were classified in [2] (see also [13]) and all others in [6]. In this work we consider the *tame* situation.

A tame algebra Λ is said to be *domestic* in at most n 1-parameters (we write $\mu_\Lambda \leqslant n$) if there are Λ-$k[T]$-bimodules $M_1, \ldots, M_n$ which are finitely generated free as right $k[T]$-modules and for every $d \in \mathbb{N}$, almost every indecomposable Λ-module of dimension d is of the form $M_i \otimes_{k[T]} L$ for some $k[T]$-module L. Our main result is the following.

THEOREM. *Let Λ be a tame algebra with a sincere directing module X. Then Λ is a domestic algebra in at most two 1-parameters, $\mu_\Lambda \leqslant 2$. If $\mu_\Lambda = 2$, then for every $k[T]$-module L we have*

$$\operatorname{Hom}_\Lambda(M_1 \otimes_{k[T]} L, X) \neq 0 \neq \operatorname{Hom}_\Lambda(X, M_2 \otimes_{k[T]} L)$$

(*up to reordering of the indices of M_1, M_2*).

The paper is organized as follows. In Section 1 we recall the main definitions and results needed in the work; in particular we explain the structure of the Auslander–Reiten quiver Γ_Λ of a tilted algebra Λ of tame representation type [8]. In Section 2 we show that the Euler form q_Λ *controls* the module category mod Λ. In Section 3, we give the proof of our

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theorem. The results of this work should be useful for the classification of all tame algebras with sincere directing modules. We hope to report soon about this.

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1. Basic Results

1.1. We say that Λ is *tame* provided for each $d \in \mathbf{N}$ there are a finite number of Λ-$k[T]$-bimodules $M_1, ..., M_n$ which are free of rank d as right $k[T]$-modules, and such that every indecomposable Λ-module of dimension d is isomorphic to $M_i \otimes_{k[T]} S$ for some i and some simple $k[T]$-module S.

For a tame algebra Λ and $d \in \mathbf{N}$, we define $\mu_\Lambda(d)$ as the smallest integer n such that there are Λ-$k[T]$-bimodules $M_1, ..., M_n$ which are free of rank d as right $k[T]$-modules and such that the set of modules

$$\{M_i \otimes_{k[T]} k[T]/(T-\lambda) : \lambda \in k,\ 1 \leqslant i \leqslant n\}$$

meets all but finitely many isomorphism classes of indecomposable Λ-modules of dimension d. Then Λ is said to be *domestic* in at most N 1-parameters if $\mu_\Lambda(d) \leqslant N$ for every d. We say that Λ is domestic in N 1-parameters (and write $\mu_\Lambda = N$) if it is domestic in at most N 1-parameters but not in at most $(N-1)$ 1-parameters. This definition is equivalent to at given in the Introduction; see [5].

Some examples of domestic algebras are discussed in [12].

1.2. For simplicity, we assume that Λ is a basic and connected algebra. We write $\Lambda = k[Q]/I$, where Q is the quiver of Λ and I is an admissible ideal of the path algebra $k[Q]$ (see [7]).

By mod Λ we denote the category of finite dimensional left Λ-modules. We view a Λ-module as a representation of Q satisfying the relations imposed by I.

By Q_0 (resp. Q_1) we denote the set of vertices (resp. arrows) of Q. For each vertex $x \in Q_0$ we denote by S_x the simple Λ-module associated with x. The projective cover P_x and injective hull I_x of S_x will be frequently used.

By Γ_Λ we denote the Auslander–Reiten quiver of Λ with translation τ ($=$Dtr, the dual of transpose operator). We do not distinguish between an indecomposable Λ-module, its isomorphism class, and the corresponding vertex in Γ_Λ.

1.3. We recall that A is said to be a *tilted algebra of type* Δ if there is a tilted module ${}_AT$ over the hereditary algebra $A=k[\Delta]$ such that $A=\mathrm{End}_A(T)$ (see [13]). The tilting module ${}_AT$ defines a torsion theory $(\mathscr{F}(T),\mathscr{G}(T))$ in $\mathrm{mod}\,A$ with $\mathscr{F}(T)=\{Y: \mathrm{Hom}_A(T,Y)=0\}$, $\mathscr{G}(T)=\{Y: \mathrm{Ext}^1_A(T,Y)=0\}$ and a torsion theory $(\mathscr{Y}(T),\mathscr{X}(T))$ in $\mathrm{mod}\,A$ with $\mathscr{Y}(T)=\{X: \mathrm{Tor}_1^A(T,X)=0\}$, $\mathscr{X}(T)=\{\mathscr{X}: T\otimes_A X=0\}$

Then the functor $F=\mathrm{Hom}_A(T,-)$ induces an equivalence between $\mathscr{G}(T)$ and $\mathscr{Y}(T)$ and $F'=\mathrm{Ext}^1_A(T,-)$ an equivalence between $\mathscr{F}(T)$ and $\mathscr{X}(T)$.

Let Σ be the set of indecomposable A-modules of the form $F(I_x)$, where I_x is an indecomposable injective A-module. Then Σ is a *slice* in Γ_A; that is, the following properties are satisfied: Σ is path closed in Γ_A; for every projective A-module P, there is some $S\in\Sigma$ with $\mathrm{Hom}_A(P,S)\neq 0$; if $S\in\Sigma$, then $\tau S\notin\Sigma$; if M and S are indecomposable, $f: M\to S$ is irreducible, and $S\in\Sigma$, then either $M\in\Sigma$ or M is not injective and $\tau^- M\in\Sigma$. Observe that the points of Σ induce a full subquiver of Γ_A isomorphic to Δ^{op}. The component $\mathscr{C}_A$ of Γ_A containing Σ is called the *connecting* component (see [14]).

The following is a central fact for our work.

PROPOSITION [12, Addendum to 4.1]. *If A has a sincere directing module X, then A is a tilted algebra. Moreover, the sets of modules*

$$\Sigma(\to X)=\{Y\in\Gamma_A: \text{there is a path from } Y \text{ to } X \text{ in } \Gamma_A \text{ and every path from } Y \text{ to } X \text{ is sectional}\}.$$

$$\Sigma(X\to\ )=\{Y\in\Gamma_A: \text{there is a path from } X \text{ to } Y \text{ in } \Gamma_A \text{ and every path from } X \text{ to } Y \text{ is sectional}\}$$

are slices in Γ_A.

1.4. If X is a directing A-module, its support $\mathrm{supp}\,X$ is convex (=path closed) in A [3]. Hence $A(X)=A/\langle s\in Q_0: X(s)=0\rangle$ is a convex subalgebra of A with a sincere directing module (namely X). This reduces the study of directing modules to that of sincere directing modules.

1.5. Let A be a tilted algebra of type Δ. Assume that A is tame but not representation-finite. Then Δ is not of Dynkin type [13].

If $A=k[\Delta]$ is tame (equivalently Δ is of Euclidean type), then A is a domestic tubular algebra [13]. In particular, A is domestic in one 1-parameter. The module category $\mathrm{mod}\,A$ and the Auslander–Reiten quiver Γ_A are completely described in [13].

Assume $A=k[\Delta]$ is *wild* (for the purposes of this work, wild means not tame; however, see [4]). Let $T=T_1\oplus\cdots\oplus T_n$ be an indecomposable decomposition of a tilting A-module such that $A = \mathrm{End}_A(T)$. Following [8], we consider the sets

$$U = \{1 \leqslant i \leqslant n : \mathrm{Ext}_A^1(T_i, X) \neq 0 \text{ for only finitely many indecomposables } X \in \mathscr{F}(T)\}$$

$$V = \{1 \leqslant i \leqslant n : \mathrm{Hom}_A(T_i, Y) \neq 0 \text{ for only finitely many indecomposables } Y \in \mathscr{G}(T)\}.$$

Let $T_\infty = \bigoplus_{i \notin U} T_i$ and ${}_\infty T = \bigoplus_{j \notin V} T_j$. Then $A_\infty = \mathrm{End}_A(T_\infty)$ is called the *right end* algebra of A and ${}_\infty A = \mathrm{End}_A({}_\infty T)$ the *left end* algebra. The following facts were proved in [8].

(a) A is an iterated one-point extension (resp. coextension) of ${}_\infty A$ (resp. A_∞).

(b) There are a factor algebra ${}_\infty A$ of A and a tilting module ${}_\infty \hat{T}$ of ${}_\infty A$ without preinjective direct summands such that $\mathrm{End}_{{}_\infty A}({}_\infty \hat{T}) = {}_\infty A$. Moreover, ${}_\infty A = A_1 \times \cdots \times A_r$ with A_i connected, ${}_\infty \hat{T} = \hat{T}_1 \oplus \cdots \oplus \hat{T}_r$, where $\hat{T}_i$ is an A_i-tilting module without preinjective direct summands and ${}_\infty A = A_1 \times \cdots \times A_r$ for $A_i = \mathrm{End}_{A_i}(\hat{T}_i)$. If we define $F_i = \mathrm{Hom}_{A_i}(\hat{T}_i, -)$: $\mathrm{mod}\, A_i \to \mathrm{mod}\, A_i$, then $\mathrm{Hom}_{{}_\infty A}({}_\infty \hat{T}, -) = (F_i)_{1 \leqslant i \leqslant n}$.

(c) Dually, there are a factor algebra $A_\infty = A'_1 \times \cdots \times A'_s$ of A with A'_i connected, and a tilting module $\hat{T}_\infty = \tilde{T}_1 \oplus \cdots \oplus \tilde{T}_s$ of A_∞ such that $A_\infty = A'_1 \times \cdots \times A'_s$ for $A'_i = \mathrm{End}_{A'_i}(\tilde{T}_i)$. If $F'_i = \mathrm{Ext}^1_{A'_i}(\tilde{T}_i, -)$: $\mathrm{mod}\, A'_i \to \mathrm{mod}\, A'_i$, then $\mathrm{Ext}^1_{A_\infty}(\hat{T}_\infty, -) = (F'_i)_{1 \leqslant i \leqslant s}$.

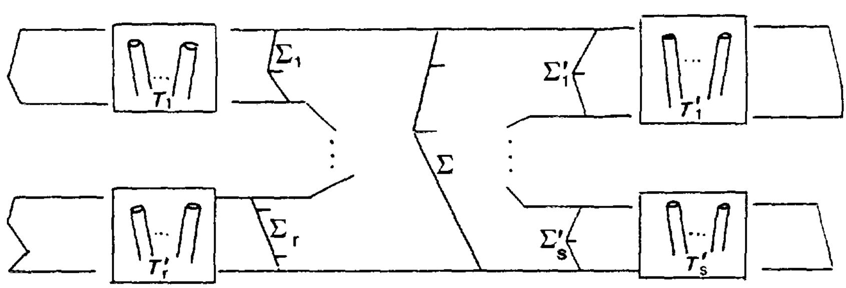
(d) Since A is tame and not representation-finite, then:

— ${}_\infty A$ or A_∞ is tame and both are not wild.

— Each A_i is representation-finite or a domestic tubular algebra.

— Each A'_i is representation-finite or a domestic cotubular algebra.

(e) The Auslander–Reiten quiver Γ_A has the shape



where Σ is the slice in Γ_A and Σ_i (resp. Σ_i') is a slice in Γ_{A_i} (resp. $\Gamma_{A_i'}$). The tubular familes $\mathscr{T}_i$ or $\mathscr{T}_i'$ may be empty, but at least one of them is not.

1.6. As a consequence of (1.5) we get:

PROPOSITION. *Let A be a tame algebra with a sincere directing module. Then*

(a) *A is a domestic algebra.*

(b) *If A is domestic in one* 1*-parameter, then A is a finite enlargement of a domestic tubular algebra or a finite coenlargement of a domestic cotubular algebra.*

(c) *If A is a domestic in at least two* 1*-parameters, then there exist only finitely many indecomposable sincere modules.*

Proof. By (1.3), A is tilted algebra of type Δ. Let $A = k[\Delta]$.

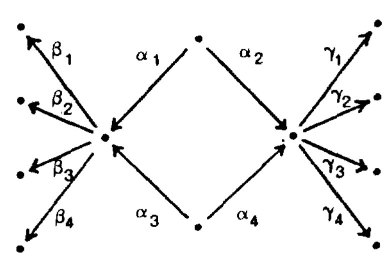
(a) With the notation of (1.5), A is domestic in at most $r+s$ 1-parameters.

(b) Suppose A_1 is not representation-finite. Then mod A_1 is cofinite in mod A.

(c) If A is domestic in at least two 1-parameters, then there are only a finite number of slices in the connecting component $\mathscr{C}_A$. But for every sincere module X, $\Sigma(X \to)$ is a slice in Γ_A (1.3). Hence the result follows. ∎

1.7. We want to remark that most of the description of Γ_A given in (1.5) also follows from the considerations on directing modules in [15].

1.8. EXAMPLES. (a) There are tame algebras which are tilted and domestic in at least three 1-parameters. We borrow the example from [8]: let A be the algebra with quiver



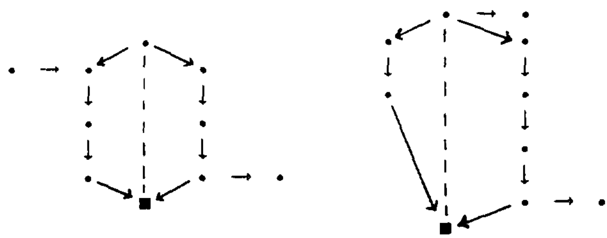
and relations

$$\alpha_1\beta_i = 0 = \alpha_2\gamma_i \qquad \text{for} \quad 1 \leqslant i \leqslant 3$$

$$\alpha_3\beta_j = 0 = \alpha_4\gamma_j \qquad \text{for} \quad 2 \leqslant j \leqslant 4.$$

Then ${}_{\infty}A$ is hereditary of type Δ, where Δ is the union of two copies of $\tilde{\mathbf{D}}_4$ and A_{∞} is a tilted algebra of type $\tilde{\mathbf{A}}_7$.

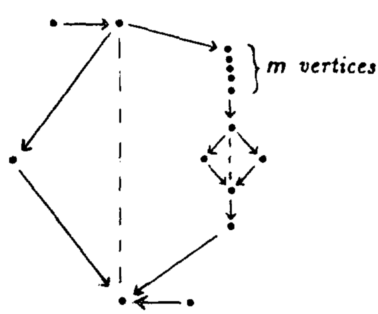
(b) The *one-relation tame algebras* in [12] denoted by $G\Delta - \Delta'$ all have a sincere directing module. The letter G stands for "gluing." Two examples are as follows:



The unique relation is the commutativity relation between the vertices joined by a dotted line. In the first case $\Delta = \tilde{\mathbf{E}}_8$, $\Delta' = \tilde{\mathbf{E}}_8$; in the second $\Delta = \tilde{\mathbf{E}}_7$, $\Delta' = \tilde{\mathbf{E}}_8$. In general, Λ is said to be the *gluing* of A and B if A and B are convex subalgebras of Λ, mod $A \cup$ mod B is a cofinite subcategory of mod Λ, and there exist a preprojective B-module N and a preinjective A-module M such that $\mathrm{Hom}_{\Lambda}(M, N) \neq 0$. Then the Auslander–Reiten quiver Γ_{Λ} is obtained from Γ_A and Γ_B by joining together the preinjective component of Γ_A with the preprojective component of Γ_B (see [12]). If A is domestic in r 1-parameters and B is domestic in s 1-parameters, then Λ is domestic in $r+s$ 1-parameters.

The two examples above are domestic in two 1-parameters.

(c) The family of algebras Λ_m given by the quivers



with the commutativity relations indicated by dotted arrows are domestic in two 1-parameters with sincere directing modules. In fact Λ_m is also a gluing of two hereditary algebras of type $\tilde{\mathbf{D}}_n$.

2. The Euler Form of a Tame Tilted Algebra

2.1. Let Λ be a tame algebra which is tilted of type Δ. Let $A = k[\Delta]$. Let $K_0(\Lambda)$ be the Grothendieck group of Λ; then $K_0(\Lambda) = \mathbf{Z}^{Q_0}$. A module $X \in \operatorname{mod} \Lambda$ has a *dimension vector* $\underline{\dim}\, X = (\dim_k X(s))_{s \in Q_0} \in K_0(\Lambda)$.

The bilinear form $\langle -, - \rangle$ on $K_0(\Lambda)$ is defined in such a way that

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle = \dim \operatorname{Hom}_\Lambda(X, Y) - \dim \operatorname{Ext}^1_\Lambda(X, Y) + \dim \operatorname{Ext}^2_\Lambda(X, Y).$$

The quadratic form $q_\Lambda(z) = \langle z, z \rangle$ is called the *Euler form* of Λ. We freely use the notation of [13].

The next proposition generalizes results in [3, 12]. Apart from its own importance, we use it in the proof of our theorem.

Proposition. *Let Λ be a tame algebra tilted of type Δ. Then the Euler form q_Λ is weakly non-negative and controls the module category* $\operatorname{mod} \Lambda$; *that is, q_Λ satisfies the following properties*:

(a) *For any indecomposable $X \in \operatorname{mod} \Lambda$, $q_\Lambda(\underline{\dim}\, X) \in \{0, 1\}$.*

(b) *For any connected vector $z \in \mathbf{N}^{Q_0}$ with $q_\Lambda(z) = 1$, there is precisely one module X up to isomorphism with $\underline{\dim}\, X = z$.*

(c) *For any connected vector $z \in \mathbf{N}^{Q_0}$ with $q_\Lambda(z) = 0$, there is an infinite family $(X_\lambda)_{\lambda \in k}$ of indecomposable modules with $X_\lambda \not\cong X_\mu$ if $\lambda \neq \mu$ and $\underline{\dim}\, X_\lambda = z$ for every λ.*

By [8, (6)] or [9], q_Λ is *weakly non-negative* (that is, $q_\Lambda(z) \geqslant 0$ for $z \in \mathbf{N}^{Q_0}$). Part (a) follows from the description of Γ_Λ given in (1.5). The proof of (c) and (b) is given in Sections (2.4) and (2.5) below.

2.2. If Δ is of Euclidean type. this result is just [12, (4.9)]. We can assume that Δ is of wild type. We keep our notation as in (1.5). We assume that $A_1, ..., A_p$ (resp. $A'_1, ..., A'_q$) are not of Dynkin type and $A_{p+1}, ..., A_r$ (resp. $A'_{q+1}, ..., A'_s$) are of Dynkin type. Hence $\mathscr{T}_1, ..., \mathscr{T}_p$ (resp. $\mathscr{T}'_1, ..., \mathscr{T}'_q$) are *tubular families* [12, (3.1)]. Let Z_i (resp. Z'_i) be a module in the mouth of a homogeneous tube of $\mathscr{T}_i$ (resp. $\mathscr{T}'_i$); we set $z_i = \underline{\dim}\, Z_i$ (resp. $z'_i = \underline{\dim}\, Z'_i$.) Therefore, the algebra Λ is domestic in $p+q$ 1-parameters.

LEMMA. (i) *If* $X \in \mathcal{T}_i$, *then* X *has projective dimension* $\operatorname{p\,dim} X \leqslant 1$.

(ii) *If* $Y \in \mathcal{T}'_i$, *then* Y *has injective dimension* $\operatorname{i\,dim} Y \leqslant 1$.

(iii) $\operatorname{Hom}_A(Z_i, Z_j) = 0 = \operatorname{Hom}_A(Z'_i, Z'_j)$, *for* $i \neq j$.

(iv) $\operatorname{Hom}_A(Z_i, Z'_j) = 0$ *if and only if* $\operatorname{supp} Z_i \cap \operatorname{supp} Z'_j = \varnothing$.

Proof. (i) If $\operatorname{p\,dim} X > 1$, then there is a morphism $0 \neq f\colon I_t \to \tau X \in \mathcal{T}_i$, which is impossible since all injective modules lie to the right of the connecting component.

(ii) is dual of (i); (iii) is obvious.

(iv) Assume $x \in \operatorname{supp} Z_i \cap \operatorname{supp} Z'_j$. Then $\operatorname{Hom}_A(Z_i, I_x) \neq 0$ and $\operatorname{Hom}_A(Z'_j, I_x) \neq 0$. Since $\mathcal{T}'_j$ is separating, then $\operatorname{Hom}_A(Z_i, Z'_j) \neq 0$. ▌

2.3. Let I_0 be a subset of $\bigcup_{x, y \in Q_0} I(x, y)$ which generates the ideal I and suppose $\operatorname{card}(I_0)$ is minimal with this property. Set $r(x, y) = \operatorname{card}(I_0 \cap I(x, y))$. By [2], we have

$$\langle z, w \rangle = \sum_{x \in Q_0} z(x)\, w(x) - \sum_{(x \mapsto y) \in Q_1} z(x)\, w(y) + \sum_{x, y \in Q_0} r(x, y)\, z(x)\, w(y).$$

We denote by $(-, -)$ the *symmetrization* of $\langle -, - \rangle$. That is, $(z, w) = \langle z, w \rangle + \langle w, z \rangle$.

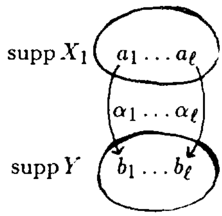
2.4. *Proof of* (2.1c). Let $0 \neq v \in \mathbb{N}^{Q_0}$ be a connected vector such that $q_A(v) = 0$. We use induction on $|v| = \sum_x v(x)$ to show that $v = mz_i$ $(1 \leqslant i \leqslant p)$ or $v = mz'_j$ $(1 \leqslant j \leqslant q)$, for some $m \in \mathbb{N}$. Let X be a A-module with $\underline{\dim}\, X = v$ and such that $\dim_k \operatorname{End}_A(X)$ is minimal. Consider the indecomposable decomposition $X = \bigoplus_{i=1}^{t} X_i$, then $\operatorname{Ext}^1_A(X_i, X_j) = 0$ for every $i \neq j$ [12, (2.3)]. Let $w_i = \underline{\dim}\, X_i$. Since q_A is weakly non-negative,

$$0 \leqslant q_A(w_i) \qquad \text{and} \qquad 0 \leqslant \langle w_i, w_j \rangle \quad \text{for every } i, j.$$

Hence $0 = \langle w_i, w_j \rangle$ for every i, j. We may assume that the vector $y := \sum_{i=2}^{t} w_i$ is connected. Since $q_A(y) = 0$, by the induction hypothesis, y is of the form mz_i or mz'_j for some $m \in \mathbb{N}$. Moreover, there is an indecomposable module Y in a homogeneous tube with $\underline{\dim}\, Y = y$. Clearly, we may choose X_1 to lie also in a homogeneous tube.

By (2.2), either $\operatorname{supp} X_1 = \operatorname{supp} Y$ or $\operatorname{supp} X_1 \cap \operatorname{supp} Y = \varnothing$. The result is proved if we show that the first situation happens. Assume that $\operatorname{supp} X_1 \cap \operatorname{supp} Y = \varnothing$.

Since v is connected, we may assume that there are arrows $\alpha_1, \ldots, \alpha_l$ from $\operatorname{supp} X_1$ to $\operatorname{supp} Y$. Hence X_1 belongs to some tubular family $\mathcal{T}_i$ and Y to some family $\mathscr{T}'_j$. In particular, there are no arrows from supp Y to supp X_1:



Let $e_x = \underline{\dim}\, S_x$. For $x = a_i$, $(e_x, y) = 0$ [if $(e_x, y) < 0$, then $q_A(2y + e_x) < 0$; contradiction. Since $0 = (w_1, y) = -\sum_x w_1(x)(e_x, y)$, then $(e_x, y) = 0$ for $x \in \operatorname{supp} X_1$]. From (2.3) we get

$$0 = (w_1, y) = -\sum_{i=1}^{l} w_1(a_i)\, y(b_i) + \sum_{i,t} r(a_i, t)\, w_1(a_i)\, y(t)$$

$$+ \sum_{a_i \neq x} \sum_t r(x, t)\, w_1(x)\, y(t)$$

$$0 = (e_{a_i}, y) = -\sum_{(a_i \to b_j) \in Q_1} y(b_j) + \sum_t r(a_i, t)\, y(t) \qquad \text{for} \quad i = 1, \dots, l.$$

Therefore, $\sum_{a_i \neq x} r(x, b_j)\, w_1(x) = 0$; that is, there is no relation joining supp $X_1 \setminus \{a_1, \dots, a_l\}$ with b_j, for any $j = 1, \dots, l$. Since there are no oriented cycles in supp X_1 we may choose a_1 to be a source in the set $\{a_1, \dots, a_l\}$. Hence, the convex subalgebra of A' of A with support supp $X_1 \cup \{b_1\}$ is wild. Indeed, the quotient $A'/\langle \alpha_i : i \neq 1, b_i = b_1 \rangle$ is a coextension of a tame concealed algebra by a preinjective module; therefore it is wild [11]. This contradiction proves our assertion. ∎

2.5. *Proof of* (2.1b). Let $0 \neq v \in \mathbb{N}^{Q_0}$ be a connected vector such that $q_A(v) = 1$.

Uniqueness. If X and Y are indecomposable A-modules with $\underline{\dim}\, X = v = \underline{\dim}\, Y$, then either X belongs to the connecting component and we apply [12, (2.4(8))] or X belongs to a domestic tubular (or cotubular) algebra and we apply [12, (4.9)].

Existence. The proof goes by induction on $|v|$. Let $X \in \operatorname{mod} A$ be a module with $\underline{\dim}\, X = v$ and such that $\dim_k \operatorname{End}_A(X)$ is minimal. Let $X = \bigoplus_{i=1}^{t} X_i$ be an indecomposable decomposition. As in (2.4), we have $\operatorname{Ext}^1_A(X_i, X_j) = 0$ for $i \neq j$. Set $w_i = \underline{\dim}\, X_i$. Then either

$$q_A(w_i) = 0 \qquad \text{for all} \quad i \quad \text{and} \quad \langle w_i, w_j \rangle \neq 0 \quad \text{for a unique pair } i, j$$

or

$$q_A(w_1)=1, \qquad q_A(w_i)=0 \quad \text{for} \quad 2 \leqslant i \leqslant t$$

and

$$\langle w_i, w_j \rangle = 0 \qquad \text{for all} \quad i \neq j.$$

We distinguish these cases:

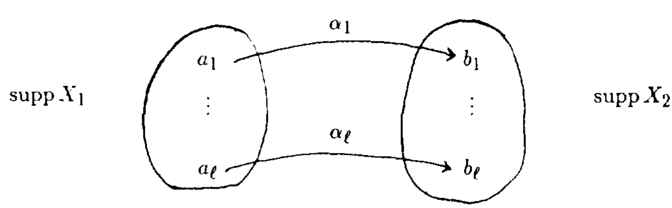
(i) Assume $q_A(w_i)=0$ for $1 \leqslant i \leqslant t$, $\langle w_1, w_2 \rangle = 1$ and $\langle w_i, w_j \rangle = 0$ for $(i, j) \neq (1, 2)$. Without lost of generality we may assume that the vector $w = \sum_{i=2}^{t} w_i$ is connected. Then $q_A(w)=0$ and by (2.4), there exists an indecomposable module Y in a homogeneous tube with $\underline{\dim}\, Y = w$. Clearly, we may choose X_1 to lie also in a homogeneous tube. Since $1 = \langle w_1, w \rangle = \dim_k \operatorname{Hom}_A(X_1, Y)$, then X_1 belongs to some family $\mathscr{T}_i$ and Y to some family $\mathscr{T}'_j$.

Since $1 = (w_1, w) = \sum_x w_1(x)(e_x, w)$ there is a vertex y with $1 = (e_y, w)$. Then $w(y)=0$. For the vector $z = w + (w_1 - e_y) = v - e_y$, with $q_A(z)=1$, there is an indecomposable module Z with $\dim Z = z$ and trivial endomorphism ring. Since $(z, e_y) = -1$, there is an indecomposable extension of Z and S_y. This proves the claim.

(ii) Assume that $q_A(w_1)=1$, $q_A(w_i)=0$ for $2 \leqslant i \leqslant t$ and $\langle w_i, w_j \rangle = 0$ for all $i \neq j$. Assume that $t \geqslant 2$. Using (2.4) and the induction hypothesis we may suppose that $t=2$ and that X_2 is a module in a homogeneous tube (in the tubular family $\mathscr{T}$).

Assume that $\operatorname{supp} X_1 \cap \operatorname{supp} X_2 \neq \varnothing$. By the argument in (2.2iv), we get that X_1 and X_2 belong to the same tubular family. Hence X_1 and X_2 are modules over a domestic tubular (or cotubular) algebra. By [12, (4.9)], there exists an indecomposable module X with $\underline{\dim}\, X = v$.

Therefore we may assume that $\operatorname{supp} X_1 \cap \operatorname{supp} X_2 = \varnothing$. Then there are arrows connecting $\operatorname{supp} X_1$ and $\operatorname{supp} X_2$:



Let B be te tame concealed algebra with $\operatorname{supp} X_2$.

Our argument in the proof of (2.4) shows that there are no relations joining points $x \in \operatorname{supp} X_1 \setminus \{a_1, ..., a_l\}$ and $y \in \operatorname{supp} X_2$. Suppose the vertices a_i are ordered in such a way that a_i is a sink in the set $\{a_i, a_{i+1}, ..., a_l\}$. Consider R_1, the restriction of $\operatorname{rad} P_{a_1}$ to the algebra B. Since $\langle \underline{\dim}\, R_1, \underline{\dim}\, X_2 \rangle = -\langle e_{a_1}, \underline{\dim}\, X_2 \rangle = -(e_{a_1}, \underline{\dim}\, X_2) = 0$, then R_1 is regular in B. Then P_{a_1} belongs to the tubular family (with ray insertions) $\mathscr{T}$.

Suppose there is an arrow $a_2 \to a_1$. Let R_2 be the restriction of $\operatorname{rad} P_{a_2}$ to the domestic algebra B_1 with vertices $\operatorname{supp} X_2 \cup \{a_1\}$. As above, R_2 belongs to the tubular family of Γ_{B_1}. Hence the one point extension $B_1[R_2]$ is a domestic tubular extension of B. In particular, there are no arrows from a_2 to $\operatorname{supp} X_2$, a contradiction. Repeating this argument, we get that there are no arrows between a_i and a_j, $i \neq j$.

Therefore there are no relations starting at $x \in \operatorname{supp} X_1 \setminus \{a_i\}$, ending at $y \in \operatorname{supp} X_2$ and using a path which passes through a_i, $i = 1, ..., l$.

First we observe that there is an indecomposable module X_2' with $\underline{\dim}\, X_2' = \underline{\dim}\, X_2 + e_{a_1}$. Moreover, $\underline{\dim}\, X_1(a_1) = 1$ [indeed, $(e_{b_1}, \underline{\dim}\, X_1) < 0$. If $(e_{b_1}, \underline{\dim}\, X_1) < -1$, then $q_A(\underline{\dim}\, X_1 + \underline{\dim}\, X_2 + e_{b_1}) = 2 + (e_{b_1}, \underline{\dim}\, X_1) \leqslant 0$, contradicting (2.4). Hence, $-1 = (e_{b_1}, \underline{\dim}\, X_1) = \langle \underline{\dim}\, X_1, e_{b_1} \rangle = -\dim X_1(a_1)$]. Therefore, there is no difficulty defining Y indecomposable with $\underline{\dim}\, Y = \underline{\dim}\, X_1 + \underline{\dim}\, X_2 = v$. This proves our result. ∎

3. The Main Theorem

3.1. Let A be a tame algebra with a sincere directing module. By (1.3), A is tilted of type Δ. If Δ is Euclidean, A is domestic in one 1-parameter (1.6). *Assume* A is domestic in at least two 1-parameters (in particular, Δ is wild). By (1.6), there exist only finitely many indecomposable sincere A-modules. By (2.1), there is a bijection $X \mapsto \underline{\dim}\, X$ between the indecomposable sincere modules and the sincere positive roots of q_A. Therefore there exists a *maximal sincere root* v of q_A.

Lemma. *Let v be a maximal sincere root of q_A. Then*

(i) $(v, e_x) \geqslant 0$ *for every* $x \in Q_0$.

(ii) *There are at most two vertices x with $(v, e_x) > 0$ (these are called exceptional indices of v [13]). Either there is a unique exceptional index a and $v(a) = 2$ or there are exceptional indices $a \neq b$ and $v(a) = 1 = v(b)$.*

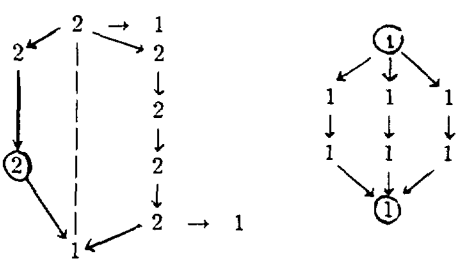
Proof. To repeat the proof in [12, (1.1)], we first need to show that for any indecomposable sincere module Y, its dimension vector $u = \underline{\dim}\, Y$ satisfies that $-1 \leqslant (u, e_x) \leqslant 1$ for every $x \in Q_0$. Indeed, if $(u, e_x) < 0$ then

$q_A(u+e_x)\leqslant 1$. By (2.1), there are no sincere null roots of q_A, therefore $(u, e_x)=-1$. Now, assume that $(u, e_x)>1$; then $q_A(u-e_x)\leqslant 0$ and $u-e_x$ is a positive vector. Hence $(u, e_x)=2$, $u-e_x$ is not sincere, and $u-e_x=\sum_{i=1}^{t} w_i$, where w_i is a null root of q_A and supp w_i, supp w_j are not connected by arrows, for $i\neq j$. Let Y_j be an indecomposable A-module with $\underline{\dim}\, Y_j=w_j$. Since Y is sincere (and hence directing), either $\langle\underline{\dim}\, Y_j, \underline{\dim}\, Y\rangle>0$ or $\langle\underline{\dim}\, Y, \underline{\dim}\, Y_j\rangle>0$. Suppose $\langle\underline{\dim}\, Y_j, \underline{\dim}\, Y\rangle>0$. Since $\langle w_j, u-e_x\rangle=0$, also $\langle w_j, e_x\rangle>0$. As supp Y_j is convex in A, $\langle e_x, w_j\rangle=0$. Thus $(w_j, e_x)>0$. But then $0<\sum_{i=1}^{t}(w_i, e_x)=(u-e_x, e_x)=0$, a contradiction.

(i) Assume $(v, e_x)=-1$, then $q_A(v+e_x)=1$ contradicting the maximality of v.

(ii) Since $2=(v, v)=\sum_x (v, e_x)$, the result follows from the first remark and (i). ∎

In two of the *examples* of (1.8) we show a maximal sincere root. The exceptional indices are encircled;



3.2. THEOREM. *Let A be a tame algebra with a sincere directing module. Assume that A is domestic in at least two 1-parameters. Then A is domestic in two 1-parameters and it is the gluing of two tilted algebras which are domestic in one 1-parameter.*

Proof. Let X be a sincere directing A-module such that $v=\underline{\dim}\, X$ is a maximal root of q_A. Assume A is tilted of type Δ; then Δ is of wild type and we keep the notation as in (2.2).

(i) Suppose v has a unique exceptional index a. Consider one of the null roots z_i; then $(v, z_i)>0$ (otherwise, $(v, z_i)=0$ contradicts the maximality of v or $(v, z_i)<0$ contradicts (2.1b)). Then $z_i(a)=\sum_x z_i(x)(v, e_x)=(v, z_i)>0$ and a is contained in supp z_i. Since for $1\leqslant i, j\leqslant p$, $i\neq j$, the supports of z_i and z_j do not intersect, then $p\leqslant 1$. Similarly, $q\leqslant 1$. Hence A is domestic in $p+q=2$ 1-parameters. In particular, $p=1=q$.

Let x be a source in supp z'_1 such that $x \notin \operatorname{supp} A_1$. Set $B = A/(x)$. Then B is a tilted algebra [8, 4.1] which is domestic in one 1-parameter. Dually, let y be a sink in supp z_1 such that $u \notin \operatorname{supp} A'_1$. Then $B' = A/(y)$ is a tilted algebra which is domestic in one 1-parameter. Clearly, A is the gluing of B and B'.

(ii) Suppose $a \neq b$ are the exceptional indices of v. By [12, (6.5)], a and b are sink or source vertices in Q (for the proof apply (3.1) instead of representation-finiteness). Suppose that a is a source in Q. Let $w_1 = z_1, \ldots, w_p = z_p, w_{p+1} = z'_1, \ldots, w_{p+q} = z'_q$. Then for each $i = 1, \ldots, p+q$,

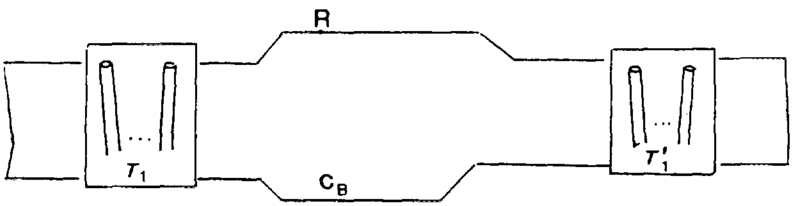
$$w_i(a) + w_i(b) = (w_i, v) > 0.$$

Then $p \leqslant 2$ and $q \leqslant 2$. Suppose $p+q=4$. Then up to reordering of the roots we have

$$w_1(a) \neq 0 = w_1(b), \qquad w_2(a) = 0 \neq w_2(b)$$

$$w_3(a) \neq 0 = w_3(b'), \qquad w_4(b) = 0 \neq w_4(b).$$

The algebra $B = A/(a)$ is a tilted algebra domestic in two 1-parameters. We write $A = B[R]$ as a one-point extension with $R = \operatorname{rad} P_a$. We do not lose generality, assuming that R is indecomposable. From the description of Γ_B in (1.5), R belongs either to the tubular family $\mathcal{T}_1$ or the connecting component $\mathcal{C}_B$ (observe in particular that ${}_\infty B$ and B_∞ are both not representation-finite):

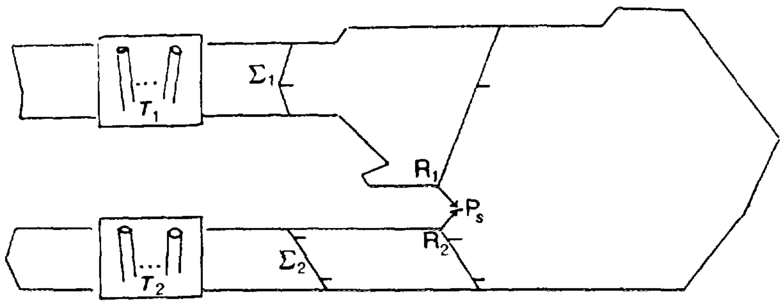


Consider indecomposable A-modules Y_i such that $\underline{\dim}\, Y_i = w_i$. Then $\operatorname{Hom}_A(R, Y_i) \neq 0$ for $i = 1, 3$. This can only happen if supp $Y_1 \cap$ supp $Y_3 = \varnothing$, which contradicts $w_1(a) \neq 0 \neq w_3(a)$.

Suppose now $p = 1$ and $q = 2$. We may assume that $w_2(a) \neq 0 \neq w_3(a)$, then $w_1(a) = 0 \neq w_1(b)$. Hence $B = A/(a)$ is a tilted algebra in one 1-parameters. Write again $A = B[R]$. We obtain a contradiction as above. Hence $p+q=2$.

Consider the case $p=2$, $q=0$. Let Σ_1, Σ_2 be the slices corresponding to the algebras A_1, A_2. It is not hard to show that there is a projective $P_s \in \mathcal{C}_A$

such that $R = \text{rad } P_s = R_1 \oplus R_2 \oplus R'$ with R_1, R_2 indecomposable modules and $\Sigma(R_i \rightarrow\ )$ containing a subgraph of type Σ_i, $i = 1, 2$:



Passing to a convenient quotient of A, we may assume that s is a source in Q. Then $A = B[R]$ with $B = A/(s)$ and the *subspace category* $\mathscr{U}(\text{Hom}_B(R, \text{mod } B))$ is representation-finite. Hence $\Sigma(R_1 \rightarrow\ )$, $\Sigma(R_2 \rightarrow\ )$ are disjoint trees. But this implies that $\text{Hom}_B(R, \text{mod } B)$ contains a subposet of type $(1, 1, 1, 1)$, which is representation-infinite. Contradiction.

The case $p = 0$, $q = 2$ is similar. Thus $p = 1 = q$ in case (ii). The last assertion follows as in case (i). ∎

3.3. For the problem of construction of the tame algebras with a sincere directing module [11], we have the following result, similar to [13, (6.5)].

PROPOSITION. *Let A be a domestic algebra in two 1-parameters with a sincere directing module. Then there exists a sink or source vertex a such that the quotient $\bar{A} = A/(a)$ also has a sincere directing module. Moreover, a may be chosen in such a way that $\bar{A}$ is domestic in one 1-parameter.*

Proof. By (1.6) and (2.1), there is a minimal sincere root u of q_A. Let Y be an indecomposable module with $\underline{\dim}\, Y = u$. By (3.1), there exists a vertex $a \in Q_0$ with $(u, e_a) = 1 = u(a)$. As in (3.2ii)), a is a sink or a source in Q. Consider the quotient $\bar{A} = A/(a)$. Since $w = u - e_a$ satisfies $q_A(w) = 1$, there is an indecomposable $\bar{A}$-module W with $\underline{\dim}\, W = w$. We claim that W is directing. Otherwise there is a cycle $W \xrightarrow{\alpha_0} W_1 \longrightarrow \cdots \longrightarrow W_t \xrightarrow{\alpha_t} W$ of non-zero non-isomorphisms between indecomposable $\bar{A}$-modules. Suppose that a is a source in Q and let $R = \text{rad } P_a$. Then the functor $\text{mod } \bar{A} \rightarrow \text{mod } A$, $M \mapsto \tilde{M} = (M, \text{Hom}_{\bar{A}}(R, M), \text{id})$ is an embedding. Hence the cycle $Y = \tilde{W} \xrightarrow{\tilde{\alpha}_0} \tilde{W}_1 \xrightarrow{\tilde{\alpha}_1} \cdots \longrightarrow \tilde{W}_t \xrightarrow{\tilde{\alpha}_t} \tilde{W} = Y$ between indecomposable A-modules yields a contradiction.

Let z be a null root of q_A and Z indecomposable with $\underline{\dim}\, Z = z$. Since Y is a sincere directing module, then

$$0 < (z, u) = \sum_x z(x)(u, e_x).$$

There exists some $a \in Q_0$ with $z(a) \neq 0$ and $(u, e_a) = 1$. Then $u(a) = 1$ and the quotient $\bar{A} = A/(a)$ is domestic in at most one 1-parameter. Then (3.2) implies that $\bar{A}$ is domestic one 1-parameter. ∎

3.4. *Remark.* The proof of Theorem (3.2) could be given using only "vector space category" arguments (in fact, we had used such arguments in the proof; a slight generalization would provide a complete proof of the result). Nevertheless, we find that the proof using "quadratic form" arguments is interesting. The additional information in (3.1), (3.3), and hence the classification problem [11] strongly depends on the use of the Euler form.

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