Rooted directed path graphs are leaf powers

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\begin{abstract}
Leaf powers are a graph class which has been introduced to model the problem of reconstructing phylogenetic trees. A graph $G = (V, E)$ is called a $k$-leaf power if it admits a $k$-leaf root, i.e., a tree $T$ with leaves $V$ such that $uv$ is an edge in $G$ if and only if the distance between $u$ and $v$ in $T$ is at most $k$. Moreover, a graph is simply called a leaf power if it is a $k$-leaf power for some $k \in \mathbb{N}$. This paper characterizes leaf powers in terms of their relation to several other known graph classes. It also addresses the problem of deciding whether a given graph is a $k$-leaf power.

We show that the class of leaf powers coincides with fixed tolerance NeST graphs, a well-known graph class with absolutely different motivations. After this, we provide the largest currently known proper subclass of leaf powers, i.e., the class of rooted directed path graphs.

Subsequently, we study the leaf rank problem, the algorithmic challenge of determining the minimum $k$ for which a given graph is a $k$-leaf power. Firstly, we give a lower bound on the leaf rank of a graph in terms of the complexity of its separators. Secondly, we use this measure to show that the leaf rank is unbounded on both the class of ptolemaic and the class of unit interval graphs. Finally, we provide efficient algorithms to compute $2|V|$-leaf roots for given ptolemaic or (unit) interval graphs $G = (V, E)$.

\end{abstract}

\section{Introduction}

The broad field of phylogenetics is the study of the evolutionary relatedness of species. Nishimura, Ragde and Thilikos \cite{30} formalized a number of phylogenetic concepts to introduce the following graph theoretic notion: Given a finite simple graph $G = (V, E)$ and an integer $k \geq 2$, a tree $T$ is a $k$-leaf root of $G$ if $V$ can be identified as the set of leaves of $T$ and, for any two distinct vertices $x, y \in V, xy \in E$ if and only if the distance of $x$ and $y$ in $T$ is at most $k$. If such a tree exists, it can be viewed as an approximate evolutionary tree that captures the distance threshold $k$ of the species data. Moreover, $G$ is called a $k$-leaf power if it has a $k$-leaf root, and $G$ is a leaf power if it is a $k$-leaf power for some $k$. Furthermore, let the leaf rank $\text{lr}(G)$ of a leaf power $G$ be the smallest $k$ for which $G$ is a $k$-leaf power.

The general problem, from a graph theoretic point of view, is to structurally characterize the class of $k$-leaf powers, for any fixed $k \geq 2$. There has been a lot of work on $k$-leaf powers recently (see e.g. \cite{5,8,10,11,12,13,16,17,18,20}). It is known that leaf powers are strongly chordal; that is, sun-free chordal \cite{20}, but not vice versa (see e.g. \cite{5,6} for details). In \cite{16,31}, 3-leaf powers and in \cite{6,10,31}, 4-leaf powers are characterized in many ways. A similar description of 5-leaf powers seems much harder. However, \cite{8} characterizes distance-hereditary 5-leaf powers. To describe the structure of $k$-leaf powers for $k \geq 6$ is an open problem. The mutual containment of $k$-leaf power classes for varying $k$ is fully conceived by \cite{12}.

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In this paper we consider the whole class of leaf powers with known graph classes. Firstly, we show that leaf powers coincide with fixed tolerance NeST (neighborhood subtree tolerance) graphs, a notion introduced by Bibelnieks and Dearing [3]. Whilst this is not a hard result, it is interesting as it identifies two entirely differently motivated graph classes. This enables the transfer of known results between both classes. Secondly, we prove that rooted directed path graphs are leaf powers. This result provides the currently best known proper subclass of leaf powers and it implies and generalizes several results of [5].

An interesting question from an algorithmic perspective is to (quickly) recognize whether a given graph is a k-leaf power. Ideally, an algorithm would provide a k-leaf root as a certificate. The structural analysis of [6,16,31] and [6,10,31] led to linear time recognition algorithms for 3- and 4-leaf powers. A linear time recognition for 5-leaf powers is given in [13]. However, there are no efficient algorithms solving this challenge for any \( k \geq 6 \).

Here we study the leaf rank problem for subclasses of leaf powers. As a first step we provide a lower bound on the leaf rank of a strongly chordal graph in means of the structural complexity in the intersection between its clique separators. By the help of this measure we are able to show that the leaf rank is unbounded on both the class of ptolemaic and the class of (unit) interval graphs. Subsequently we show that any ptolemaic or interval graph \( G = (V, E) \) has a leaf rank of at most \( 2 |V| \). Obtaining this upper bound is not straightforward, but it seems to be much harder to determine the value of leaf rank. Accordingly we develop two efficient algorithms to compute a \( 2 |V| \)-leaf root for a given ptolemaic or interval graph, respectively. These algorithms are first steps in the algorithmic accomplishment of leaf power recognition and leaf rank optimization.

The paper is organized as follows. Following this introduction, we provide some basic notions and facts in Section 2. In Section 4 we present the methods used to prove our results. In Section 5 we give a lower bound on the leaf rank of strongly chordal graphs and we deal with the leaf rank problem for ptolemaic and (unit) interval graphs. We conclude with a summary and a discussion of open problems in Section 6.

2. Basic notions and facts

Throughout this paper, let \( G = (V, E) \) be a finite undirected graph without self-loops and multiple edges with vertex set \( V \) and edge set \( E \), and let \( |V| = n, |E| = m \). For a vertex \( v \in V \), let \( N(v) = \{ u \mid uv \in E \} \) denote the open neighborhood of \( v \) in \( G \), and let \( N[x] = N(x) \cup \{ x \} \). If \( xy \in E \) then we say that \( x \) sees \( y \), and if \( xy \notin E \) then we say that \( x \) misses \( y \).

A clique is a set of mutually adjacent vertices. An independent vertex set is a set of mutually nonadjacent vertices. A vertex is simplicial in \( G \) if its neighborhood \( N(v) \) is a clique. A vertex set \( M \subseteq V \) is a module if for all vertices \( z \in V \setminus M \), either \( z \) sees all vertices in \( M \) or misses all of them. Two vertices \( x, y \in V \) are true twins if \( N[x] = N[y] \), and they are false twins if \( N(x) = N(y) \).

For a subset \( U \subseteq V \), let \( G[U] = (U, E_U) \) denote the induced subgraph of \( G \) where \( E_U \) consists of all edges in \( E \) with both end vertices in \( U \). Let \( \mathcal{F} \) denote a set of graphs. A graph \( G \) is \( \mathcal{F} \)-free if none of its induced subgraphs is in \( \mathcal{F} \).

A sequence \( P = (v_1, \ldots, v_k) \) of pairwise distinct vertices is a (simple) path if for all \( i \) with \( 1 \leq i \leq k-1 \), \( v_i v_{i+1} \in E \). Moreover, \( P \) is an induced path if \( G[P] \) contains only these edges; such a path with \( k \) vertices will be denoted by \( P_k \), and its length \( |P| \) is \( k-1 \). An induced cycle \( C_k \) with \( k \) vertices \( v_1, \ldots, v_k \) has exactly the edges \( v_1 v_{k+1}, 1 \leq k < v_k v_1 \).

If \( P = (v_1, \ldots, v_k) \) and \( Q = (v_k, \ldots, v_1) \) are induced paths with \( \{ v_1, \ldots, v_{k-1} \} \cap \{ v_{k+1}, \ldots, v_1 \} = \emptyset \) and there are no edges between the vertices in \( \{ v_1, \ldots, v_{k-1} \} \) and those in \( \{ v_{k+1}, \ldots, v_1 \} \), then \( P \parallel Q \) denotes the concatenation of \( P \) and \( Q \), i.e., the induced path with vertex set \( \{ v_1, \ldots, v_k \} \).

A 2-connected component (or block) of \( G \) and a cut vertex of \( G \) are defined in the usual way. If \( xy \notin E \) then a vertex set \( S \) is an \( x \)-\( y \)-separator if \( x \) and \( y \) are in different connected components of \( G[V \setminus S] \). \( S \) is a minimal \( x \)-\( y \)-separator if it is an \( x \)-\( y \)-separator and minimal with respect to set inclusion. \( S \) is a (minimal) separator if it is a (minimal) \( x \)-\( y \)-separator for some \( x \) and \( y \).

The distance \( d_G(x, y) \) is the length of a shortest path in \( G \) between \( x \) and \( y \). For \( k \geq 1 \), the k-th power \( G^k \) of \( G = (V, E) \) is \( G^k = (V, E^k) \) with \( xy \in E^k \) if and only if \( d_G(x, y) \leq k \).

An edge-weighted graph \( G = (V, E, \omega) \) consists of a graph \( G = (V, E) \) and an edge weight function \( \omega : E \to \mathbb{R} \). Now the distance \( d_G(x, y) \) between any two nodes \( x \) and \( y \) is the minimum weight sum on any (not necessarily induced) path between \( x \) and \( y \).

Fig. 2 contains some graphs which are referred to in this paper.

For \( k \geq 3 \), a (complete) k-sun is a graph with 2k vertices \( v_1, \ldots, v_k \) and \( u_1, \ldots, u_k \) such that \( \{ v_1, \ldots, v_k \} \) is a clique, \( \{ u_1, \ldots, u_k \} \) is an independent set and for all \( i \in \{ 1, \ldots, k-1 \}, N(u_i) = \{ v_i, v_{i+1} \} \) and \( N(u_i) = \{ v_k, v_1 \} \). For convenience we call a graph sun if it is a k-sun for some \( k \geq 3 \). Fig. 2 shows the 3-sun.

In Section 4, we use the notion of k-planets given in [4]: For \( k \geq 4 \), a k-planet, denoted by \( L_k \), consists of an induced path \( (u_1, \ldots, u_k) \) and a triangle \( u_1, u_2, u_3 \) such that \( u_1 \) sees exactly \( v_1, \ldots, v_k-1 \) and \( u_2 \) sees exactly \( v_2, \ldots, v_k \). A 5-planet \( L_5 \) is displayed in Fig. 1.

A graph is
- **chordal** if it is \( C_k \)-free for any \( k \geq 4 \),
- **strongly chordal** if it is chordal and sun-free [20] (see also [9] for various characterizations of (strongly) chordal graphs).
It is easy to see that each of the above graph classes is properly contained in the preceding one. Fagin [18] describes the close relationship between chordal graphs and $\alpha$-acyclicity (strongly chordal graphs and $\beta$-acyclicity, ptolemaic graphs and $\gamma$-acyclicity, block graphs and Berge-acyclicity, respectively) motivated by desirable properties of relational database schemes.

Moreover we will also need the following graph classes:

- interval graphs,
- rooted directed path graphs,
- distance-hereditary graphs, and
- neighborhood subtree tolerance (NeST) graphs. In particular, we need the fixed tolerance NeST graphs.
A graph $G = (V, E)$ is an interval graph, if it has an intersection model of intervals on the real line, i.e., there exists a collection $I$ of intervals and a bijection $\phi : V \to I$ such that $uv \in E$ if and only if the two paths $\phi(u)$ and $\phi(v)$ intersect. Interval graphs are another important subclass of strongly chordal graphs (see e.g. [9]). If there is an intersection model $I$ in which all intervals have unit length, then their intersection graph $G$ is called a unit interval graph.

As it is well known, Gilmore and Hoffman [21] have shown that every interval graph has also an intersection model of subpaths $\mathcal{P}$ of a path $P$:

**Lemma 1.** The maximal cliques of an interval graph can be linearly ordered such that for each vertex $v$, the cliques containing $v$ occur consecutively.

Hence, interval graphs have a clique tree $P$ which is a path and thus, we will call $P$ a clique path. There is a bijection $\phi : V \to I$ such that $uv \in E$ if and only if the two paths $\phi(u)$ and $\phi(v)$ intersect, i.e., share a node of $P$.

A graph $G = (V, E)$ is a rooted directed path graph if it has an intersection model of subpaths $\mathcal{P}$ in a rooted directed tree $T = (N, A)$: $T$ has a root $r$, every path from $r$ to some leaf $v$ is directed from $r$ to $v$, and $\mathcal{P}$ is a collection of directed subpaths of $T$ with a bijection $\phi : V \to \mathcal{P}$, such that $uv \in E$ if and only if the paths $\phi(u)$ and $\phi(v)$ share a node, i.e., their node sets intersect. By [19], every rooted directed path graph is strongly chordal. For later reference, we state the following simple fact as a proposition.

**Proposition 1.** Every interval graph is a rooted directed path graph.

A graph $G$ is distance hereditary if for every connected induced subgraph $H$ of $G$, the distance function in $H$ is the same as in $G$. For various characterizations of distance-hereditary graphs see [2,25]. It is well known that a chordal graph is distance hereditary if and only if it is gem-free chordal (i.e., ptolomaic) (see [2,9,25,26]). Ptolemaic graphs $G$ were characterized by Bandelt and Mulder [2] in terms of three operations adding a new vertex $y$ to $G$ with respect to an existing vertex $x$:

- pendant vertex ($pv$): add $y$ adjacent only to $x$.
- true twin ($tt$): add $y$ as a true twin to $x$.
- restricted false twin ($rft$): add $y$ as a false twin to $x$ if $x$ is simplicial.

**Theorem 1 ([2]).** A graph is ptolomaic if and only if it can be obtained from a single vertex by recursively applying the operations $pv$, $tt$, and $rft$.

In this paper we will also use the following characterization of ptolomaic graphs found by Kloks [28].

**Lemma 2 ([28]).** A graph $G$ is ptolomaic if and only if all its connected induced subgraphs $H$ have the following property: If $H$ has no cut vertex then $H$ is a clique or contains true twins.

**Proof.** Obviously, $G$ is gem-free chordal if and only if its blocks are gem-free chordal. Thus, if every block of $G$ is a clique or contains true twins, then $G$ is gem-free chordal by Theorem 1.

For the other direction, suppose that the induced subgraph $H$ of $G$ is 2-connected, gem-free chordal, and is not a clique. Let $x, y$ be vertices which miss each other, and let $S_{xy}$ be a minimal $x - y$-separator. Since $H$ is chordal, $S_{xy}$ is a clique, and since $H$ is 2-connected, $|S_{xy}| \geq 2$. We claim that $S_{xy}$ is a module (this easily follows by the fact that $H$ is 2-connected and gem-free). Then any two vertices in $S_{xy}$ are true twins in $H$. □

Dahlhaus [14] showed that ptolomaic graphs are rooted directed path graphs. In order to keep this paper self-contained as much as possible, we give a proof of this fact.

**Lemma 3.** Every ptolomaic graph is a rooted directed path graph but not vice versa.

**Proof.** We use structural induction on the recursive construction in Theorem 1. Trivially, a single vertex is a rooted directed path graph. Now let $G' = (V, E)$ be ptolomaic with rooted directed path model $(T', \mathcal{P}', \phi')$ and let $x \in V$. Let $G = G' + y$ denote the resulting graph by adding a new vertex $y$ to $G'$ in one of the three ways as in Theorem 1.

Moreover let $t$ be the terminal node of path $P_x = \phi'(x)$ in $T'$. If a vertex $y$ is attached to $x$ as $pv$ then we obtain a model $(T, \mathcal{P}, \phi)$ by adding a new branch $tu, uv$ at node $t$, extending $P_x$ to $tu, uv$ and creating a path $P_y = \phi(y) = uv$ on the new branch, $tt$ then $(T, \mathcal{P}, \phi)$ yields from extending $\mathcal{P}'$ with a copy $P_y = \phi(y)$ of path $P_x$, $rft$ then $N(x)$ is a clique and we can assume that $t \in \phi'(c)$ for all $c \in C$ and $t \notin \phi'(r), r \in V \setminus C$. Otherwise we may extend the relevant paths until they reach $t$. Now we obtain $(T, \mathcal{P}, \phi)$ if we replace node $t$ in $T'$ and all paths except $P_x$ by two new edges $tu, uv$ and add $P_y = \phi(y) = uv$.

Hence, $G = G' + y$ is a rooted directed path graph.

The gem is a rooted directed path graph (it is even a unit interval graph) but not ptolomaic. □
Subsequently, we will also need the following notions: A leaf root $T$ is basic if at most one leaf is attached to each internal node of $T$, and a leaf power $G$ is basic if it has a basic leaf root. Obviously, any set of leaves with the same parent is a clique module whenever $k \geq 2$, and thus, every $k$-leaf power $G$ results from a basic $k$-leaf power $G'$ by substituting cliques into the vertices of $G'$. Moreover, by [11] a graph $G = (V, E)$ is a $(k, \ell)$-leaf power if it has a $(k, \ell)$-leaf root, i.e., a $k$-leaf root $T$ such that all $xy \notin E$ fulfill $d_T(x, y) \geq \ell$. The following theorem relates to strictly chordal graphs, block graphs and a very small subclass of leaf powers:

**Theorem 2 ([11]).** For graph $G$, the following conditions are equivalent:

(i) $G$ is strictly chordal, i.e., (dart, gem)-free chordal;

(ii) $G$ is a $(4, 6)$-leaf power;

(iii) $G$ results from a block graph by substituting cliques into its vertices.

Obviously, the leaf rank of block graphs and strictly chordal graphs is at most four. We conclude this section with useful facts on leaf powers (see e.g. [6,10]):

**Proposition 2.** (i) Every induced subgraph of a $k$-leaf power, $k \geq 2$, is a $k$-leaf power.

(ii) A graph is a $k$-leaf power if and only if each of its connected components is a $k$-leaf power.

(iii) Graph $G$ is a basic $(k + 2)$-leaf power if and only if $G$ is an induced subgraph of the $k$-th power $T^k$ of a tree $T$.

(iv) For every $k \geq 1$, graph $G$ is a $k$-leaf power if and only if $G$ results from a basic $k$-leaf power $G'$ by substituting cliques into the vertices of $G'$.

### 3. Leaf powers coincide with fixed tolerance NeST graphs

Neighborhood subtree tolerance (NeST) graphs were introduced by Bibelnieks and Dearing [3] and were also studied in [23,24]. In this section we show that fixed tolerance NeST graphs and leaf powers are exactly the same graph class. We avoid defining NeST graphs and instead use the characterization of fixed tolerance NeST graphs as neighborhood subtree intersection graphs which has been proved in [23] (see also a corresponding remark in [24]):

**Theorem 3 ([23]).** A graph $G = (V, E)$ is a fixed tolerance NeST graph if and only if there is a positive constant $k > 0$ and an undirected weighted tree $T = (N, A, \omega)$ with $V \subseteq N$ and positive weights $\omega : A \rightarrow \mathbb{R}$ on the edges such that for all $u, v \in V$

$$uv \in E \iff d_T(u, v) \leq k.$$  

Note that in the original formulation of Theorem 3, instead of using positive weights $\omega$ in trees, in [3,23,33], trees are embedded into the plane (and the length $d_T(u, v)$ of the path between $u$ and $v$ is given by their distance in the plane embedding of $T$); in fact, these two models are equivalent. For us it is simpler to consider weighted undirected trees. We will refer to $k$ in Theorem 3 as the diameter of the representation, due to the original characterization of neighborhood subtree intersection graphs.

By the results of [3] on fixed tolerance NeST graphs (called constant NeST graphs in [3]), we know that these graphs are properly contained in the class of strongly chordal graphs. However, characterizing fixed tolerance NeST graphs is an open problem.

In Theorem 4 we show that fixed tolerance NeST graphs and leaf powers are exactly the same graph class. Besides relating two entirely differently motivated graph classes, this will simplify some proofs in the rest of the paper. Theorem 3 obviously implies that leaf powers are fixed tolerance NeST graphs but the opposite implication, although not too difficult, requires some careful argumentation.

**Theorem 4.** A graph is a fixed tolerance NeST graph if and only if it is a leaf power.

**Proof.** (1) "$\Rightarrow$": Let $G = (V, E)$ be a $k$-leaf power for some $k$, and let $T = (N, A)$ be a $k$-leaf root of $G$, i.e.:

1. $T$ is an undirected tree without edge weights; let $\omega(e) = 1$ for all $e \in A$.
2. The set of leaves of $T$ is the vertex set $V$ of $G$.
3. $uv \in E$ if and only if $d_T(u, v) \leq k$.

Clearly $T = (N, A, \omega, k)$ is also a neighborhood subtree representation of $G$, and therefore $G$ is a fixed tolerance NeST graph by Theorem 3.

(2) "$\Leftarrow$": For the direction we need to turn the neighborhood subtree representation $T = (N, A, \omega, k)$ of a given fixed tolerance NeST graph $G = (V, E)$ into a leaf root. That is, we need to remove the real weights on the edges, keeping the distance relations unaltered. For that we define some constants to make the proof more readable. Let $a = \min_{e \in A} \omega(e)$ and $b = \min_{v \in V} \min_{x, y \notin E} d_T(x, y) - k$. Moreover, we choose $c = \min\{a, b\}$, namely, the smallest distance we should subtract between two nodes of $V$, so that the corresponding vertices of $G$ can become adjacent. Notice that $a, b, c > 0$. 

Now we are ready to argue our statement. For each edge \( e \in A \), replace \( \omega(e) \) with \( \omega'(e) = \left\lfloor \frac{\omega(e) \cdot |A| + 1}{c} \right\rfloor \). This defines a new tree \( T'' = (N', A, \omega') \) where all weights are integers and different from zero. Then we define the new diameter \( k' \) with \( k' = k \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor \). We call \( P_{uv} \) the unique path between two nodes \( u, v \) in \( T \) or \( T' \), respectively.

First, for any \( u, v \in V \) we show that \( d_{T''}(u, v) > k' \) if \( d_{T}(u, v) > k \). By definition we have that, if \( d_{T}(u, v) > k \), then \( d_{T}(u, v) \geq k + b \). Then, since every path has at most \( |A| \) edges we get

\[
d_{T''}(u, v) = \sum_{e \in P_{uv}} \omega'(e) = \sum_{e \in P_{uv}} \left\lfloor \frac{\omega(e) \cdot |A| + 1}{c} \right\rfloor
g \geq \sum_{e \in P_{uv}} \left\lfloor \frac{\omega(e) \cdot |A| + 1}{c} \right\rfloor - |A| = d_{T}(u, v) \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor - m
\geq (k + b) \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor - m = k' + b \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor - m > k'.
\]

Conversely, if \( d_{T}(u, v) \leq k \), we show that \( d_{T''}(u, v) \leq k' \). This follows from

\[
d_{T''}(u, v) = \sum_{e \in P_{uv}} \omega'(e) = \sum_{e \in P_{uv}} \left\lfloor \frac{\omega(e) \cdot |A| + 1}{c} \right\rfloor \leq \sum_{e \in P_{uv}} \omega(e) \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor \leq d_{T}(u, v) \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor \leq k \cdot \left\lfloor \frac{|A| + 1}{c} \right\rfloor = k'.
\]

Now we can construct an unweighted tree \( T''' \) from \( T' \) by the following three operations:

1. Construct from \( T' \) the tree \( T'_1 = (N'_1, A'_1) \) by replacing each edge \( uv \in A \) with an unweighted path of \( \omega'(e) \) edges starting at \( u \) and ending in \( v \).
2. Derive from \( T'_1 \) the tree \( T'_2 = (N'_2, A'_2) \) by replacing each node \( v \in V \) with a node \( v' \) and adding the edge \( v'v \) to \( A'_1 \).
3. Take from \( T'_2 \) the connected subtree \( T''' = (N'', A'') \) which is spanned by \( V \).

For vertices \( u, v \in V \) the distance functions fulfill \( d_{T'''}(u, v) = d_{T''}(u, v) + 2 \). Finally, to obtain an integral diameter, we take

\[
k'' = 2 + \max_{u, v \in V} \{ d_{T''}(u, v) \mid d_{T''}(u, v) \leq k' \}.
\]

Since all distances are integers, the new diameter \( k'' \) is an integer and for all \( u, v \in V \) it is true \( d_{T'''}(u, v) \leq k'' \) if and only if \( d_{T''}(u, v) \leq k' \). Thus, \( G \) is a \( k'' \)-leaf power with \( k'' \)-leaf root \( T''' \). \( \Box \)

By Theorem 4, all known results for fixed tolerance NeST graphs will now apply to leaf powers and vice versa. It has been shown in [3] that fixed tolerance NeST graphs are strongly chordal but not vice versa (see Fig. 1 for an example of a strongly chordal graph which is not a fixed tolerance NeST graph). Therefore, we obtain the following

**Proposition 3. Leaf powers are a proper subclass of strongly chordal graphs.**

It follows from Theorem 4 that proper subclasses of fixed tolerance NeST graphs such as interval graphs [3] and unit interval graphs, are also proper subclasses of leaf powers which will be discussed in more detail in Section 5.4.

### 4. Rooted directed path graphs are leaf powers

Recall from Section 1 that the class of leaf powers is properly contained in the class of strongly chordal graphs. In order to compare the class of leaf powers with known graph classes, other subclasses of strongly chordal graphs are of interest. Recall from Section 2 that the class of rooted directed path graphs is such a subclass. We are going to show that the class of rooted directed path graphs is a proper subclass of the class of leaf powers and, thereby, is the largest currently known graph class of that kind. Implied consequences are briefly discussed at the end of this section.

**Theorem 5. Every rooted directed path graph is a leaf power but not vice versa.**

**Proof.** Let \( G = (V, E) \) be a rooted directed path graph and let \( T = (N, A) \) be a rooted directed tree representation of \( G \) with root \( r \in N \), a collection of directed paths \( P \) in \( T \) and a bijection \( \phi : V \to P \), such that, for all distinct vertices \( u, v \in V \), we have \( uv \in E \) if and only if the node sets of \( \phi(u) \) and \( \phi(v) \) intersect. In order to show that \( G \) is a leaf power we construct a leaf root \( T'' \) of \( G \) which is based on the rooted directed tree representation \( T \) of \( G \).

In a first step construct a new directed tree \( T'' = (N', A') \) which replaces every directed edge \( (x, y) \) of \( T \) by a directed path \( P_{xy} = (x, u_1, \ldots, u_{2^{|(x,y)|}-1}, y) \) of length \( 2^{|(x,y)|} \), i.e., it subdivides the edge correspondingly by new nodes. Moreover, we construct the adapted set \( P' \) of paths in \( T'' \) and the corresponding bijection \( \phi' \) as follows: For each vertex \( v \in V \) gain from path \( \phi(v) = (u_1, u_2, u_3, \ldots) \) of \( P \) a new path \( \phi'(v) = P_{u_1u_2} \parallel P_{u_2u_3} \parallel \ldots \) of \( P' \). Roughly speaking, we obtain stretched
The classes of ptolemaic graphs and of interval graphs are properly contained in the class of leaf powers.

5. The leaf rank problem

Recall that the leaf rank \( lr(G) \) of a leaf power is the smallest integer \( k \) such that \( G \) has a \( k \)-leaf root. If \( G \) is not a leaf power then \( lr(G) = \infty \). For a class \( \mathcal{G} \) of leaf powers, let \( lr(\mathcal{G}) \) be the maximum \( lr(G) \) over all \( G \in \mathcal{G} \), and \( lr(\mathcal{G}) = \infty \) if it is unbounded.

It seems to be a very challenging problem to determine the leaf rank for leaf powers as well as for subclasses of them. It is even not clear how to compute \( k \)-leaf roots for leaf powers \( G \) such that \( k = \text{as close as possible to} \) \( lr(G) \), i.e., in the best case the value \( k = lr(G) \).

Thus, for rooted directed path graphs, the proof of Theorem 5 gives \( k \)-leaf roots with exponential \( k \). We leave it as an open problem to determine better upper bounds on their leaf rank.

Subsequently we will show that for ptolemaic and interval graphs, the leaf ranks are unbounded. Hence, the leaf rank of rooted directed path graphs is unbounded, too.
5.1. Unit interval graphs

For the next result we implicitly use the concept of clique-width which will not be defined here (see e.g. [22] for details).

**Proposition 4.** The leaf rank of unit interval graphs is unbounded.

**Proof.** Golubic and Rotics [22] showed that unit interval graphs have unbounded clique-width. By a result of Todinca [34], for every fixed $k$, the class of $k$-leaf powers has bounded clique-width since $k$-th powers of a graph class of bounded clique-width (such as trees) have bounded clique-width.

Assuming that unit interval graphs have bounded leaf rank would thus imply bounded clique-width for unit interval graphs, which is a contradiction. The claim follows. □

Since unit interval graphs are interval graphs and rooted directed path graphs, these two classes also have unbounded leaf rank. Another consequence of the fact that unit interval graphs are leaf powers, is the following:

**Corollary 2.** Leaf powers have unbounded clique-width.

Unit interval graphs are leaf powers with caterpillars as leaf roots: A *caterpillar* $T$ is a tree consisting of a path (the backbone of $T$) and some leaves attached to the backbone. It turns out that this characterizes unit interval graphs:

**Theorem 6.** For a graph $G$, the following conditions are equivalent:

(i) $G$ has a leaf root which is a caterpillar.

(ii) $G$ is an induced subgraph of the power of some induced path.

(iii) $G$ is a unit interval graph.

**Proof.** Let $G = (V, E)$ be a graph.

(i) $\implies$ (ii): Let caterpillar $T$ be a $k$-leaf root of $G$ for some $k \geq 2$ and let $B$ be the backbone path of $T$. First we assume that every leaf $v$ of $T$ has a unique parent node $b_v$ on $B$. Now, $uv \in E$ if and only if $d_T(u, v) \leq k$ if and only if $d_B(b_u, b_v) \leq k - 2$. This shows that $G$ is an induced subgraph of $B^{k-2}$.

If the number of leaves which share parent nodes is positive, then the repeated execution of the following procedure gives caterpillar leaf roots which successively reduce this number: If $u$ and $v_1, \ldots, v_t$ have the same parent $b$, then subdivide every edge on $B$ except those adjacent to $b$. We obtain a caterpillar $T'$ after we replace $b$ by an edge $b_1b_2$ and attach $u$ to $b_1$ and $v_1, \ldots, v_t$ to $b_2$. For leaves $x$ on the left of $b_1$ the $T'$-distance to $u$ is now $d_{T'}(x, u) = 2d_T(x, u) - 3$ and for leaves $y$ on the right of $b_2$ it is $d_{T'}(y, u) = 2d_T(y, u) - 2$. The same holds mirror-inverted for $v_1, \ldots, v_t$. Moreover, $d_{T'}(x, y) = 2d_T(x, y) - 3$. Then $T'$ is a caterpillar $(2k-2)$-leaf root of $G$ with less parent node collisions.

(ii) $\implies$ (iii): Assume that $G$ is an induced subgraph of the $k$-th power of some path. In [29, Theorem 3.8], says that a graph is a proper interval graph if and only if it is a unit interval graph, and Theorem 3.10 says that $G$ is a proper interval graph if and only if the vertex-maxclique incidence matrix $M(G)$ has the consecutive ones property for both rows and columns (which is mentioned in [15] and goes back to [32]). Obviously, powers of paths have the last property, and the property is hereditary for induced subgraphs.

(iii) $\implies$ (i): Let $G = (V, E)$ be a unit interval graph with interval model $(I_v)_{v \in V}$, and let $m_v$ denote the midpoint of interval $I_v$. Without loss of generality, we can assume that the midpoints have rational but not necessarily distinct values. Let $M$ be the least common multiple of the denominators of $m_v$, $v \in V$. By definition, $uv \in E$ if and only if $|m_u - m_v| \leq 1$. Thus, by multiplying the midpoints by $M$, we obtain a path $B$ containing $n$ midpoints as nodes $m'_u$ such that $uv \in E$ if and only if $d_B(m'_u, m'_v) \leq M$. Note that two midpoints $m'_u$ and $m'_v$ for, $u, v \in V$ may be the same node on $B$. The path $B$ is the backbone of a caterpillar $T$ where we attach a leaf $v$ to midpoint $m'_u$ such that $uv \in E$ if and only if $d_T(u, v) \leq M + 2$. □

Recall that Proposition 4 states that the leaf rank is unbounded on the class of unit interval graphs. We can be more specific by showing the value of leaf rank of some unit interval graphs to be roughly half the number of the vertices. Theorem 2 of [12] states that, for every $k \geq 3$, $P_{2k-3}^{k-2}$, the $(k-2)^{nd}$ power of the path on $2k-3$ vertices, is a $k$-leaf power which is not a $k'$-leaf power, for any $2 \leq k' < k$; that is, the leaf rank of $P_{2k-3}^{k-2}$ is precisely $k$. By Theorem 6, $P_{2k-3}^{k-2}$ is a unit interval graph, and we obtain the following proposition.

**Proposition 5.** For every $k \geq 3$, $P_{2k-3}^{k-2}$, the $(k-2)^{nd}$ power of the path on $2k-3$ vertices, is a unit interval graph with leaf rank $k$. 
5.2. A lower bound for leaf rank

Recall from Section 2 the definition of separators. The following notion of separator depth leads to a lower bound for leaf rank \( \text{lr}(G) \):

**Definition 1.** Let \( G \) be a strongly chordal graph and \( \mathcal{S} \) the set of minimal clique separators of \( G \). We say that \( G \) has separator depth \( d \) if there exists a sequence \( S_1, S_2, \ldots, S_d \) of separators from \( \mathcal{S} \) such that for all \( i \in \{1, \ldots, d-1\} \), \( S_i \subseteq S_{i+1} \) holds and there is no longer sequence of this kind. If \( \mathcal{S} \) is empty then \( G \) has a separator depth of zero.

By the new notion and the following lemma we can easily get a rough lower bound on the leaf rank of a strongly chordal graph:

**Lemma 4.** A strongly chordal graph \( G = (V, E) \) of separator depth \( d \) has a leaf rank of at least \( \text{lr}(G) \geq d + 2 \).

**Proof.** Clearly, we may assume that \( G \) is a leaf power. If \( d = 0 \) then \( G \) is a collection of cliques and every \( k \)-leaf root of \( G \) must have \( k \geq 2 \).

Otherwise, by definition, there is a sequence \( S_1, S_2, \ldots, S_d \) of separators such that, for all \( i \in \{1, \ldots, d-1\} \), \( S_i \subseteq S_{i+1} \).

Since \( S_d \) is a separator it is a subset of another clique which we denote \( S_{d+1} \) for convenience. In \( G \) there must be vertices \( x_1, \ldots, x_d, x_{d+1} \) with \( x_1 \in S_1 \) and \( x_i \in S_i \setminus S_{i-1} \), for all \( 2 \leq i \leq d+1 \). Moreover, because all \( S_i, i \in \{1, \ldots, d\} \), are separators, there exists an independent set of vertices \( y_1, \ldots, y_d \) in \( G \) such that, for all \( i \in \{1, \ldots, d\} \), the vertex \( y_i \) is adjacent to \( x_1, \ldots, x_i \) but not to \( x_{i+1}, \ldots, x_{d+1} \).

Take any \( k \)-leaf root \( T \) of \( G \) with \( k \in \mathbb{N} \). We will show by induction on \( d' \in \{1, \ldots, d\} \) that there are \( i, j \in \{1, \ldots, d' + 1\} \) such that \( d_T(x_i, x_j) \geq d' + 2 \).

Let \( d' = 1 \). If \( d_T(x_1, x_2) = 1 \) then \( y_1 \) would not be connected to \( T \) because \( x_1 \) and \( x_2 \) are leaves. If \( d_T(x_1, x_2) = 2 \) then \( x_1 \) and \( x_2 \) have the same parent node and would become a module in \( G \). However, they are not because \( y_1 \) distinguishes them. Hence, \( d_T(x_1, x_2) \geq 3 \).

Now assume \( d' > 1 \). By induction hypothesis there are \( i, j \in \{1, \ldots, d'\} \) such that \( d_T(x_i, x_j) \geq d' + 1 \). There are paths \( P \) from \( x_i \) to \( x_j \) and \( Q \) from \( x_d \) to \( x_d' \). Then, \( |P| \geq d' + 2 \) and \( |Q| > k \).

We will show that the distance between \( x_d \) and either \( x_i \) or \( x_j \) is at least \( d' + 2 \). For that we consider the following two cases:

Case 1. \( P \) and \( Q \) are disjoint: Because \( T \) is a tree there is a path \( P \) connecting \( P \) and \( Q \). Then one can subdivide \( P = P_1 \parallel P_2 \) and \( Q = Q_1 \parallel Q_2 \) such that \( Q_1 \parallel R \parallel P_1 \) is the path from \( x_d \) to \( x_i \) and \( Q_1 \parallel R \parallel P_2 \) the path from \( x_d \) to \( x_j \). Assume \( |P_1| \geq |Q_1| \). Then, because \( d_T(x_d, y_d) > k \) and \( |R| > 1 \), we obtain \( d_T(x_i, y_d) > k \), a contradiction. Hence, \( |P_1| < |Q_1| \). Now \( d_T(x_i, x_{d+1}) \) is clearly greater than \( d_T(x_i, x_j) \), and we are done.

Case 2. \( P \) and \( Q \) share a subpath \( R \): Then one can subdivide \( P = P_1 \parallel R \parallel P_2 \) and \( Q = Q_1 \parallel R \parallel Q_2 \). Without loss of generality, we may assume that \( Q_1 \parallel R \parallel P_2 \) is the \( T \)-path from \( x_d \) to \( x_i \) and \( Q_1 \parallel P_1 \) is the \( T \)-path from \( x_d \) to \( x_j \). Because \( d_T(x_d, y_d) > k \) and \( d_T(x_i, y_d) \leq k \), we obtain \( |P_1| < |Q_1| \). Now, as in the previous case, \( d_T(x_i, x_{d+1}) \) is greater than \( d_T(x_i, x_j) \), and we are done.

For \( d' = d \), we get two adjacent vertices \( x_i \) and \( x_j \) in \( G \) with \( d_T(x_i, x_j) \geq d + 2 \), implying \( k \geq d + 2 \), which finishes the proof.

5.3. Leaf rank of ptolemaic graphs

By **Corollary 1**, ptolemaic graphs are leaf powers. In [22], it was shown that the clique-width of ptolemaic graphs is at most three and hence, it is a natural question to ask whether their leaf rank is also bounded. The lower bound in **Lemma 4**, however allows to conclude:

**Corollary 3.** Ptolemaic graphs have unbounded leaf rank. In particular, for any \( i \geq 1 \), there is a ptolemaic graph \( G_i \) with \( 2i + 1 \) vertices and \( \text{lr}(G_i) \geq i + 2 \).

**Proof.** Let \( G_1 = (V = \{x_1, x_2, y_1\}, E = \{x_1x_2, x_1y_1\}) \). Obviously \( G_1 \) is a ptolemaic graph with the clique \( \{x_1, x_2\} \) and simplicial vertex \( x_2 \). Its separator depth is one by the only separator \( S_1 = \{x_1\} \).

We use structural induction to obtain ptolemaic graphs \( G_i \) from \( G_1 \), with \( 2i + 1 \) vertices and separator depth \( i \). Then we are done by **Lemma 4**.

For all \( i > 1 \), we may assume that the induction hypothesis is true, i.e., \( G_{i-1} \) is a ptolemaic graph with clique \( C_{i-1} = \{x_1, \ldots, x_{i-1}\} \) and vertex \( x_i \) simplicial by its adjacency only to \( C_{i-1} \). Moreover, the separator depth of \( G_{i-1} \) is \( i - 1 \).

Now we use two of the three operations allowed for ptolemaic graphs to obtain the new graph \( G_i \): We

1. add a true twin \( y_i \) of \( x_i \) and
2. add a false twin \( x_{i+1} \) of \( y_i \).
Vertex $y_i$ becomes adjacent to the clique $C_i = C_{i-1} \cup \{x_i\}$ and thus, it is simplicial, too. Since $y_i$ is simplicial, it is allowed to add $x_{i+1}$ as a restricted false twin to $y_i$, which also means that $x_{i+1}$ becomes simplicial by the adjacent clique $C_i$. Hence, $G_i$ is ptolemaic with clique $C_i$ and simplicial vertex $x_{i+1}$.

It remains to show that $G_i$ has separator depth $i$. First, since $G_i$ introduces no edges between vertices of $G_{i-1}$ it takes over all minimal separators $S_1 = \{x_1\}, S_2 = \{x_1, x_2\}, \ldots, S_{i-2} = \{x_1, \ldots, x_{i-1}\}$ contained already in $G_{i-1}$. However, $S_i = \{x_1, \ldots, x_i\}$ is a new minimal separator because it cuts between the two false twins $y_i$ and $x_{i+1}$. The claim follows.

Subsequently, we give the simple algorithm depicted in Fig. 3 to compute a $k$-leaf root for any given ptolemaic graph $G = (V, E)$ where $k$ is only linear in $|V|$. Basically the algorithm uses Lemma 2 to split the problem recursively. This means that the input graph is either a clique, contains a cut vertex or a pair of true twins. For the trivial case of a clique, the algorithm performs no recursion and simply returns the leaf root which has the shape of a star. Otherwise the algorithm searches for true twins $x, y$ in $G$ and computes recursively a leaf root $T'$ for $G - \{y\}$. Since $x$ and $y$ behave the same way it is easy to obtain a leaf root $T$ for $G$ from $T'$ by simply attaching $y$ to the parent of $x$. If $G$ is neither a clique nor contains true twins then it has to have a cut vertex $c$ by Lemma 2. In that case the graph $G$ is split down into its components $G'_i$ to $G'_c$ adjacent to $c$ and the algorithm computes recursively their leaf roots $T'_i$ to $T'_c$. Since every tree $T'_i$ to $T'_c$ contains a leaf corresponding to $c$ the algorithm can merge the branches adjacent to $c$ to obtain a single leaf root $T$ for $G$.

The details of constructing a $(2|V|, 2|V| + 2)$-leaf root $T = (N, A)$ of a ptolemaic graph $G = (V, E)$ are given in Fig. 3 and in the proof of the following theorem.

**Theorem 7.** Every ptolemaic graph $G = (V, E)$ is a basic $(2|V|, 2|V| + 2)$-leaf power, and a basic $(2|V|, 2|V| + 2)$-leaf root $T = (N, A)$ of $G$ can be obtained by Algorithm PtolemaicLeafRoot depicted in Fig. 3.

**Proof.** Let $G = (V, E)$ be a ptolemaic graph and $k \geq |V|$. We start by showing that the function call $T = (N, A, \omega) = \text{NeSTModel} (G = (V, E), k)$ gives a fixed tolerance NeST model $(T, k)$ for $G$ where

1. $\omega$ contains only integer weights,
2. the set of $T$’s leaves is exactly $V$, and
3. for any leaf $v \in V$ it is true that $k - |V| + 2 \leq \omega(p_v, v) \leq k$, where $p_v$ is the parent node of $v$ in $T$.

This is trivially true for the base cases of $G$ being a clique. Otherwise, if $G$ contains a pair $x, y$ of true twins, then by induction hypothesis $(T', k)$ is a fixed tolerance NeST model for $G' = G[V \setminus \{y\}]$ which fulfills the above weight criterion. Now consider the tree $T$. Since $\omega(px) = \omega(py) \leq k$, it follows that $x$ and $y$ see each other and have equal $T$-distances to all other leaves. Therefore, $(T, k)$ is a fixed tolerance NeST model for $G$ which fulfills the criteria (even for $y$).

**cut vertex $c$.** Then by induction hypothesis $(T'_i, k), \ldots, (T'_c, k)$ are fixed tolerance NeST models for the components $G_1 = (V_1, E_1), \ldots, G_\ell = (V_\ell, E_\ell)$ of $G$ incident to $c$ which all fulfill the given criteria. In particular, for all $i \in \{1, \ldots, \ell\}$ and all leaves $v$ in $V_i$ with parent $p$ it is true $k - |V_i| + 2 \leq \omega_i(p_v, v) \leq k$. Now consider the tree $T$. Trivially, for all $i \in \{1, \ldots, \ell\}$ the $T$-distance between nodes $u, v \in V_i \setminus \{c\}$ equals the $T'_i$-distance between $u$ and $v$.

For $(T, k)$ being a fixed tolerance NeST model for $G$ we need to show that for all $1 \leq i < j \leq \ell$, all $u \in V_i \setminus \{c\}$ and all $v \in V_j \setminus \{c\}$ the $T$-distance between $u$ and $v$ is at least $2k + 2$. Since $(|V_u| + |V_v|) \geq (|V_u| + |V_v|)$ it follows:

$$d_T(u, v) = d_T(u, p_v) + \omega(p_v, p) + \omega(p, v) + d_T(p_v, v) \geq \omega(p_v, p) + \omega_i(p_c, v) + \omega_i(p_v, v) - 2(k - (|V_u| + |V_v|) + 3) \geq 2(k - |V_i| + 2) + 2(k - |V_j| + 2) - 2(k - (|V_u| + |V_v|) + 3) = 2k - 2(|V_i| + |V_j|) + 2(|V_u| + |V_v|) + 2 \geq 2k + 2.$$

Moreover, $T$ fulfills the weight criterion for pendant edges since $\omega(pc) = k - (|V_u| + |V_v|) + 3 \geq k - |V| + 2$ which follows from $(|V_u| + |V_v|) \leq |V|.$

Hence, the function call of Row 30 gives a fixed tolerance NeST model $(T' = (N', A', \omega'), 2|V|)$ for any given ptolemaic graph $G = (V, E)$. Moreover, it is true for all leaves $v \in V$ that $2 \leq \omega'(p_v, v) \leq |V|$. Then $T$ is obtained from $T'$ by replacing all edges $e$ by a path of length $\omega(e)$. Obviously $T$ is a $(2|V|, 2|V| + 2)$-leaf root. However, $T$ is also basic since all pendant edges of $T'$ have a length of at least two and thus, every leaf of $T$ is adjacent to a unique parent node.

Now, by Theorem 7, we know that ptolemaic graphs have relatively “compact” leaf roots. It is quite easy to check that the algorithm of Fig. 3 runs in polynomial time with respect to the number of nodes in $G$. We can observer that faster algorithms exist, but our intention was to show an intuitive approach.

Unfortunately, the computed leaf root is not always optimal with respect to the leaf rank of $G$. In fact, in comparison with the lower bound in Lemma 4, the algorithm does not perform well. In particular, cliques with an arbitrary number of vertices have a separator depth of zero; they have 2-leaf roots but our algorithm still computes $2|V|$-leaf roots. Consequently, we state as an open problem to find an algorithm which computes leaf rank-optimal leaf roots for ptolemaic graphs.
Fig. 3. An algorithm which computes a basic \((2|V|, 2|V| + 2)\)-leaf root for given Ptolemaic graph \(G = (V, E)\). It recursively computes a fixed tolerance NeST model with \(V\) as its set of leaves. The main procedure replaces the integer weights by paths of corresponding length.

5.4. Leaf rank of interval graphs

The ways of computing leaf roots from the fixed tolerance neighborhood model or the rooted directed path model of graphs \(G\) as described in the proofs of Theorems 4 and 5 are not satisfying enough when it comes to interval graphs. The result depends heavily on the particular model and due to the use of arbitrary real numbers it may be far away from the optimum with respect to the leaf rank of \(G\). Therefore in this section, Fig. 4 presents a simple algorithm based on the following idea:

According to Lemma 1 we can obtain an integer interval model for input graphs \(G\). Starting form an integer center point every interval spreads an equal number of integer units in both directions of the real line. We find a leaf root \(T\) for \(G\) by simulating this behavior. Firstly, a path becomes the backbone of the leaf root to resemble the real line by representing each
Algorithm IntervalLeafRoot /* Leaf root for interval graphs */
Input: Interval graph $G = (V, E)$.
Output: $(2|V|, 2|V| + 2)$-leaf root $T = (N, A)$ of $G$.

1. begin /* Main() for leaf root computation */
2. compute a clique path $P = C_1, C_2, \ldots, C_{\ell}$;
3. for all $v \in V$ do begin
4. identify the subpath $P_v = C_{s(v)} C_{s(v)+1} \ldots C_{s(v)+\ell(v)}$ induced by $v$;
5. let $I(v) = 2s(v) + \ell(v)$;
6. end;
7. let $L = \max\{\ell(v) : v \in V\}$;
8. compute a list $L = v_1, \ldots, v_n$ of all vertices in $V$ sorted by $I(v)$;
9. compute $N' = \{p_1, \ldots, p_n, v_1, \ldots, v_n\}$;
10. compute $A' = \{p_1 p_2, \ldots, p_{n-1} p_n, p_1 v_1, \ldots, p_n v_n\}$;
11. for $i = 1$ to $n - 1$ do begin
12. let $\omega'(p_i p_{i+1}) = I(v_{i+1}) - I(v_i)$;
13. let $\omega'(p_i v_i) = L - \ell(v_i) + 1$;
14. end;
15. let $\omega'(p_n v_n) = L - \ell(v_n) + 1$;
16. let $T' = (N', A', \omega')$; /* get weighted leaf root */
17. let $N = N'$ and $A = \emptyset$;
18. for all $uv \in A'$ do /* replace weight of $uv$ by $\omega(uv)$-length path */
19. if $\omega'(uv) = 1$ then $A = A \cup \{uv\}$;
20. else begin
21. let $\ell = \omega'(uv) - 1$ and $N = N \cup \{v_i, \ldots, v_{s,\ell}\}$;
22. $A = A \cup \{uv, v_i v_{s,1}, v_{s,1} v_{s,2}, \ldots, v_{s,\ell-1} v_{s,\ell}, v_{s,\ell} v_{s,1}\}$;
23. end;
24. return $T = (N, A)$;
25. end.

Fig. 4. An algorithm to compute a $(2k, 2k + 2)$-leaf root for interval graphs $G = (V, E)$ with $k < |V|$. First we compute a fixed tolerance NeST model and then we replace the integer weights by paths of according length.

integer unit by an edge. Then every vertex $v$ of $G$ is attached to the node on the backbone that represents the center of the interval corresponding to $v$. To realize the different lengths of intervals, the vertices are attached in varying distances to the backbone; longer intervals are connected in short distance, and vice versa. Using this idea the algorithm in Fig. 4 computes $k$-leaf roots for interval graphs $G = (V, E)$ where $k < 2|V|$.

**Theorem 8.** Every interval graph $G = (V, E)$ is a $(k, k + 2)$-leaf power for $k < 2|V|$, and a corresponding $(k, k + 2)$-leaf root $T = (N, A)$ can be obtained by Algorithm IntervalLeafRoot depicted in Fig. 4.

**Proof.** The algorithm computes a weighted tree $T'$ which has as backbone a path $(p_1, p_2, \ldots, p_n)$ and for all $i \in \{1, \ldots, n\}$ the node $p_i$ is adjacent to the leaf $v_i$.

Let $v_i, v_j \in V$ be two nodes of the list $L$ with $i < j$. Their $T'$-distance is

$$d_{T'}(v_i, v_j) = \omega'(p_i, v_i) + d_{T'}(p_i, p_j) + \omega'(p_j, v_j)$$

$$= L - \ell(v_i) + 1 + I(v_j) - I(v_i) + L - \ell(v_j) + 1$$

$$= 2L + 2 + (I(v_j) - I(v_i)) - (\ell(v_i) + \ell(v_j)).$$

Assume that $uv \in E$. Then there is a clique $C_r, r \in \{1, \ldots, \ell\}$ such that $u, v \in C_r$, i.e., their intervals intersect. By $r \geq s(v_j)$ and $r \leq s(v_i) + \ell(v_i)$ it follows

$$I(v_j) - I(v_i) = 2s(v_i) + \ell(v_i) - 2s(v_i) - \ell(v_i)$$

$$\leq r + \ell(v_i) - s(v_i) - r$$

$$= s(v_j) + \ell(v_j) - s(v_i)$$

$$\leq \ell(v_i) + \ell(v_j) - 2.$$

Thus, $d_{T'}(v_i, v_j) \leq 2L$. 


Reversely, if $uv \notin E$, then $s(v_j) > s(v_i) + \ell(v_i)$ and

$$I(v_j) - I(v_i) = 2s(v_j) + \ell(v_j) - 2s(v_i) - \ell(v_i)$$

$$\geq s(v_i) + \ell(v_i) + s(v_j) + \ell(v_j) - 2s(v_i) - \ell(v_i) + 1$$

$$= s(v_j) + \ell(v_j) - s(v_i) + 1$$

$$\geq \ell(v_i) + \ell(v_j).$$

Thus, $d_{T'}(v_i, v_j) \geq 2L + 2$. Consequently, $(T', 2L)$ is a fixed tolerance NeST model of $G$ where

1. $\omega'$ gives only integer weights,
2. the set of leaves is exactly $V$ and
3. non-adjacent vertices $x, y \in V$ have $d_{T'}(x, y) \geq 2L + 2$.

Finally, we obtain a $(2L, 2L + 2)$-leaf root by replacing each edge $e$ with integer weight $\omega'(e)$ by a path of length $\omega'(e)$. Since obviously $L < |V|$, the claim follows. $\square$

Again, we have shown that interval graphs have “compact” leaf roots. The running time of the algorithm in Fig. 4 is at most quadratic with respect to the number of nodes in $G$. However, it is again not true that computed leaf roots are optimal with respect to the leaf rank of $G$. Certain interval graphs do not obey entirely the linear clique structure. In fact, sometimes it is profitable when certain sets of leaves are not arranged incident to the backbone but instead are sourced out to some additional branch of the leaf root. Thus, another open problem remains, i.e., to find an algorithm which computes leaf rank-optimal leaf roots for interval graphs.

6. Discussion and outlook

In this paper, we have shown that leaf powers coincide with fixed tolerance NeST graphs. This implies that interval graphs and ptolemaic graphs are leaf powers. Moreover, we have shown that rooted directed path graphs are leaf powers. It remains as an open problem whether leaf powers and $k$-leaf powers for $k \geq 6$ can be efficiently recognized.

We also give upper bounds for the leaf rank of ptolemaic graphs and of interval graphs by introducing efficient algorithms for the computation of leaf roots for ptolemaic and interval graphs. However, these algorithms do not guarantee to find the leaf rank of the given graph. The complexity of determining the leaf rank of a given leaf power is an open problem. A first step might be to determine the leaf rank of unit interval graphs.

In [1], it is shown that for rooted directed path graphs, the isomorphism problem can be solved in polynomial time. In contrast, [35] shows that the isomorphism problem on strongly chordal graphs is as hard as on arbitrary graphs. Thus, it is a challenging open problem to determine the complexity of the isomorphism problem on leaf powers.

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