The Lack of Definable Witnesses and Provably Recursive Functions in Intuitionistic Set Theories

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Let $ZFI_R$ ($ZFI_C$) be intuitionistic $ZF$ set theory formulated with Replacement (resp. Collection). It is known that if $ZFI_R$ proves a sentence $\exists x A(x)$, then there is a formula $C(z)$ so that $ZFI_R$ proves $\exists! z C(z)$ and $\exists x (C(x) \land A(x))$, the existence property. It is shown that $ZFI_C$ does not have the existence property, and thus $ZFI_R \subseteq ZFI_C$. This remains true even if one adds Dependent Choice and all true $\Sigma_1$ sentence of $ZF$. It is known that $ZF$ and $ZFI_C$ have the same provably recursive functions. It is also shown that this is not true for $ZFI_C$ and $ZFI_R$.

INTRODUCTION

It is known that most considered theories $T$ in Heyting's predicate calculus with equality have the existence property: if $T$ proves a sentence $\exists x A(x)$, then there is a formula $C(z)$ with exactly $z$ free, so that $T$ proves $\exists! z C(z)$ and $\exists x (C(x) \land A(x))$. In classical logic, Zermelo set theory, $ZF$, and $ZFC$ are known not to have the existence property. In particular, Feferman [3] and Levy [14] gave forcing extensions which show the failure of the existence property for $\Pi^1_2$ sets. Nevertheless, one has the fragment of the existence property as a consequence of the uniformization property for $\Sigma^1_2$ sets. This is important in point-set topology and descriptive set theory [15], where given a definable relation, one wants a definable

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function with the same domain. ZF + Projective Determinacy has the projective existence property (with \( A(x), C(z) \) projective). Kunen observes that an extension of ZF has the existence property iff it proves \( V = OD \).

Freyd [6] proved the category-theoretic equivalent of the existence property for various free categories.

Going back to intuitionistic theories, let us conveniently consider the language with \( \in \) only, where \( x = y \) is defined as \( \forall z (z \in x \leftrightarrow z \in y) \). Consider the following axioms of set theory: (1) Extensionality, (2) Pairing, (3) Separation, (4) Infinity, (5) Union, (6) Power Set, and (7) \( \epsilon \)-induction (because Regularity implies Excluded Middle [16]).

Let \( ZF_L \) be the result of adding (the scheme of) Replacement:

\[
\forall x \in a. \exists ! y A(x, y) \rightarrow \exists u. \forall x \in a. \exists y \in u. A(x, y),
\]

and let \( ZF_C \) be the result of adding (the scheme of) Collection to (1)–(7):

\[
\exists u. \forall x \in a. (\exists y A(x, y) \rightarrow \exists y \in u. A(x, y)),
\]

or, equivalently as a scheme (by Separation):

\[
\forall x \in a. \exists y A(x, y) \rightarrow \exists u. \forall x \in a. \exists y \in u. A(x, y).
\]

Clearly, Collection proves Replacement. In classical logic, the converse holds as well.

It is a joint result of the first author and Myhill [16] that \( ZF_L \) has the existence property (by an extension of the Kleene slash [7]). Only a very restricted version, namely the numerical existence property is known for \( ZF_C \) (i.e., if a sentence \( \exists x \in \omega. A(x) \) is provable, then there is a numeral \( \bar{n} \) so that \( A(\bar{n}) \) is provable). It was shown by a kind of recursive realizability by Beeson [11].

On the other hand, Gödel's negative interpretation goes through for Collection, showing that \( ZF_C \) is equiconsistent with classical ZF set theory [8]. Furthermore, \( ZF_C \) and ZF have the same provably recursive functions [9]. By the work of D. Scott, Fourman, and Grayson, \( ZF_C \) suffices to interpret (any bounded fragment of) \( ZF_L \) in any Heyting-valued model, indeed in any Grothendieck topos [4, 12, 5]. The same holds again for various versions of recursive realizability [7, 11]. None of the counterparts of these results are known for \( ZF_R \). In particular, it is not known whether \( ZF_R \) is equiconsistent with \( ZF_R + \text{"Every } f \in \omega^0 \text{ is recursive."} \)

Recently, Goodman suggested that Kripke models might show that \( ZF_R \) augmented with a unary predicate symbol does not prove \( ZF_C \) augmented with the same unary predicate symbol. We use Kripke models here in a simpler way to show that \( ZF_C \) does not have the existence property. Therefore, \( ZF_R \) does not prove \( ZF_C \).
THE LACK OF DEFINABLE WITNESSES

This basic construction, due to the first author, is given in Section 1. Section 2 contains joint work on a considerable strengthening of the basic result, showing that even ZFI\(_R\) with Dependent Choice and all classically true \(\Sigma_1\) sentences of ZF still does not prove Collection. This is accomplished by introducing the concept of relative existence property (due to the second author). It is the unique definability from parameters satisfying finitely many formulae of a certain class. This fails for Collection, but holds (in a weak form) for Replacement.

In Section 3 (due to the first author), it is proved that ZFI\(_C\) and ZFI\(_R\) do not have the same provably recursive functions.

We refer the reader now to [13, 2, 17] for a body of knowledge in set theory, model theory, and recursion theory used especially in Sections 2 and 3. In Section 2 we shall also refer to the methods developed in [10].

1. COLLECTION LACKS THE EXISTENCE PROPERTY

Our argument takes place in \(\text{ZFC} + \text{Con(ZFC)}\). Let \(\mathcal{N}\) be a countable model of \(\text{ZFC} + V = L\). Let \(\lambda\) be the least ordinal in \(\mathcal{N}\) that is greater than all definable ordinals in \(\mathcal{N}\) (cf. remark at the end of this section). Collapse \(\lambda\) by forcing, to make \(\lambda\) countable. Let \(\mathcal{M}\) be the resulting forcing extension. In \(\mathcal{M}\), \(\omega_1\) is greater than \(\lambda\). Thus we can work in \(\text{ZFC} + \text{Con(\exists ordinal definable in } V, \text{ greater than all ordinals definable in } L)\).

Now let \(\mathcal{M}^*\) be the structure defined as follows. Elements of \(\mathcal{M}^*\) are all pairs \(\langle x, y \rangle\), where \(x, y \in \mathcal{M}\). Let \(R(\langle x, y \rangle, \langle u, v \rangle) \iff \mathcal{M} = \langle x, y \rangle \in u \cdot \langle \mathcal{M}^*, R \rangle\) satisfies all axioms of ZF except Extensionality and Power Set. Let \(\equiv\) be the equivalence relation on \(\mathcal{M}^*\) given by \((\forall c)(R(c, a) \iff R(c, b))\).

Let \(F : \mathcal{M}^* \rightarrow \mathcal{N}\) be a definable function such that

1. \(R(x, y) \implies F(x) \in F(y)\),
2. \(F\) maps each equivalence class \([x]\) w.r.t. \(\equiv\) one-to-one onto \(\{z \in \mathcal{N} \mid \forall y(R(y, x) \implies F(y) \in z)\}\).

Thus the range of \(F\) is as large as possible, subject to the constraint of (1). Such an \(F\) is unique up to isomorphism and is defined by transfinite recursion on the rank of elements of \(\mathcal{M}^*\).

We now form the following Kripke structure. There are two moments 1 < 2. The objects at both moments are the elements of \(\mathcal{M}^*\) if \(A(x_1, \ldots, x_n)\) is a formula of ZFI with exactly \(x_1, \ldots, x_n\) free, we let

\[2 \models A(x_1, \ldots, x_n) \iff \mathcal{N} \models A(F(x_1), \ldots, F(x_n)),\]

with the ordinary abuse of notation. Forcing at 1 is defined by
One readily shows by induction on the complexity of $A(x_1,\ldots, x_n)$ that $1 \models A(x_1,\ldots, x_n)$ implies $2 \models A(x_1,\ldots, x_n)$.

**Lemma 1.1.** All axioms of ZFC are forced at 1. In particular, $1 \models \forall z(z \in x \leftrightarrow z \in y)$ iff $x = y$.

**Proof.** This semantics is a special case of Kripke models [20], but it can easily be verified directly that Heyting's predicate calculus is forced at 1.

**Extensionality.** $\forall z(z \in x \leftrightarrow z \in y) \rightarrow (A(x) \leftrightarrow A(y))$. Let $1 \models \forall z(z \in x \leftrightarrow z \in y)$, i.e., for all $z$, $R(z, x)$ iff $R(z, y)$, and $F(z) \in F(x)$ iff $F(z) \in F(y)$. Because $F$ is onto, $F(x) = F(y)$. Because $F$ is one-to-one on equivalence classes, $x = y$. Then the conclusion is forced.

**Pairing.** $\exists x(y \in x \land z \in x)$. Obvious.

**Separation.** $\exists x. \forall y(y \in x \leftrightarrow y \in z \land A(y))$. Consider $u = \{v \in F(z) \mid A(v)\}$ in $\mathcal{N}$, and let $x' \in M^*$ be such that for each $y \in M^*$, $R(y, x')$ iff $R(y, z)$ and $1 \models A(y)$. Because $R(y, x')$ implies $F(y) \in u$, let $x \in [x']$ be such that $F(x) = u$.

**Infinity.** $\exists x. \exists y. \forall z \forall v(y \in v \land v \in u \rightarrow y \in x)$. Consider $\omega$ in $\mathcal{N}$ and successively choose $0, 1, \ldots, \tilde{n}, \ldots$, in $M^*$ so that $F(\tilde{n}) = n$, each $n \in \omega$. Let $x' \in M^*$ be such that $R(y, x')$ iff $y = \tilde{n}$, some $n \in \omega$. Let $x \in [x']$ be such that $F(x) = \omega$.

**Union.** $\exists x. \forall y. \forall u(y \in v \land v \in u \rightarrow y \in x)$. Let $x' \in M^*$ be such that $R(y, x')$ if $R(y, v)$ and $R(v, u)$, some $v \in M^*$. Then $R(y, x')$ implies $F(y) \in F(u)$, so let $x \in [x']$ be such that $F(x) = \bigcup F(u)$.

**Power Set.** $\exists x. \forall y. \forall z \forall v(y \in z \land u \in v \rightarrow y \in x)$. This is rather delicate because Power Set does not hold in $M^*$. We need, however, only $x \in M^*$ so that for all $y \in M^*$, $F(y) \subseteq F(u)$ implies $F(y) \subseteq F(x)$, and that $F(y) \subseteq F(u)$ and $\forall z(R(z, y) \rightarrow R(z, u))$ imply $R(y, x)$. Recall that $F$ maps $[y]$ 1–1 onto $\{w \in \mathcal{N} \mid \forall z(R(z, y) \rightarrow F(z) \subseteq w)\}$. Suppose $F(y) \subseteq F(u)$ and $\forall z(R(z, y) \rightarrow R(z, u))$; so we need consider only $w$ in the power set of $F(u)$ in $\mathcal{N}$.
\( x' \in \mathcal{M}^* \) be a set of all such \( y \), i.e., \( R(y, x') \) iff \( F(y) \subseteq F(u) \) and \( \forall z (R(z, y) \rightarrow R(z, u)) \). Now choose \( x \in [x'] \) so that \( F(x) \) is the power set of \( F(u) \) in \( \mathcal{N} \).

**Foundation.** \( \forall x (\forall y \in x. A(y) \rightarrow A'(x)) \rightarrow \forall x A(x) \). We wish to use Foundation w.r.t. formula \( 1 \models A(x) \); but this is not possible right away because the antecedent involves nested implication, and \( 1 \models B \rightarrow C \) involves \( 2 \models B \rightarrow C \). Note, however, that because the antecedent is universal, if \( 1 \) forces it, then so does \( 2 \); so \( \forall x A(x) \) is true in \( \mathcal{N} \) by Foundation. Thus \( 1 \models \forall y \in x. A(y) \) is equivalent to \( \forall y (R(y, x) \rightarrow 1 \models A(y)) \), and \( 1 \models (\forall y \in x. A(y) \rightarrow A(x)) \) is equivalent to

\[
\forall y (R(y, x) \rightarrow 1 \models A(y)) \rightarrow 1 \models A(x).
\]

Use Foundation w.r.t. formula \( 1 \models A(x) \) to get \( 1 \models A(x) \) for each \( x \in \mathcal{M}^* \).

**Collection.** \( \exists y. \forall x \in u (\exists y. A(x, y) \rightarrow \exists y \in v. A(x, y)) \). Let \( v' \in \mathcal{M}^* \) be such that for each \( x \in \mathcal{M}^* \), \( R(x, u) \) implies

\[
\exists y. 1 \models A(x, y) \rightarrow \exists y (R(y, v') \land 1 \models A(x, y)),
\]

by Collection w.r.t. formula \( 1 \models A(x, y) \). By Collection in \( \mathcal{N} \), let \( w \in \mathcal{N} \) be such that for each \( x \in \mathcal{M}^* \), \( F(x) \subseteq F(u) \) implies

\[
\exists y. A(F(x), F(y)) \rightarrow \exists y (F(y) \in w \land A(F(x), F(y))).
\]

Recall that \( F \) maps \([v'] \) 1-1 onto \( \{s \in \mathcal{N} | \forall y (R(y, v') \rightarrow F(y) \in s)\} \) and choose \( v \in [v'] \) so that \( F(v) = F(v') \cup w \).

**Lemma 1.2.** Let Dec be the decidability axiom \( \forall x. \forall y (x \in y \lor x \notin y) \). For any formula \( A \), let \( A^* \) be the result of replacing every atomic subformula \( R \) of \( A \) by \( R \lor \text{Dec} \). Then \( 1 \models A^* \iff \mathcal{M}^* \models A \). Moreover, \( 2 \models A^* \).

**Proof.** Let \( x \in \mathcal{M}^* \) be such that for no \( y \in \mathcal{M}^* \) do we have \( R(y, x) \). Recall that \( F \) maps \([x]\) 1-1 onto \( \mathcal{N} \). Thus \( 1 \models \text{Dec} \), i.e., \( 1 \models (\bot^*) \). Of course \( 2 \models \text{Dec} \), so \( 2 \models A^* \), any \( A \). Also, \( 1 \models (u \in v) \lor \text{Dec} \) iff \( 1 \models u \in v \), i.e., \( R(u, v) \), i.e., \( \mathcal{M}^* \models u \in v \). Proceed by induction on the complexity of \( A \). Only \( \rightarrow \) and \( \forall \) are interesting. \( 1 \models (A \rightarrow B)^* \), i.e., \( 1 \models (A \rightarrow B^*) \) iff \( 1 \models A^* \) implies \( 1 \models B^* \), and \( 2 \models (A \rightarrow B)^* \). The second condition holds, and by induction hypothesis, \( 1 \models A^* \iff \mathcal{M}^* \models A, 1 \models B^* \iff \mathcal{M}^* \models B \). Thus \( 1 \models (A^* \rightarrow B^*) \) iff \( \mathcal{M}^* \models (A \rightarrow B) \). \( \forall \) is treated similarly.

**Lemma 1.3.** Suppose \( 1 \models \exists ! x A(x) \), where \( A \) has no other free variable. Then for the unique \( x \in \mathcal{M}^* \) such that \( 1 \models A(x) \), it is the case that \( F(x) \) is the unique solution to \( A \) in \( \mathcal{N} \). In particular, \( rk(x) \) is bounded by some definable ordinal of \( \mathcal{N} \).
Proof. \( \exists ! x A(x) \) says \( \exists x (A(x) \land \forall y (A(y) \rightarrow \forall z (z \in x \iff z \in y)) \). By Lemma 1.1, \( 1 \models \forall z (z \in x \iff z \in y) \) if \( x = y \), thus \( 1 \models \exists ! x A(x) \) indeed implies that there is a unique \( x \in \mathcal{M}^* \) so that \( 1 \models A(x) \). Now \( 1 \models B(y) \) implies \( 2 \models B(F(y)) \) for any formula \( B \), so \( 2 \models \exists ! x A(x) \), and on the other hand \( 2 \models A(F(x)) \) for the unique \( x \in \mathcal{M}^* \) so that \( 1 \models A(x) \). The second part of the statement follows because \( \text{rk}(x) \leq \text{rk}(F(x)) \).

**Lemma 1.4.** Let \( A(y) \) be a formula stating "\( y \) is an uncountable ordinal," e.g., "\( y \) is transitive, linearly ordered by \( \in \), and for some set \( x \) satisfying the axiom of Infinity, there is no 1-1 onto map from \( x \) to \( y \)." Then there is no formula \( C(z) \) with exactly \( z \) free so that

\[
\text{ZF}_C \models \exists ! z C(z) \land \exists v (C(v) \land (\exists y . A^*(y) \rightarrow \exists y \in v . A^*(y))).
\]

**Proof.** Suppose there is such a formula \( C(z) \). Note that

\[
\exists v (\exists y . A^*(y) \rightarrow \exists y \in v . A^*(y))
\]

is an instance of Collection (with \( u \) a singleton), and thus forced at 1, by Lemma 1.1. Because \( \mathcal{M}^* \models \exists y . A(y) \), we see that \( 1 \models \exists y . A^*(y) \) by Lemma 1.2, and thus \( 1 \models \exists y \in v . A^*(y) \). Again by Lemma 1.2, \( \mathcal{M}^* \models A(y) \) for some \( y \in \mathcal{M}^* \) with \( R(y, v) \). Therefore \( \text{rk}(y) \leq \text{rk}(v) \), so it is bounded by a definable ordinal \( < \lambda \) of \( \mathcal{N} \) (Lemma 1.3), that has been made countable in \( \mathcal{M}^* \). On the other hand, \( y \) is an uncountable ordinal.

**Remark.** All we need is \( \lambda > \text{rk}(v) \), where \( v \) is definable by the formula \( C(v) \).

We thus have

**Theorem 1.1.** Assume \( \text{Con}(ZFC) \). Then \( \text{ZF}_C \) does not have the existence property. In particular, it does not have the existence property for some sentence of the form

\[
\exists v (\exists y . B(y) \rightarrow \exists y \in v . B(y)).
\]

**Corollary 1.1.** There is an instance of Collection that is not provable in \( \text{ZF}_R \).

**Proof.** \( \text{ZF}_R \) has the existence property [16].
2. ADDING DEPENDENT CHOICE AND ALL TRUE $\Sigma_1$-SENTENCES

We consider the schema of Relativized Dependent Choice (RDC):

$$\forall x (A(x) \rightarrow \exists y (A(y) \land B(x, y)))$$

$$\rightarrow \exists f (\text{"$f$ is a function with domain } \omega \text{"} \land f(0) = z$$

$$\land \forall x \in \omega . (A(f(x)) \land B(f(x), f(x + 1))))).$$

Let $\Sigma_1$ be the collection of all classically true $\Sigma_1$ sentences in the language of $ZF$. In this section we prove

**Theorem 2.1.** Let $\text{Con}(ZF + \Sigma_1)$. Then there is an instance of Collection of the form $\exists v (\exists y. A(y) \rightarrow \exists y \in v. A(y))$ that is not provable in $ZF + RDC + \Sigma_1$.

We first modify the Kripke structure given in Section 1 to show that $ZF + RDC + \Sigma_1$ lacks even a weak form of the existence property that asserts definability in parameters satisfying finitely many conditions of a certain kind. Then we prove this weak form of the existence property for $ZF + RDC + \Sigma_1$.

The proofs of both facts are facilitated by

**Lemma 2.1.** Let $\sigma_1$ be the collection of all true sentences of the form

$$\exists x \subseteq \omega \times \omega. \exists y \subseteq \omega. \text{"}x \text{ is well founded"}$$

$$\land \text{"x and y are decidable"} \land A(x, y),$$

where $A(x, y)$ is an arithmetic formula with parameters $x, y$, and where “$x$ is well founded” is stated as the transfinite induction over $x$ w.r.t. sets. “$x$ is decidable” reads $\forall n, m \in \omega . ((n, m) \in x \lor (n, m) \notin x)$, similarly for $y$. Then

$$ZF + RDC + \Sigma_1 = ZF + RDC + \sigma_1.$$

Remark. Decidability condition may be dropped.

**Proof.** Every true $\sigma_1$-sentence is intuitionistically equivalent to a true $\Sigma_1$-sentence, because $\omega$ is $A_0$-definable, and “$x$ is well founded” can be reformulated as “there is a function $f$ into ordinals such that $(n, m) \in x$ implies $f(n) < f(m)$.” Function $f$ is defined by transfinite recursion on $x$. For the other direction, it is crucial to observe that one does not need an intuitionistic proof that every true $\Sigma_1$-sentence is equivalent to a true $\sigma_1$-sentence. Rather, it suffices to show that every true $\Sigma_1$-sentence is provable in $ZF + \sigma_1$. We use the Gödel condensation argument. Given a true $\Sigma_1$-
sentence $B$, use the reflection principle in $ZFC$ for $B \land \text{Extensionality}$, and let $\langle y, x \rangle$ be a countable well-founded extensional structure so that $\langle y, x \rangle \models B$. The existence of $x, y$ is a true $\sigma_1$-sentence that implies $B$ in $ZFI_R$ by the Mostowski transitive collapse. 

**Lemma 2.2.** Let $A(y)$ be the formula stating \"$y$ is transitive, linearly ordered by $\in$, and there is an uncountable ordinal $x$ such that there is no 1-1 onto map from $x$ to $y$.\" Then there are no true $\sigma_1$-sentences $\exists r_i, \exists s_i, B(r_i, s_i), 1 \leq i \leq n$, and a formula $C(z, r_1, s_1, \ldots, r_n, s_n)$ with the free variables as exhibited, so that $ZFI_C + \text{RDC} + \sigma_1$ proves

$$\forall r, s \left( \bigwedge_{i=1}^n B_i(r_i, s_i) \rightarrow \exists! z C(z, r, s) \land \exists v (C(v, r, s)) \right) \land (\exists y. A^*(y) \rightarrow \exists v. A^*(y)) \right) \tag{\dagger}$$

**Proof.** We modify the Kripke structure given in Section 1 so that RDC is Kripke-forced at moment 1. Further analysis of the structure will show that every true $\sigma_1$-sentence is Kripke-forced at moment 1. By the Levy–Shoenfield Absoluteness Lemma, we can have a countable model of $ZFC + (V=L) + \sigma_1$ at moment 2. Again, let $\lambda$ be an ordinal $> \omega_1$ greater than any definable ordinal. Now, however, collapse $\lambda$ to $\omega_1$ by $\omega$-closed forcing, so every new $\omega$-sequence of constructible sets is constructible. Build the nonextensional structure $M^*$ at moment 1 and the transition function $F$ as in Section 1. Let us check RDC. Assume its antecedent is Kripke-forced at 1, in particular $1 \models A(z)$, and for each $x$ in $M^*$, $1 \models A(x)$ implies $1 \models A(y)$ and $1 \models B(x, y)$ for some $y$ in $M^*$. Choose a sequence $x_0, x_1, \ldots, x_n, \ldots$ so that $x_0 = z$ and $1 \models A(x_n), 1 \models B(x_n, x_{n+1})$, each $n$. The internal $\omega$ in the Kripke structure was defined in the proof of Lemma 1.1. by choosing $0, 1, \ldots, \hat{n}, \ldots$, in $M^*$ so that $F(\hat{n}) = n$, all $n$, and $F\{\hat{n}\}_n = \omega$. Choose ordered pairs $\langle \hat{n}, x_n \rangle$ in $M^*$ so that $F(\langle \hat{n}, x_n \rangle) = \langle n, F(x_n) \rangle$. Now we want to choose a copy $f$ in $M^*$ of the sequence $\langle \hat{n}, x_n \rangle$ so that $F(f) = \{ \langle n, F(x_n) \rangle \}_n$. We can certainly get $F(f)$ to include this set, because for any $x$ in $M^*$, $F(x)$ satisfies $\forall y (R(y, x) \rightarrow F(y) \in F(x))$ by definition of $F$ (Section 1). To finish the proof, we must show that $\{ \langle n, F(x_n) \rangle \}_n$ is constructible. This holds because we collapsed $\lambda$ to $\omega_1$ by $\omega$-closed forcing.

Now let $B$ be a true $\sigma_1$-sentence. By the Levy Shoenfield Absoluteness Lemma, $B$ is true in $L$, and because it is upward absolute, it is true in the forcing extension. Choose copies $x_0, y_0$ in $M^*$ so that $F(x_0) \subseteq \omega \times \omega, F(y_0) \subseteq \omega$ are witnesses for $B$ at 2. $1 \models (x$ is well founded), by an argument similar to the validation of Foundation in Lemma 1.1. $1 \models A(x_0, y_0)$ because $A(x, y)$ is arithmetic. Note that $1 \models \langle x_0, y_0 \rangle$ are decidable.”
Furthermore, $1 \models "x \subseteq \omega \times \omega$ and $y \subseteq \omega$ are decidable” requires that $R(\bar{a}, y)$ iff $n \in F(y)$, and similarly for $x$. Assume $1 \models (\dagger)$, with $(\dagger)$ as in the statement of Lemma 2.2. Because all true $\sigma_1$-sentences are Kripke-forced at moment 1, there are some $r_i, s_i \in M^*$, $1 \leq i \leq n$ so that

$$1 \models \bigwedge_{i=1}^{n} ("r_i \subseteq \omega \times \omega \text{ and } s_i \subseteq \omega \text{ are decidable")}$$

and thus

$$1 \models \exists z C(z, r, s) \land \exists v (C(v, r, s) \land (\exists y. A^*(y) \to \exists v \in v. A^*(y))).$$

Transition function $F$ has to be the identity on each $r_i, s_i$, $1 \leq i \leq n$. By the Kondo–Addison Uniformization Theorem (at 2), we can choose $r_i, s_i$, $1 \leq i \leq n$ definable at 2. Now reason as in the proof of Lemma 1.4.

\[\text{Lemma 2.3.}\]
Let $\exists x B(x)$ be a sentence provable in $ZFI_{R} + RDC + \sigma_1$. Then there are true $\sigma_1$-sentences $\exists r_i, \exists s_i, B(r_i, s_i)$, $1 \leq i \leq n$, and a formula $C(z, r_1, s_1, \ldots, r_n, s_n)$ with all free variables exhibited, so that $ZFI_C \models RDC + \sigma_1$ proves

$$\forall r, s \left( \bigwedge_{i=1}^{n} B(r_i, s_i) \to \exists z C(z, r, s) \land \exists x (C(x, r, s) \land B(x)) \right).$$

Remark. The conclusion of the lemma does not stipulate provability in $ZFI_{R} + RDC + \sigma_1$.

Proof. We follow the methods of [10]. Suppose $ZFI_{R} + RDC + \sigma_1$ proves a sentence $\exists x B(x)$. Let $TC_0$ be a finite fragment that proves $\exists x B(x)$. In particular, $TC_0$ involves only finitely many true $\sigma_1$-sentences, say $k$ of them. Build the bounded theory $TC_0'$ as in [10], but adding also a new kind of set constants $c_1, c_2, \ldots, c_k$ together with the new axioms requiring that $c_{2i-1}$, $c_{2i}$ witness the $i$th true $\sigma_1$-sentence in $TC_0$, together with finitely many true conditions (obtained from its subformulae) allowing each of the $k$ true $\sigma_1$-sentences of $TC_0$ to be slashed. If $\beta$ is a bounded fragment of the collection of all true $\sigma_1$-sentences, together with the conditions on parameters just described, let $CT(\beta)$ require that “Every $f: \omega \to \omega$ is recursive in $\beta$.” Choose $\beta$ large enough so that $TC_0$ slashes $TC_0$ in the metatheory $ZFI_{R} + RDC + \beta + CT(\beta)$. Applying 1945-realizability relative to $\beta$, we obtain a bounded fragment $\gamma \equiv \beta$ so that $ZFI_C + RDC + \gamma$ proves that for some set constant $\tau$, $TC_0$ proves $B(\tau)$. By a semantic interpretation of $TC_0$, $ZFI_C + RDC + \gamma$ proves that there is a term $\xi$ (in finitely many parameters) so that $B(\xi)$. Use $q$-realizability relative to $\gamma$ to obtain a (gödelnumeral of a) term $\xi$ (in finitely many parameters) so that $ZFI_C + RDC + \delta$ proves $B(\xi)$, with $\delta \equiv \gamma$. The well-foundedness conditions
in \( \gamma \) are \( \tau \)-realized as in \([20, \text{p. 199}] \) (cf. also \([1, 11]\)). Finally, the conclusion of the lemma follows by compactness.

**Proof of Theorem 2.1.** Let \( A(\gamma) \) be as in Lemma 2.2. The sentence \( \exists v(\exists y A^*(\gamma) \rightarrow \exists y \in v. A^*(\gamma)) \) is provable in \( ZF_{\mathcal{C}} \). We claim that it is not provable in \( ZF_{\mathcal{R}} + \text{RDC} + \Sigma_1 \). By Lemma 2.1, it suffices to show that it is not provable in \( ZF_{\mathcal{R}} + \text{RDC} + \sigma_1 \). But its provability in \( ZF_{\mathcal{R}} + \text{RDC} + \sigma_1 \) would make Lemma 2.2 contradict Lemma 2.3.

**Remarks.** (1) Because \( \forall x \forall y(x \in y \lor x \not\in y) \) is \( \Pi_1 \), \( ZF_{\mathcal{R}} + \Pi_1 = ZF_{\mathcal{C}} + \Pi_1 = ZF + \Pi_1 \). One cannot even add all (classically) true sentences about the power set of \( \{0\} \), because the Separation would imply \( \forall x \forall y(x \in y \lor x \not\in y) \). By the same reason, we need so many copies of \( \{0\} \) at moment \( 1 \) in our Kripke structures.

(2) Let \( \text{T}_I \) be the collection of sentences expressing transfinite induction over each (true) primitive recursive well-ordering. Let \( \text{MP} \) be the schema

\[
\forall x, y \in \omega(A(x, y) \lor \neg A(x, y) \rightarrow \forall x \in \omega(\neg \neg \exists y \in \omega. A(x, y) \\
\rightarrow \exists y \in \omega. A(x, y)),
\]

where \( A(x, y) \) is any formula. Both \( \text{T}_I \) and \( \text{MP} \) are self-slashing, and both hold in our Kripke structure. Therefore, all the results of this section extend to \( \text{T}_I \) and \( \text{MP} \).

3. Replacement and Collection Do Not Have the Same Provably Recursive Functions

\( ZF_{\mathcal{C}} \) and \( ZF \) are equiconsistent \([8]\), and prove the same \( \Pi_2^0 \) sentences \([9]\). Here we prove, assuming \( \text{Con}(ZF) \), that every provably recursive function (i.e., a provable \( \Pi_2^0 \) sentence) of \( ZF_{\mathcal{R}} \) is a provably recursive function of a particular weak fragment of \( ZFC \). We still do not know the proof-theoretic strength of \( ZF_{\mathcal{R}} \). We conjecture that \( ZF \) proves the consistency of \( ZF_{\mathcal{R}} \).

Let \( T \) be the theory in classical logic in the language of \( ZF \), consisting of the axioms

(1) Extensionality,
(2) Pairing,
(3) Full Separation,
(4) Infinity,
(5) Union,
(6) Power Set,
(7) Full Foundation,
(8) Axiom of Choice,
(9) Every well-ordering is isomorphic to an ordinal,
(10) There is a cumulative hierarchy on every ordinal.

Let $T' = T + \text{Con}(ZFC)$. Because $T'$ is obtained from $T$ by adding a true $\Pi^0_1$ sentence, provably recursive functions of $T$ and $T'$ are the same. Let $T''$ be $T' + \text{"There is no fixed point of the } \exists \text{ function."}$ Note that by truncation, $T''$ is conservative over $T'$ for arithmetic sentences.

**Theorem 3.1.** Every prenex arithmetic consequence of $ZFI_R$ is provable in $T'$. In particular, every provably recursive function of $ZFI_R$ is a provably recursive function of $T$.

Working in $T''$, we give a two-moment Kripke structure for $ZFI_R$. The key idea is that it suffices for Replacement to hold only in the model at moment 2 to be Kripke-forced at moment 1. The model at moment 2 is built from a model of $T''$ by using indiscernibles. Similar techniques have been utilized in [18, 19] using large cardinals, but with stronger conclusions. The construction here is the first author's version of the construction of Kunen that uses only $\text{Con}(ZFC)$ to show the existence of $\varphi$-like models of $ZFC$, for any singular cardinal $\varphi$.

**Lemma 3.1.** $T''$ proves the existence of a model of $ZFC + V = L$ that is a proper class and whose bounded initial segments form sets.

**Proof.** Let $M \models T''$. We need a model $N$ of $ZFC + V = L$ with $\omega$ blocks of $\omega$-sequences of ordinals such that

(a) For each $i$, any finite sequence of ordinals $\alpha_1, \ldots, \alpha_m$ below the sup of the block $i$, and any finite sequence $u_{j_1}, \ldots, u_{j_k}$ from the block $i + 1$ and above,

$$N \models (\phi(\alpha_1, \ldots, \alpha_m, u_{j_1}, \ldots, u_{j_k}) \leftrightarrow \phi(\alpha_1, \ldots, \alpha_m, v_{j_1}, \ldots, v_{j_k}))$$

for any formula $\phi$ of the given bounded complexity, and any finite sequence $v_{j_1}, \ldots, v_{j_k}$ such that $|v_{j_k}| = |u_{j_k}|$ and the elements of the finite sequences $u_{j_i}$ and $v_{j_i}$ lie in the same block, say $j_i$, in which they are strictly increasing.

(b) For any element $a$ of any block, $L_a$ is an elementary substructure of $N$ w.r.t. formulae of the given bounded complexity.

Now take the model generated by these $\omega$ blocks of $\omega$-sequences. Then their sup is cofinal and the properties (a), (b) are preserved, because they are 1st order properties. Stretch the block 1 to $\mathbb{N}_0$, block 2 to $\mathbb{N}_{\kappa_0}$, block 3 to $\mathbb{N}_{2\kappa_0}$, etc. The model $K$ thus obtained satisfies the desired properties.

We construct $N$ as follows. Let $S = \{ \alpha | L_\alpha \preceq L \}$ w.r.t. formulae of the
given complexity \}. \(S\) is closed unbounded. Let \(a_\alpha\) be the \(\alpha\)th element of \(S\). Choose \(\omega\) increasing elements \(b_1, b_2, \ldots\), as follows. Let \(b_1 = a_\omega\), \(b_2 = a_{b_1}, \ldots, b_{i+1} = a_{b_i}, \ldots\). Observe that \(|S \cap (b_i, b_{i+1})| \geq b_i\). Now choose \(n\)-tuples from each of \(k\) sets of the form \(S \cap (b_i, b_{i+1})\) as follows. By induction on \(k\), as in the Erdős–Rado Theorem (for \(n\)-tuples for arbitrary partitions) for each \(\omega\)-sequence of subsets \(A_i \subseteq (b_{i-1}, b_i)\) with \(|A_i| = l\)th cardinal beyond \(b_{i-1}\), there exist subsets \(B_i \subseteq A_i\) such that \(|B_i| = (l - f(k))\)th cardinal beyond \(b_{i-1}\), and elements of \(B_i\)'s are indiscernibles for \((n, k)\)-tuples w.r.t. formulae of the given complexity. Here \(f: \omega \to \omega\) is a slow-growing function, say, \(f(k) \approx k^2n^2\).

Let \(M \models T^\omega\), and let \(K\) be a model of \(\text{ZFC} + V = L\) as described in Lemma 3.1. Within \(M\), define the nonextensional version \(M^*\), and the transition function \(F: M^* \to K\), as in Section 1. Define the Kripke structure whose objects are elements of \(M^*\), so that \(1 \models x \in y\) iff \(R(y, x)\), and \(2 \models y \in x\) iff \(K \models F(y) \in F(x)\).

**Lemma 3.2.** \(T^\omega\) proves that \(1 \models \text{Replacement}\).

**Proof.** \(M^*, K, \text{and } F\) are defined in \(M\). Suppose \(1 \models \forall x \in a. \exists! y A(x, y)\). Then for each \(x \in M^*\) with \(R(x, a)\), there exists a unique \(y \in M^*\) with \(1 \models A(x, y)\), and for each \(F(x) \in F(a)\) there exists a unique \(w \in K\) with \(K \models A(F(x), w)\). As in Lemma 1.3, \(w = F(y)\). By Replacement in \(K\), let \(v \in K\) be such that \(K \models \forall s \in F(a). \exists z \in v. A(s, z)\). By Lemma 3.1, \(\{y \in M^* \mid F(y) \in v\}\) is a set in \(M\), and hence an element of \(M^*\), along with its copies. By the properties of \(F\), there exists its copy \(u\) with \(v = F(u)\). Then \(1 \models \forall x \in a. \exists y \in u. A(x, y)\).

**Proof of Theorem 3.1.** Suppose \(\text{ZF}1_R\) proves \(Qx_1 \in \omega. Qx_2 \in \omega. \ldots Qx_n \in \omega. A\), where \(A\) is quantifier-free. Then this sentence is Kripke-forced at moment 1 in \((M^*, f, K)\), and so this sentence holds in \(M\).

**References**