On the thermodynamic stability of the generalized Chaplygin gas

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Abstract

The main purpose of this Letter is to discuss the temperature behavior and the thermodynamic stability of an exotic fluid known as generalized Chaplygin gas considering only general thermodynamics. This fluid is considered a perfect fluid which obeys an adiabatic equation of state like \( P = -A/\rho^\alpha \), where \( P \) and \( \rho \) are respectively the pressure and energy density; the parameter \( A \) is a positive universal constant and \( \alpha > 0 \). It is remarked that if the energy density of the fluid is a function of volume only, the temperature of the fluid remains zero at any pressure or volume, violating the third law of thermodynamics. We have determined a scenario where its thermal equation of state depends on temperature only and the fluid presents thermodynamic stability during any expansion process. Such a scenario also reveals that the fluid cools down through the expansion without facing any critical point or phase transition.

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1. Introduction

The generalized Chaplygin gas is an exotic fluid that has been explored by cosmologists recently. Observations of redshift and luminosity-distance relations of type-Ia supernovae indicate \([1,2]\) an universe in accelerated expansion and this is in disagreement with the standard picture of a matter dominated universe. Among other possibilities, different authors sustain that these observations can be explained theoretically by postulating that a certain fluid with negative pressure dominates the present epoch of our universe. Such an exotic fluid, named generalized Chaplygin gas, is considered a perfect fluid which obeys the following adiabatic equation of state (EoS):

\[
P = \frac{-A}{\rho^\alpha}.
\]

where \( \rho \) corresponds to energy density of the fluid

\[
\rho = \frac{U}{V}.
\]

\( U \) corresponding to its internal energy and \( V \) the volume filled by the fluid. The parameter \( \alpha \) is constant and positive: \( \alpha > 0 \). The parameter \( A \) is also positive and considered as an universal constant. The same equation of state, with \( \alpha = 1 \), is known as EoS of Chaplygin gas in honor of the Russian physicist Chaplygin who introduced it in 1904 as a convenient soluble model to study the lifting force on a plane wing in aerodynamics \([3]\); its generalization for \( \alpha > 0 \) was originally proposed in Ref. \([4]\) and the ensuing cosmology has been analyzed in Refs. \([4–8]\).

This generalized Chaplygin gas (GCG) behaves as pressureless at high energy densities and essentially constant negative pressure at low energy densities.

Although there have been a number of papers discussing various aspects of GCG behavior \([9–14]\) in order to reconcile the standard cosmological model with observations, there has not been, to our knowledge, a full analysis of the thermodynamic constraints that can be placed on such a fluid. To accomplish
this task we have determined, in this work, the corresponding thermal equation of state for the GCG and analyzed its temperature behavior as well as its thermodynamic stability considering both adiabatic and thermal equations of state.

This Letter is organized as follows. In Section 2 we calculate the internal energy $U$ for the Chaplygin fluid as a function of its natural variables, the entropy $S$ and volume $V$. Its adiabatic equation of state $P$ is also determined as a function of the natural variables of $U$. From these results, in Section 3, we determine the thermal EoS and finally investigate how behaves the temperature of the gas and its thermodynamic stability.

2. Adiabatic generalized Chaplygin gas EoS

In this section we consider the GCG as an one component thermodynamic system with constant particle number $N$ and determine the internal energy $U$ and pressure $P$ of GCG as a function of its entropy $S$ and volume $V$. From general thermodynamics [15,16] we have the following relationship:

$$\left(\frac{\partial U}{\partial V}\right)_S = -P.$$  

(3)

Combining Eqs. (1)–(3), we get the differential equation

$$\left(\frac{\partial U}{\partial V}\right)_S = A \frac{V^\alpha}{U^\alpha},$$  

(4)

whose solution gives the internal energy $U$ of the GCG in its natural variables, $S$ and $V$:

$$U = \left[A V^{\alpha+1} + b\right]^{1/(\alpha+1)}.$$  

(5)

The parameter $b$ is an integration constant of Eq. (4), which may be a universal constant or a function of entropy $S$ only: $b = b(S)$. It is interesting to remark that even in the case when $A$ is not a universal constant but a function of entropy only, $A = A(S)$, Eq. (5) remains valid. To further discussion, it is convenient to write Eq. (5) as

$$U = A^{1/(\alpha+1)} V \left[1 + \left(\frac{\delta}{V}\right)^{\alpha+1}\right]^{1/(\alpha+1)},$$  

(6)

where, in this equation, $\delta^{\alpha+1}$ represents the ratio

$$\delta^{\alpha+1} = \frac{b}{A}.$$  

(7)

Therefore, the energy density $\rho$ of the Chaplygin gas reduces as its volume expands adiabatically:

$$\rho = A^{1/(\alpha+1)} \left[1 + \left(\frac{\delta}{V}\right)^{\alpha+1}\right]^{1/(\alpha+1)}.$$  

(8)

For small volumes, the energy density of the generalized Chaplygin gas behaves like

$$\rho \approx A^{1/(\alpha+1)} \left(\frac{\delta}{V}\right),$$  

(9)

and can be very high (one remarks that Eq. (9) also imposes that $b > 0$). For large volumes, the energy density behaves as a mixing of two fluids:

$$\rho \approx A^{1/(\alpha+1)} + A^{1/(\alpha+1)} \left(\delta\right)^{\alpha+1}.$$  

(10)

In the equation above one has a term with constant energy density and another one whose density depends on volume.

After replacing Eq. (8) into Eq. (1), the pressure $P$ of the GCG is determined as a function of entropy $S$ and volume $V$:

$$P = -\frac{A^{1/(\alpha+1)}}{[1 + (\frac{\delta}{V})^{\alpha+1}]/(\alpha+1)}.$$  

(11)

One observes from the equation above that the pressure may be nearly zero or have negative values. Moreover, at small volumes, Eq. (11) suggests that the pressure $P$ of the Chaplygin gas behaves like

$$P \approx -A^{1/(\alpha+1)}.$$  

(12)

For large volumes, the pressure is like

$$P \approx -\rho.$$  

(13)

Thus, the generalized Chaplygin gas is essentially pressureless at small volumes and it has constant negative pressure at large volumes. It is interesting to see that at low energy densities its equation of state can be expressed as

$$V \propto a^3,$$  

(15)

the energy density $\rho$ and the pressure at small $a$ are approximated by

$$\rho \approx \frac{b^{1/(\alpha+1)}}{a^3}, \quad P \approx 0,$$  

(16)

that corresponds to a universe dominated by dust-like matter. For large values of the cosmological factor $a$ it follows that

$$\rho \approx A^{1/(\alpha+1)},$$  

(17)

which, in turn, corresponds to an universe with a cosmological constant $A^{1/(\alpha+1)}$ (i.e., a de Sitter universe). Therefore, the generalized Chaplygin EoS is able to interpolate between a dust dominated phase of the universe and a de Sitter universe through its evolution.

To verify the thermodynamic stability conditions of this fluid along its evolution, it is necessary to determine if the pressure is reduced through an adiabatic expansion

$$\left(\frac{\partial P}{\partial V}\right)_S < 0.$$  

(18)
Deriving Eq. (11) with respect to the volume we get

\[
\left( \frac{\partial P}{\partial V} \right)_S = \alpha \left( \frac{P}{V} \right) \left\{ 1 - \frac{1}{\left[ 1 + \left( \frac{P}{V} \right)^{\alpha+1} \right]} \right\}. \tag{19}
\]

Therefore, the condition \( \alpha = 0 \) must be discarded because it will place a severe constraint on the stability of this fluid: in such a case, \((\partial P/\partial V)_S = 0\), and the pressure will remain the same through any adiabatic change of volume. For \( \alpha > 0 \) this derivative is always negative. Then, if we consider Eq. (19) as the only criterion of stability, we must conclude that the Chaplygin fluid is not prohibited to attain all pressures defined by Eq. (11). However, to discuss the stability of the fluid extensively, it is not enough to consider the Eq. (18) only. It is also necessary to verify if the thermal capacity at constant volume \( c_V \) is positive:

\[ c_V > 0. \tag{20} \]

In addition, it is also necessary to determine if the pressure reduces as the fluid expands at constant temperature \( T \), in the same region where Eq. (19) is negative or at least the pressure remains the same during the expansion

\[
\left( \frac{\partial P}{\partial V} \right)_T \leq 0. \tag{21}
\]

Moreover, to discuss the existence of critical points it is necessary to verify if

\[
\left( \frac{\partial P}{\partial V} \right)_T = 0, \quad \left( \frac{\partial^2 P}{\partial V^2} \right)_T = 0, \quad \left( \frac{\partial^3 P}{\partial V^3} \right)_T < 0. \tag{22}
\]

If the equations in (22) are fulfilled, then the GCG presents a critical point [16].

To discuss these questions it is necessary at first to determine the thermal equation of state of the fluid, that means, how the pressure \( P \) of the fluid depends on volume \( V \) and temperature \( T \). This is considered in the next section.

3. Thermal generalized Chaplygin gas EoS

In this section we set up initially how the temperature \( T \) of the generalized Chaplygin fluid depends on its volume \( V \) and its entropy \( S \) and after we determine its thermal equation of state \( P = P(T, V) \). From general thermodynamics [15,16], the temperature \( T \) of the fluid is determined from the following equation:

\[
T = \left( \frac{\partial U}{\partial S} \right)_V. \tag{23}
\]

Replacing Eq. (5) into Eq. (23), we get

\[
T = \frac{1}{\alpha + 1} \left[ A V^{\alpha+1} + b \right]^{-\alpha/(\alpha+1)} \left[ V^{\alpha+1} \frac{dA}{dS} + \frac{db}{dS} \right]. \tag{24}
\]

Thus, to determine completely the temperature of the fluid at given volume \( V \), it is necessary the knowledge of \( A \) and \( b \) as a function of entropy \( S \). If these parameters are assumed universal constants, then

\[
\frac{dA}{dS} = 0 \quad \text{and} \quad \frac{db}{dS} = 0. \tag{25}
\]

In such conditions, the temperature will be zero for any value of the volume or pressure of the gas. Thus, the isotherm \( T = 0 \) is simultaneously an isentropic curve (adiabatic) at \( S = \text{const} \) and this violates the third law of thermodynamics. Therefore, to discuss extensively the thermodynamic stability of the generalized Chaplygin gas, it is necessary to assume that at least Eq. (25) or Eq. (26) is not satisfied.

Let us, initially, consider the limits of the (24). For large volumes, i.e.,

\[
V \gg \left( \frac{b}{A} \right)^{1/(\alpha+1)}, \tag{27}
\]

the temperature \( T \) is reduced to:

\[
T \approx \frac{1}{(\alpha + 1)A^{\alpha/(\alpha+1)}} \left[ V \frac{dA}{dS} + \frac{V}{V^{\alpha+1}} \frac{db}{dS} \right]. \tag{28}
\]

Therefore, in order to have positive temperatures and cooling along an adiabatic expansion, we must impose

\[
\frac{dA}{dS} = 0 \quad \text{and} \quad \frac{db}{dS} > 0. \tag{29}
\]

Unfortunately, the analytical determination of a unique thermal equation of state \( P = P(T, V) \) for the Chaplygin fluid is not possible without knowledge of \( b = b(S) \) in Eq. (5). However, from dimensional analysis we observe that the dimension of \( b \) is

\[
[b]^{1/(\alpha+1)} = [U]. \tag{31}
\]

Thus, it is a natural choice to study the case where

\[
b = \frac{1}{\beta^{\alpha+1}} S^{\alpha+1}, \tag{32}
\]

where \( \beta^{-1} \) is a universal constant with dimension of temperature: \( \beta^{-1} = T_* \). (This ansatz can be justified only from experience or from its theoretical consequences.) Deriving Eq. (32) we get

\[
\frac{db}{dS} = (\alpha + 1)(T_*)^{\alpha+1} S^\alpha. \tag{33}
\]

After replacing Eqs. (25), (29), (32), and (33) into Eq. (24) the temperature \( T \) of the fluid is determined:

\[
T = (T_*)^{\alpha+1} S^\alpha \left[ A V^{\alpha+1} + (T_*)^{\alpha+1} S^{\alpha+1} \right]^{-\alpha/(\alpha+1)}. \tag{34}
\]

In such a case, the Eq. (34) can be solved for the entropy \( S \) as

\[
S = A^{1/(\alpha+1)} T_* \left[ 1 - \left( \frac{V}{T_*} \right)^{(\alpha+1)/\alpha} \right]^{1/(\alpha+1)} V. \tag{35}
\]
We remark that, in such a scenario, in order to have the entropy always positive and finite, the temperatures and pressures as well as the Chaplygin fluid may attain must be in the range
\[ 0 < T < T*. \]

Using the general thermodynamic relationship [15,16] for the thermal capacity at constant volume \( c_v \),
\[ c_v = T \left( \frac{\partial S}{\partial T} \right)_V \]
and Eq. (35), the thermal capacity \( c_v \) can be determined
\[ c_v = \frac{1}{\alpha T^* \left[ 1 - (T/T^*)^{(a+1)/\alpha} \right]} \left( \frac{T}{T^*} \right)^{1/\alpha}. \]  
(37)

Since \( \alpha > 0 \) and \( 0 < T < T* \), the Chaplygin fluid will have its thermal capacity at constant volume positive and the second condition of stability \( c_v > 0 \) is always satisfied.

The thermal equation of state of the fluid, \( P = P(T, V) \), can also be determined. Combining Eqs. (1), (5), (32), and (35) yields
\[ P = -A^{1/(\alpha+1)} \left[ 1 - \left( \frac{T}{T^*} \right)^{(a+1)/\alpha} \right]^{1/\alpha}. \] 
(39)

We observe from Eq. (39) that the thermal equation of state of the GCG is dependent on temperature only. That means the isobaric curves for the GCG are coincident with its isotherms in the diagram of thermodynamic states, in a similar behavior like the radiation pressure. However, differently from the radiation pressure, when the temperature goes to zero the pressure of the GCG is essentially constant:
\[ T \rightarrow 0, \quad P \sim -A^{1/(\alpha+1)}. \] 
(40)

— that means at very low temperatures the universe filled with GCG behaves as a de Sitter universe — and when the temperature grows up until near \( T^* \), the pressure goes to zero
\[ T \rightarrow T^*, \quad P \sim 0. \] 
(41)

In such a case, the universe behaves as a dust-like or a pressureless universe. Therefore, in the adiabatic evolution from a dust-like to a de Sitter cosmological model, the temperature of the universe filled with the GCG cools down as expected.

We also observe from Eq. (39) that all derivatives \( \left( \frac{\partial^n P}{\partial V^n} \right)_T, \quad n = 1, 2, 3, \ldots, \) are equal to zero for any volume \( V \). Therefore, in the scenario built up from the condition \( b = \left( T^* / \rho_0 \right)^{a+1} \), the Chaplygin fluid presents stable behavior through any expansion. Thus, the GCG is a perfect fluid which cools down during an adiabatic expansion. It can also attain the negative and essentially constant pressure suggested by Eq. (13) without breaking its thermodynamic stability and without facing any critical point in the process.

We can also determine the temperature \( T^* \) as a function of the initial conditions of the expansion. If we consider that the initial conditions at \( V = V_0 \) are \( \rho = \rho_0, \quad P = P_0, \) and \( T = T_0 \), then we can write the integration constant \( b \) in Eq. (5) as
\[ b = \left( \rho_0^{a+1} - A \right) V_0^{a+1}. \] 
(42)

After replacing Eq. (42) into Eqs. (8) and (11), we obtain the energy density \( \rho \) as a function of the volume \( V \):
\[ \rho = \rho_0 \left[ \frac{A}{\rho_0^{a+1}} + (1 - \frac{A}{\rho_0^{a+1}}) \left( \frac{V_0}{V} \right)^{a+1} \right]^{1/(a+1)} \] 
(43)

and the adiabatic equation of state Eq. (11) as a function of \( \rho_0 \) and \( V \):
\[ P = -\frac{A}{\rho^{a}} = -\frac{A^{1/(\alpha+1)} (A/\rho_0^{a+1})^{1/\alpha}}{\left[ \left( \frac{A}{\rho_0^{a+1}} \right)^{1/\alpha} + \left( 1 - \frac{A}{\rho_0^{a+1}} \right) \left( \frac{V_0}{V} \right)^{a+1} \right]^{\alpha/(\alpha+1)}} \] 
(44)

When Eqs. (39), (43) and (44) are expressed as functions of the reduced parameters \( \epsilon, \nu, \gamma, \) and \( t \) such that:
\[ \epsilon = \frac{\rho}{\rho_0}, \quad \nu = \frac{V}{V_0}, \quad P = \frac{P}{A^{1/(\alpha+1)}}, \]
\[ \gamma = \frac{A}{\rho_0^{a+1}}, \quad t = \frac{T}{T_0}, \quad t^* = \frac{T^*}{T_0}, \] 
(45)

the thermal equation of state (Eq. (39)) can be written in reduced units like
\[ p = \left[ 1 - \left( \frac{t}{t^*} \right)^{(a+1)/\alpha} \right]^{1/\alpha}, \] 
(46)

the energy density (Eq. (43)) in reduced units like
\[ \epsilon = \left[ \gamma + (1 - \gamma) \right]^{1/(\alpha+1)} , \] 
(47)

and the adiabatic equation of state (Eq. (39)) in reduced units like
\[ p = \frac{P}{A^{1/(\alpha+1)}} = -\frac{\gamma^{a/(\alpha+1)}}{\left[ \gamma + (1 - \gamma) \right]^{a/(\alpha+1)}}. \] 
(48)

At \( P = P_0, \quad V = V_0 \) and \( T = T_0 \), we have \( t = 1 \) and \( v = 1 \), and we get from Eqs. (46) and (48) that
\[ p_0 = -\gamma^{a/(\alpha+1)} = -\left( \frac{1}{t} \right)^{(a+1)/\alpha} \] 
(49)

and \( t^* \) can be determined from the equation above:
\[ t^* = \left( \frac{1}{1 - \gamma} \right)^{a/(\alpha+1)}. \] 
(50)

Eq. (50) relates the temperature limit \( t^* \) with the initial conditions of the expansion. For instance, for \( \gamma = 0.98 \) and \( \alpha = 1 \), that means for an initial energy density near its lower value, says \( \rho = 1.1 A^{1/2}, \) the temperature \( t^* \) is \( t^* \approx 7.1. \) Therefore, from Eq. (49), if we consider \( \alpha > 0 \) and the maximum temperature of the fluid as \( T^* = 10^{32} \) K, the temperature of the Planck era, and \( T_0 = 2.7 \) K, [17] the temperature of the gas at the present epoch, the ratio \( \gamma \) should be
\[ \gamma \approx 1 - (10^{-32})^{(\alpha+1)/\alpha} \sim 1. \] 
(51)

Thus, in the scenario set up from Eq. (32), the energy density \( \rho \) of the universe filled with the GCG at present epoch must be very close to its limit value \( A^{1/(\alpha+1)}. \)

The internal energy \( U \) as a function of the volume \( V \) and temperature \( T \) can also be determined. Combination of
Eqs. (1)–(3) and (39) yields

\[
U = \frac{A^{1/(\alpha+1)}}{[1 - (\frac{T}{T_n})^{(\alpha+1)/\alpha}]^{1/(\alpha+1)}},
\]

(52)

and

\[
\rho = \frac{A^{1/(\alpha+1)}}{[1 - (\frac{T}{T_n})^{(\alpha+1)/\alpha}]^{1/(\alpha+1)}},
\]

(53)

and the energy density is dependent on temperature only. It is interesting to remark that, if we assume at the starting point that the energy density is a function of temperature only, then

\[
P = \frac{A}{\rho^{\alpha+1}} \left( \frac{\partial P}{\partial T} \right)_V = \alpha \frac{A}{\rho^{\alpha+1}} \frac{d\rho}{dT}.
\]

(54)

Replacing the results from Eq. (54) into the relationship \[15,16\]

\[
\left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial P}{\partial T} \right)_V - P
\]

(55)

we get the following differential equation

\[
\frac{dT}{T} = \frac{\alpha A d\rho}{\rho(\rho^{\alpha+1} - A)},
\]

(56)

whose solution is

\[
\frac{T}{T_0} = \left[ \frac{\rho^{\alpha+1} - A}{\rho_0^{\alpha+1} - A} \right]^{\alpha/(\alpha+1)}.
\]

(57)

The result found in Eq. (57) is the same expression for the energy density as that one in Eq. (53).

4. Conclusions

Using only general thermodynamics, one has shown in this Letter that the generalized Chaplygin fluid behaves essentially as a pressureless gas at small volumes and it has constant negative pressure at large volumes, in agreement with the results obtained in the framework of Friedmann–Robertson–Walker cosmology. We have extended the general thermodynamics analysis of the GCG and we have set up conditions were the thermal equation of state of the fluid \(P = P(T, V)\) is determined. In such a scenario its thermal EoS is dependent on temperature only and the temperature of the fluid remains in the range \(0 < T < T^*\), where \(T^*\) is the maximum temperature the generalized Chaplygin gas may sustain when it fills small volumes. All conditions for thermodynamic stability have been discussed in this scenario and it is remarked that this fluid can be thermodynamically stable in all range of pressures through any expansion process. The scenario also shows that there is no critical point in its diagram of thermodynamic states.

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